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Chaos and coherence: a new framework for interest-rate modelling

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A set of elementary axioms for stochastic finance is presented wherein a prominent role is played by the state-price density, which in turn determines the stochastic dynamics of the interest-rate term structure. The fact that the state-price density is a potential implies the existence of an asymptotic random variable X_∞ with the property that its conditional variance is the state-price density. The Wiener chaos expansion technique can then be applied to X_∞, thus enabling us to 'parametrize' the dynamics of the discount-bond system in terms of the deterministic coefficients of the chaos expansion. Using this method, we find that there is a natural map from the space of all admissible term-structure trajectories to the symmetric Fock space $\mathfrak{F}$ naturally associated with the space of square-integrable random variables on the underlying probability space. An element of $\mathfrak{F}$ is either coherent or incoherent, and a stochastic bond-price system is necessarily represented by an incoherent element of $\mathfrak{F}$. Making use of the linearity of $\mathfrak{F}$, we derive simple analytic formulae for the bond-price system, the volatility structure, the short rate, and the risk premium associated with an arbitrary admissible term-structure model. Extensions to foreign-exchange markets and general asset systems are also developed.

Keywords: interest-rate models; Heath–Jarrow–Morton theory; Flesaker–Hughston framework; Wiener chaos; coherent-state method

1. Introduction

Over the past three decades, steady progress has been made in understanding the arbitrage-free dynamics for systems of asset prices. In particular, widespread applications have been developed that are based on the 'standard model' of asset price dynamics, according to which asset prices are modelled by continuous semi-martingales on a fixed probability space equipped with a Brownian filtration (see, for example, Karatzas 1997; Karatzas & Shreve 1998). In this respect it is widely held that financial modelling has been one of the most significant applications that has come out of the theory of stochastic differential equations (SDEs).

The dynamical theory of yield curve evolution plays a particularly important role in asset pricing. This is because interest rates influence the price trajectories of all assets. For that reason, it has for a long time been recognized that the discount-bond system should be regarded as a 'primitive' element in the mathematical theory of finance. The challenging feature of the discount-bond system is that one is modelling the dynamics of a one-parameter family of assets, rather than simply a finite set of assets, and it was not until the development of the HJM framework (Heath *et al.* 1992) that a consistent general treatment of the pricing of interest-rate derivatives became available within the context of the standard model.

The HJM approach is to model the system of instantaneous forward rates as a parametric family of Itô processes defined over some fixed time horizon, and to impose conditions on these rates sufficient to ensure no arbitrage. The instantaneous forward rate volatilities and the market price of risk are then left as essentially freely specifiable processes. Thus, within the HJM approach, for a specific model what is required is an exogenous stipulation of these freely specifiable inputs, which, once fed into the SDEs for the instantaneous forward rates, along with initial conditions, completely determine the random dynamics of the interest-rate system. The so-called 'market model' approach to interest-rate derivatives pricing, which has gained considerable popularity in recent years, can be viewed as a further development of the HJM methodology, with the imposition of some additional assumptions on the nature of the volatility processes (see, for example, Rebonato (2002) and references cited therein).

There is, however, another approach to interest-rate dynamics, still consistent with the general tenets of the standard model, that in some respects is rather more mathematically satisfying, and at the same time is somewhat more in tune with modern economic thinking. This is what one might call the 'pricing kernel' approach, the elements of which are variously represented for example in the work of Foldes (1979), Constantinides (1992), Flesaker & Hughston (1996, 1997a, b), Rogers (1997), Rutkowski (1997), Musiela & Rutkowski (1997), Hunt & Kennedy (2000), Jin & Glasserman (2001), Hughston & Rafailidis (2002), Cairns (2004) and others. According to the pricing kernel idea, the primary ingredient in the specification of an interest-rate model is the state-price density (or 'deflator' in the language of Duffie (2001)). Under reasonable economic assumptions, one can show that the state-price density is a positive supermartingale, and indeed, if the time horizon of the model is extended to infinity, a potential.

Intuitively speaking, the attraction of the pricing kernel approach is that one is in some sense working closer to the solution set of the stochastic system. This point is made in more specific terms in Hughston & Rafailidis (2002), where it is shown that the solution of the SDEs corresponding to the arbitrage-free dynamics of a system of discount bonds, including the choice of initial conditions, can be achieved in effect with the specification of an element of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of square-integrable functions on Wiener space.

Here we carry this line of work further by demonstrating the special significance of the interest-rate models associated with 'coherent' vectors in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, in terms of which the general positive interest HJM model can be constructed in a remarkable way by a process of superposition.

The article is organized as follows. An axiomatic formulation of the general arbitrage-free stochastic dynamics of a system of financial assets is reviewed in §§ 2–3 in the context of the standard model. In § 4 we consider the SDE satisfied by the state-price density. As an example, we study the class of interest-rate models for which the bond price can be expressed in the rational form. We show, in particular, that, for rational lognormal models, the associated short rate obeys a simple diffusion equation with polynomial drift and volatility. In § 5 we introduce the idea of an admissible term-structure model, based on the requirements of interest-rate positivity and good asymptotic behaviour. In § 6 we show that there exists a natural representation of the state-price density in terms of the conditional variance of an asymptotic random variable. This idea is developed further in § 7, and the idea of the use of the Wiener chaos expansion for the modelling of term structures is outlined. Examples of finite chaos models are presented in § 8, where we consider the first and second chaos models. The first chaos models, in particular, are deterministic: the space of these models is in one-to-one correspondence with the space of admissible yield curves. A methodology for term-structure analysis based on the notion of coherent states is developed in § 9, and its generalization to the construction of an arbitrary admissible term-structure model, by means of a simple analytical representation, is established in § 10. Specifically, we derive closed form expressions for the bond price, the risk premium, the short rate and the bond volatility. A natural extension to the foreign-exchange market is considered in § 11, where we examine the general relationship between interest-rate processes and exchange-rate volatilities. We conclude in § 12 with a brief remark about the applicability of the chaos expansion and coherent state methods to more general classes of assets.

2. Axiomatic formalism

Let us begin with a summary of the theoretical framework within which we shall examine the dynamics of interest rates. The axiomatic scheme adopted here will be sufficient to support the existence of an arbitrage-free system of discount bonds over all time horizons, while at the same time admitting other categories of assets as well.

We model the economy by a fixed probability space $\varPi = (\varOmega, \mathcal{F}, \mathbb{P})$ equipped with the standard filtration $\varPhi = (\mathcal{F}_t)_{0\leqslant t<\infty}$ generated by a system of one or more independent Wiener processes $(W_t^\alpha)_{0\leqslant t<\infty}$ $(\alpha = 1, \dots, k)$. The probability measure $\mathbb{P}$ is to be interpreted as the natural measure, and filtration-dependent concepts, such as the martingale property, are defined relative to $\varPhi$. We assume that the random trajectories followed by asset prices are continuous, and are described by Itô processes on $\varPi$. The absence of arbitrage in the economy will be characterized by the existence of an Itô process V_t, which we call the *state-price density*, satisfying $V_t > 0$ for all $t \in [0, \infty)$, with the property that the following three axioms hold.

(i) There exists a strictly increasing absolutely continuous asset, denoted B_t, that represents the money-market account.

(ii) Given the price process S_t and the dividend rate D_t of any asset, the process M_t defined by

$$M_t = V_t S_t + \int_0^t V_s D_s \, \mathrm{d}s \tag{2.1}$$

is a martingale.

(iii) There exists a floating-rate note that offers a continuous dividend rate sufficient to keep the value of the note constant.

The bond market is then introduced with the following additional assumption.

(iv) There exists a system of discount bonds P_{tT} for all $0 \leqslant t \leqslant T < \infty$ satisfying $\lim_{T\to\infty} P_{tT} = 0$.

The axiomatic scheme considered here is essentially that of Hughston & Rafailidis (2002). We deduce from axiom (i) that there exists an adapted short-rate process $r_t > 0$ such that the money-market account satisfies

$$\mathrm{d}B_t = r_t B_t \, \mathrm{d}t, \tag{2.2}$$

and thus is given by

$$B_t = B_0 \exp\biggl(\int_0^t r_s \, \mathrm{d}s\biggr). \tag{2.3}$$

Because the money-market account pays no dividend, it follows from axiom (ii) that the process Λ_t defined by

$$\Lambda_t = V_t B_t \tag{2.4}$$

is a positive martingale, and hence satisfies a stochastic equation of the form

$$\mathrm{d}\Lambda_t = -\Lambda_t \lambda_t \, \mathrm{d}W_t \tag{2.5}$$

for some adapted vector-valued process λ_t. Note that in (2.5) and in what follows we use a compact notation and there is an implied dot product over vectorial quantities in the sense that

$$\lambda_t \, \mathrm{d}W_t = \sum_{\alpha=1}^{k} \lambda_t^\alpha \, \mathrm{d}W_t^\alpha. \tag{2.6}$$

Integrating the SDE (2.5), we obtain

$$\Lambda_t = \Lambda_0 \exp\biggl(-\int_0^t \lambda_s \, \mathrm{d}W_s - \frac{1}{2}\int_0^t \lambda_s^2 \, \mathrm{d}s\biggr), \tag{2.7}$$

provided that $\int_0^\infty \lambda_s^2 \, \mathrm{d}s < \infty$ almost surely.

If a risky asset S_t has positive value and pays no dividend, then it follows from axiom (ii) that the price process can be expressed in the form $S_t = M_t/V_t$, where M_t is a positive martingale. Therefore, there exists an adapted vector-valued process σ_t such that $\mathrm{d}M_t = M_t(\sigma_t - \lambda_t)\,\mathrm{d}W_t$. By combining this equation with the relations (2.2) and (2.5), we deduce that the price process of a risky asset that pays no dividend obeys

$$\mathrm{d}S_t = S_t(r_t + \lambda_t \sigma_t)\,\mathrm{d}t + S_t \sigma_t \, \mathrm{d}W_t. \tag{2.8}$$

This is the stochastic equation that governs the dynamics of a risky asset with limited liability in a market absent of arbitrage opportunities. The vector process λ_t has the interpretation of being the market price of risk per unit of volatility, and thus the inner product $\lambda_t \sigma_t$ is the extra rate of return offered by the asset above the risk free rate r_t.

When the dividend rate is non-zero, the dynamics can be obtained by making the substitution $r_t \to r_t - \delta_t$ in (2.8), where $\delta_t = D_t/S_t$ is the proportional dividend rate, and thus in this case we have

$$\mathrm{d}S_t = S_t(r_t - \delta_t + \lambda_t \sigma_t)\,\mathrm{d}t + S_t \sigma_t \, \mathrm{d}W_t. \tag{2.9}$$

In the case when S_t can also assume negative values (for example, short positions or swap contracts), a further generalization exists that eliminates the 'proportional' volatilities and dividends and involves only the corresponding absolute quantities (see Hughston & Rafailidis 2002).

Let us now examine axiom (iii). Such an asset with constant value has the interpretation of being a floating rate note. Equation (2.9) shows that if we set $S_t = 1$ for all $t \geqslant 0$, then the dividend rate offered by this instrument must be r_t. It follows from (2.1) that the process

$$M_t = V_t + \int_0^t r_s V_s \, \mathrm{d}s \tag{2.10}$$

is a martingale. In particular, because r_t and V_t are both assumed to be positive processes we deduce as a consequence of axiom (iii) that

$$\mathbb{E}[V_t] < \infty \tag{2.11}$$

and that

$$\mathbb{E}\left[\int_0^t r_s V_s \, \mathrm{d}s\right] < \infty \tag{2.12}$$

for all $t \geqslant 0$. We remark, incidentally, that because 'cash' clearly has a constant value in units of the given base currency, it follows that the convenience yield associated with the possession of cash is identical to the short rate.

3. Price processes for discount bonds

We now introduce a system of discount bonds associated with the base currency in terms of which other assets on Π are priced. We shall make it a condition that the economy supports such a system of discount bonds over all time horizons. The T-maturity discount-bond price processes will be denoted P_{tT}, where $0 \leqslant t \leqslant T < \infty$.

The zero-coupon bond for a given value of T is a contract that pays one unit of the base currency at time T. Thus, by the definition of the contract we require $P_{TT} = 1$ for all $T \geqslant 0$. For the moment we make no other assumptions concerning the discount-bond processes apart from those properties implicit in axioms (i)–(iii), though later we shall bring into play the additional assumption, stipulated in axiom (iv), concerning the asymptotic behaviour of the bond prices.

A discount bond pays no dividend. It therefore follows from axiom (ii) that $V_t P_{tT}$ is a martingale, and hence that there exists a family of positive martingales M_{tT} such that $P_{tT} = M_{tT} B_t / \Lambda_t$. Since M_{tT} is a positive martingale for each maturity date T, we infer the existence of a vector-valued process Ω_{tT} such that M_{tT} satisfies the following stochastic equation

$$\mathrm{d}M_{tT} = M_{tT}(\Omega_{tT} - \lambda_t) \, \mathrm{d}W_t. \tag{3.1}$$

It then follows from equations (2.5) and (3.1) that the dynamics of the discount-bond system are given by

$$\mathrm{d}P_{tT} = P_{tT}(r_t + \lambda_t \Omega_{tT}) \, \mathrm{d}t + P_{tT} \Omega_{tT} \, \mathrm{d}W_t. \tag{3.2}$$

By integrating (3.2) and making use of the relation $P_{TT}=1$, then, providing that

$$\int_0^T \Omega_{sT}^2 \,\mathrm{d}s < \infty$$

almost surely, we are able to deduce that the discount-bond price processes can be represented in the form

$$P_{tT} = P_{0T}\exp\biggl(\int_0^t \lambda_s \Omega_{sT}\,\mathrm{d}s + \int_0^t \Omega_{sT}\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \Omega_{sT}^2\,\mathrm{d}s\biggr) \times \biggl[P_{0t}\exp\biggl(\int_0^t \lambda_s \Omega_{st}\,\mathrm{d}s + \int_0^t \Omega_{st}\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \Omega_{st}^2\,\mathrm{d}s\biggr)\biggr]^{-1}, \quad (3.3)$$

and that the money-market account process is given by a corresponding expression of the form

$$B_t = B_0\biggl[P_{0t}\exp\biggl(\int_0^t \lambda_s \Omega_{st}\,\mathrm{d}s + \int_0^t \Omega_{st}\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \Omega_{st}^2\,\mathrm{d}s\biggr)\biggr]^{-1}. \quad (3.4)$$

Formulae (3.3) and (3.4) are well known and can be found, for example, in Hughston (1996) and Flesaker & Hughston (1997*a*, *b*). It is evident by inspection that (3.3) satisfies the boundary condition $P_{TT}=1$ for all $T \geqslant 0$, and also correctly incorporates the initial discount function.

Note that in formulae (3.3) and (3.4) both the discount-bond system and the money-market account are given directly in terms of the market risk premium process λ_t and the bond volatility process Ω_{tT}, together with the initial discount function P_{0T}, without direct reference to the short rate r_t. It is therefore possible, at least in principle, to regard λ_t and Ω_{tT} as being *exogenous* variables. Historically, this observation is of significance because it forms the basis of the approach to interest-rate derivatives pricing that has until recently been most frequently used by practitioners. In such an approach one typically assumes market completeness, then transforms to the risk-neutral measure to eliminate the market risk premium, and then models the bond volatility process exogenously, calibrating it to a suitable given set of market interest-rate option data. It is a problematic feature of the volatility modelling approach that if λ_t and Ω_{tT} are specified exogenously, then there is no guarantee that axiom (i) is satisfied—that is to say, the resulting interest rates need not be positive. The problem of finding an explicit expression for the bond volatility Ω_{tT} necessary and sufficient to ensure positive interest is addressed in Brody & Hughston (2002), where the most general form of the volatility structure compatible with axiom (i) is derived.

Because discount bonds pay no dividends, it follows as a consequence of axiom (ii) that the martingale relations

$$V_t P_{tT} = \mathbb{E}_t[V_u P_{uT}] \quad (3.5)$$

hold for all $0 \leqslant t \leqslant u \leqslant T < \infty$. Here $\mathbb{E}_t[-]$ denotes conditional expectation with respect to the σ-algebra $\mathcal{F}_t$. Setting $u=T$ in (3.5) we then obtain the bond-pricing formula

$$P_{tT} = \frac{1}{V_t}\mathbb{E}_t[V_T]. \quad (3.6)$$

Thus, once axioms (i)–(iii) have been specified, the associated discount-bond system is also fully determined.

It is interesting to note, as was shown by Baxter (1997), that the inequality (2.12), which follows from axiom (iii), is the additional assumption required to ensure the differentiability of the bond-price system with respect to the maturity date. In other words, as a consequence of (2.12) there exists a family of Itô processes f_{tu}, adapted to the filtration Φ, for all $0 \leqslant t \leqslant u < \infty$, such that

$$P_{tT} = \exp\biggl(-\int_t^T f_{tu}\,\mathrm{d}u\biggr). \tag{3.7}$$

It then follows that $-\partial_T \ln P_{tT} = f_{tT}$, where ∂_T denotes differentiation with respect to T, and also that $\lim_{t\to T} f_{tT} = r_T$ and $\lim_{t\to T} \Omega_{tT} = 0$. The existence of the instantaneous forward rates f_{tT}, as thus defined, means that the class of interest-rate models under consideration here is equivalent to the family of all positive interest HJM models (Heath *et al*. 1992) defined over all time horizons.

4. Properties of the state-price density

Let us consider in more detail the properties of the state-price density that hold under our axiomatic scheme. According to (2.4), V_t is given by the expression

$$V_t = \Lambda_t \exp\biggl(-\int_0^t r_s\,\mathrm{d}s\biggr). \tag{4.1}$$

As a consequence of equations (2.5) and (4.1), we deduce that the stochastic equation satisfied by the state-price density is

$$\mathrm{d}V_t = -r_t V_t\,\mathrm{d}t - \lambda_t V_t\,\mathrm{d}W_t. \tag{4.2}$$

Thus, the specification of V_t is sufficient to determine both the short rate r_t and the risk-premium vector λ_t. The associated discount-bond system is then given by equation (3.6).

Example 4.1 (rational models). A simple class of interest-rate models can be obtained by choosing the state-price density to be of the form $V_t = \alpha_t + \beta_t m_t$, where m_t is a positive martingale with $m_0 = 1$, and α_t and β_t are positive decreasing functions such that $\alpha_T + \beta_T = P_{0T}$ for some prescribed initial discount function. A calculation then shows that the short rate in this model is given by

$$r_t = -\frac{\dot{\alpha}_t + \dot{\beta}_t m_t}{\alpha_t + \beta_t m_t}, \tag{4.3}$$

where $\dot{\alpha}_t = \partial\alpha_t/\partial t$, $\dot{\beta}_t = \partial\beta_t/\partial t$, and that for the risk premium we have

$$\lambda_t = -\frac{\beta_t m_t \nu_t}{\alpha_t + \beta_t m_t}, \tag{4.4}$$

where ν_t is given by $\mathrm{d}m_t = m_t \nu_t\,\mathrm{d}W_t$. It follows directly from equation (3.6) that the associated discount-bond system can be expressed in the *rational* form

$$P_{tT} = \frac{\alpha_T + \beta_T m_t}{\alpha_t + \beta_t m_t}. \tag{4.5}$$

Returning now to the discussion of the state-price density, we observe that if we integrate the SDE (4.2) from t to T, we obtain

$$V_T = V_t - \int_t^T r_s V_s \, \mathrm{d}s - \int_t^T \lambda_s V_s \, \mathrm{d}W_s. \tag{4.6}$$

Taking the conditional expectation of each side of this formula then by use of (2.10) gives us the relation

$$\mathbb{E}_t[V_T] = V_t - \mathbb{E}_t\left[\int_t^T r_s V_s \, \mathrm{d}s\right]. \tag{4.7}$$

Because the second term on the right-hand side of (4.7) is necessarily positive for $T > t$, we deduce that

$$\mathbb{E}_t[V_T] < V_t, \tag{4.8}$$

which shows that V_t is a *supermartingale*. In particular, it follows as a consequence of the bond-pricing formula (3.6) together with equation (4.7) that P_{tT} satisfies the *positive interest* conditions

$$0 < P_{tT} \leqslant 1 \quad \text{and} \quad \frac{\partial}{\partial T} P_{tT} < 0. \tag{4.9}$$

Example 4.2 (rational lognormal models). In the context of example 4.1 above, if ν_t is chosen to be deterministic, then m_t has a lognormal distribution for each value of t. The resulting *rational lognormal model* is Markovian, and the short-rate process r_t is the relevant state variable (Flesaker & Hughston 1996; see also Rutkowski (1997) and Musiela & Rutkowski (1997)). It is interesting to note more generally that if z_t is any rational process

$$z_t = \frac{A_t + B_t m_t}{C_t + D_t m_t}, \tag{4.10}$$

where A_t, B_t, C_t and D_t are deterministic, then we have

$$m_t = -\frac{A_t - C_t z_t}{B_t - D_t z_t}. \tag{4.11}$$

If m_t is lognormally distributed martingale, i.e. it satisfies $\mathrm{d}m_t = m_t \nu_t \, \mathrm{d}W_t$, where ν_t is deterministic, then it is a straightforward calculation to establish a lemma to the effect that z_t is a *diffusion* process. In fact, z_t satisfies the following diffusion-type SDE:

$$\begin{aligned}\mathrm{d}z_t = \Bigg[&\frac{\dot{A}_t B_t - A_t \dot{B}_t}{B_t C_t - A_t D_t} + \left(\frac{\dot{B}_t C_t - B_t \dot{C}_t + A_t \dot{D}_t - \dot{A}_t D_t}{B_t C_t - A_t D_t}\right) z_t + \left(\frac{\dot{C}_t D_t - C_t \dot{D}_t}{B_t C_t - A_t D_t}\right) z_t^2 \\ &+ \frac{D_t \nu_t^2}{(B_t C_t - A_t D_t)^2} \begin{pmatrix} D_t C_t^2 z_t^3 - (2A_t C_t D_t + B_t C_t^2) z_t^2 \\ + (A_t^2 D_t + 2A_t B_t C_t) z_t - A_t^2 B_t \end{pmatrix}\Bigg] \mathrm{d}t \\ &+ \left[\frac{C_t D_t z_t^2 - (A_t D_t + B_t C_t) z_t + A_t B_t}{A_t D_t - B_t C_t}\right] \nu_t \, \mathrm{d}W_t.\end{aligned} \tag{4.12}$$

We note that the absolute drift of z_t is a cubic polynomial in z_t with deterministic coefficients, and the absolute volatility is quadratic in z_t. In particular, by examination of the roots of the quadratic polynomial, we infer that the volatility term vanishes if $z_t = B_t/D_t$ or $z_t = A_t/C_t$, corresponding to the two limiting values for $m_t \to \infty$ and $m_t \to 0$, respectively. For rational lognormal interest-rate models, we make the substitutions $A_t \to -\dot{\alpha}_t$, $B_t \to -\dot{\beta}_t$, $C_t \to \alpha_t$, $D_t \to \beta_t$, and $z_t \to r_t$ to obtain the diffusion equation satisfied by the short-rate process. A similar substitution gives the differential equation satisfied by the bond prices. Rational diffusions are of considerable interest in their own right and may have applications to areas outside of finance, for example, to mathematical biology.

5. Admissible term structures

The quotient $K_{tT} = V_T/V_t$ can be regarded as the *pricing kernel* for the given financial market (cf. Constantinides 1992). For example, if H_t for $t \in [0,T]$ is the price process of a European-style derivative that has the random pay-out H_T at maturity T, then it follows from axiom (ii) that H_t is given by

$$H_t = \mathbb{E}_t[K_{tT}H_T]. \tag{5.1}$$

In particular, if the pay-off of the derivative is one unit of base currency at time T, then we recover the bond-price formula (3.6).

Note, incidentally, that if we divide both sides of (4.7) by V_t, then we obtain an interesting expression for the discount bond, which carries an economic interpretation. This is given by

$$P_{tT} = 1 - \mathbb{E}_t\left[\int_t^T K_{ts}r_s\,\mathrm{d}s\right]. \tag{5.2}$$

Thus, we see that the value of the discount bond at time t is equal to unity, less the value, at that time, of the dividend flow between times t and T.

We now consider the limit $T \to \infty$. It should be intuitively evident on economic grounds that we wish to impose a condition of the form

$$\lim_{T\to\infty} P_{tT} = 0 \tag{5.3}$$

almost surely, on the discount-bond system, in accordance with axiom (iv), corresponding to the idea that a unit of cash received at an infinite time has no value. Equivalently, we see by virtue of (5.2) in the limit $T \to \infty$ that the present value of a floating-rate note on a unit principle always has value unity:

$$\mathbb{E}_t\left[\int_t^\infty K_{ts}r_s\,\mathrm{d}s\right] = 1. \tag{5.4}$$

A term structure that satisfies the positive interest conditions (4.9) and the asymptotic condition (5.3) will be called an *admissible term structure*. It is a feature of the axiomatic formalism we have adopted here that the resulting interest-rate term structure is necessarily admissible.

The space of admissible term structures turns out to have a fascinating *geometry* associated with it. This comes about as follows. Suppose we introduce the so-called

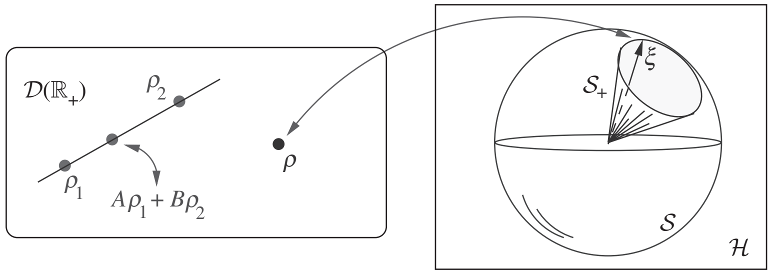

Figure 1. The system of admissible term structures. A positive interest term structure can be regarded as a point in $\mathcal{D}(\mathbb{R}^1_+)$, the convex space of density functions on $\mathbb{R}^1_+$. There is a ray ξ, associated with each point $\rho \in \mathcal{D}(\mathbb{R}^1_+)$, in the positive orthant $\mathcal{S}_+$ of the unit sphere $\mathcal{S}$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^1_+)$. A dynamical trajectory on $\mathcal{D}(\mathbb{R}^1_+)$ can then be mapped to a corresponding trajectory in $\mathcal{S}_+$.

Musiela parameter $x = T - t$ for the tenor of a discount bond. The process $\rho_t(x)$ defined for each value of $x \geqslant 0$ by the relation

$$\rho_t(x) = -\frac{\partial}{\partial x} P_{t,t+x} \tag{5.5}$$

then has the characteristics of a *density function*. In particular, one easily verifies on account of (5.3) and (4.9) that

$$\rho_t(x) > 0 \quad \text{and} \quad \int_{x=0}^{\infty} \rho_t(x)\,\mathrm{d}x = 1. \tag{5.6}$$

The bond price P_{tT} can then be interpreted as the *tail distribution* of $\rho_t(x)$, in the sense that

$$\int_{x=0}^{T-t} \rho_t(x) = 1 - P_{tT}. \tag{5.7}$$

It follows from the defining equation (5.5) that the term structure density $\rho_t(x)$ is the product of the instantaneous forward rate and the discount function itself. Now, clearly, if $\rho_1(x)$ and $\rho_2(x)$ are admissible term-structure densities, and if A and B are non-negative constants satisfying $A + B = 1$, then $A\rho_1(x) + B\rho_2(x)$ is also an admissible term-structure density. Putting these ingredients together, we see that the term structure of interest rates can be given the following general characterization: the system of admissible term structures is isomorphic to the convex space $\mathcal{D}(\mathbb{R}^1_+)$ of density functions on the positive real line. Furthermore, by considering the square-root map $\rho_t(x) \to \xi_t(x) = \sqrt{\rho_t(x)}$, we see that the dynamical evolution of the term structure can be cast as a stochastic process taking values on the unit sphere in Hilbert space (see figure 1).

See Brody & Hughston (2001*a*, *b*, 2002) for further details of the 'term-structure density' approach and the information-theoretic yield curve calibration methodology that arises naturally in association with this idea. Related approaches to characterizing the space of yield curves as a geometric entity have been pursued by Björk and collaborators (Björk & Christensen 1999; Björk & Gombani 1999; Björk 2001), Filipović (2000) and Filipović & Teichmann (2004).

6. The conditional variance representation

As we have seen, for an admissible term structure we require that the value of a discount bond must vanish in the limit that the maturity date approaches infinity. As a consequence of relation (5.3), this implies that $\lim_{T\to\infty} \mathbb{E}_t[V_T] = 0$, almost surely, and thus also that $\mathbb{E}[V_T] \to 0$ as $T \to \infty$. A positive supermartingale with the property that its expectation approaches zero asymptotically is known as a *potential* (see, for example, Meyer 1966). Thus, for every admissible term structure, the corresponding state-price density is a potential (Rogers 1997).

Interestingly, one can take this scheme a step further by noting that the state-price density can be expressed as a conditional variance of a square-integrable random variable X_∞, as is pointed out in Hughston & Rafailidis (2002). The Wiener chaos expansion can then be used to model the random variable X_∞ in order to formulate a wide class of admissible interest-rate models. The idea can be sketched as follows. If we rearrange (5.4) slightly we obtain the relation

$$V_t = \mathbb{E}_t\left[\int_t^\infty r_s V_s \, \mathrm{d}s\right]. \tag{6.1}$$

Therefore, if we let η_t be a vector-valued process satisfying $\eta_t^2 = r_t V_t$ and write

$$X_\infty = \int_0^\infty \eta_s \, \mathrm{d}W_s, \tag{6.2}$$

then a short calculation shows that the state-price density is given by the *conditional variance* of X_∞, defined by

$$V_t = \mathbb{E}_t[(X_\infty - \mathbb{E}_t[X_\infty])^2]. \tag{6.3}$$

Thus, given the random variable X_∞, we can recover the state-price density and hence the corresponding admissible discount-bond system. Conversely, for a given state-price density V_t there is an equivalence class of random variables X_∞ with the given conditional variance V_t. We shall refer any element of such an equivalence class as a *generator* for the corresponding term-structure model.

7. Chaos expansion for term structures

As we indicated above, a particularly useful way of modelling the random variable X_∞ is by means of the Wiener chaos expansion. The idea of the chaos expansion technique (Wiener 1938; Itô 1951) can be informally sketched as follows. Let X_T be an $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[X_T^2] < \infty$. Then from the martingale representation theorem we know that there exists a square-integrable process $\theta_1(s_1)$, $0 \leqslant s_1 \leqslant T$, satisfying

$$\mathbb{E}\left[\int_0^T \theta_1^2(s_1) \, \mathrm{d}s_1\right] \leqslant \mathbb{E}[X_T^2], \tag{7.1}$$

such that X_T can be expressed in the form

$$X_T = \mathbb{E}[X_T] + \int_0^T \theta_1(s_1) \, \mathrm{d}W_{s_1}. \tag{7.2}$$

We now apply the representation theorem once again to the process $\theta_1(s_1)$, which implies the existence of a square-integrable process $\theta_2(s_1, s_2)$, $0 \leqslant s_2 \leqslant s_1$, satisfying

$$\mathbb{E}\left[\int_0^T \theta_2^2(s_1, s_2)\,\mathrm{d}s_2\right] \leqslant \mathbb{E}[\theta_1^2(s_1)], \tag{7.3}$$

such that

$$\theta_1(s_1) = \mathbb{E}[\theta_1(s_1)] + \int_0^{s_1} \theta_2(s_1, s_2)\,\mathrm{d}W_{s_2}. \tag{7.4}$$

Another application of the representation theorem then expresses $\theta_2(s_1, s_2)$ in terms of the process $\theta_3(s_1, s_2, s_3)$, and so on. By iteration we thus find that, if we write

$$\phi_{s_1 s_2 \cdots s_n} = \mathbb{E}[\theta_n(s_1, s_2, \ldots, s_n)], \tag{7.5}$$

then the random variable X_T can be expanded as a series in the form

$$X_T = \phi + \int_0^T \phi_{s_1}\,\mathrm{d}W_{s_1} + \int_0^T \left(\int_0^{s_1} \phi_{s_1 s_2}\,\mathrm{d}W_{s_2}\right)\mathrm{d}W_{s_1} + \cdots , \tag{7.6}$$

where $\phi = \mathbb{E}[X_T]$, and it can be shown that this series converges to X_T.

The deterministic quantities ϕ, ϕ_{s_1}, $\phi_{s_1 s_2}$, and so on, appearing in (7.6) are called the *Wiener chaos coefficients* for the expansion of the given random variable X_T. The point is that the 'information' in X_T is captured entirely by this series of deterministic functions. For further discussion of the chaos expansion technique and its role in stochastic analysis, see, for example, Ikeda & Watanabe (1989), Nualart (1995), Janson (1997), Malliavin (1997) and Øksendal (1997).

In the context of interest-rate theory, the random variable X_∞ is given by (6.2), and for its conditional expectation $X_t = \mathbb{E}_t[X_\infty]$ we have

$$X_t = \int_0^t \eta_s\,\mathrm{d}W_s. \tag{7.7}$$

Note, in particular, that from (5.2) we obtain

$$\frac{1}{V_0}\mathbb{E}[X_T^2] = 1 - P_{0T}, \tag{7.8}$$

from which it follows, by taking the limit $T \to \infty$, that $\mathbb{E}[X_\infty^2] = V_0$. Thus, for any representative choice of the process η_t satisfying $\eta_s^2 = r_s V_s$, there is an expansion of the form

$$\eta_{s_1} = \phi_{s_1} + \int_0^{s_1} \phi_{s_1 s_2}\,\mathrm{d}W_{s_2} + \int_0^{s_1} \left(\int_0^{s_2} \phi_{s_1 s_2 s_3}\,\mathrm{d}W_{s_3}\right)\mathrm{d}W_{s_2} + \cdots . \tag{7.9}$$

Let us now define the martingale M_{ts} for each fixed $s > t$ by the conditional expectation

$$M_{ts} = \mathbb{E}_t[\eta_s^2]. \tag{7.10}$$

From (6.3) the state-price density is given by the integral

$$V_t = \int_t^\infty M_{ts}\,\mathrm{d}s, \tag{7.11}$$

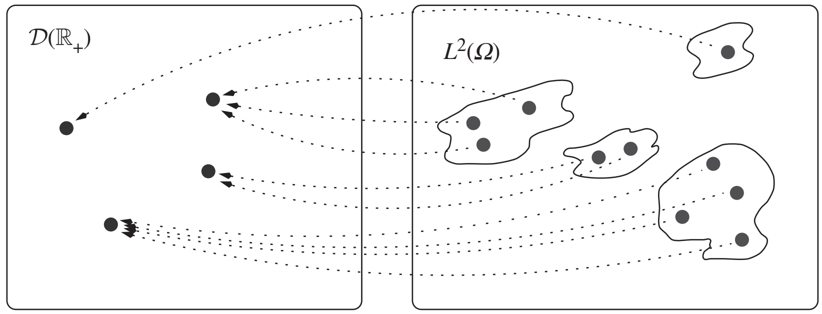

Figure 2. Every element of the 'quantized' Wiener space $\mathfrak{F} = L^2(\Omega)$ corresponds to a random variable X_∞. The conditional variance of X_∞ then determines the state-price density V_t. Each state-price density thus obtained determines an admissible initial term structure, represented by a term-structure density function in the convex space $\mathcal{D}(\mathbb{R}_+)$ of density functions on $\mathbb{R}^1_+$. The mapping $L^2(\Omega) \to \mathcal{D}(\mathbb{R}_+)$ is many-to-one; that is to say, different points of $L^2(\Omega)$ can give rise to the same term structure in $\mathcal{D}(\mathbb{R}_+)$.

and we see, as a consequence, that the bond-price system admits a representation of the Flesaker–Hughston form

$$P_{tT} = \left(\int_T^\infty M_{ts}\,\mathrm{d}s\right)\left(\int_t^\infty M_{ts}\,\mathrm{d}s\right)^{-1}. \tag{7.12}$$

Therefore, once the deterministic functions $\phi_{s_1 s_2 \cdots s_n}$ that form the expansion coefficients are specified, they uniquely determine the martingales M_{ts}, which in turn determine the state-price density and the discount-bond system. For further details of the representation (7.12), see, for example, Flesaker & Hughston (1996, 1998), Rutkowski (1997), Musiela & Rutkowski (1997), Hunt & Kennedy (2000) or Jin & Glasserman (2001).

8. First and second chaos models

A series of specific interest-rate models of varying degrees of complexity can be constructed in the situation where there are only a finite number of non-vanishing chaos coefficients. The first chaos models, in particular, are completely deterministic in the sense that both the state-price density and the discount-bond system are deterministic. The generating random variable arising in the case of a first chaos model has a centred normal distribution. The second chaos models are no longer deterministic; in this case the random variable admits a distribution given by a linear combination of χ^2-distributed variables.

Example 8.1 (first chaos models). If only the first chaos coefficient α_s is non-vanishing, so that the random variable is given by

$$X_\infty = \int_0^\infty \alpha_s\,\mathrm{d}W_s, \tag{8.1}$$

then it follows by virtue of the Itô isometry that the state-price density is deterministic and is given by

$$V_t = \int_t^\infty \alpha_s^2\,\mathrm{d}s. \tag{8.2}$$

The corresponding bond-price system

$$P_{tT} = \left(\int_T^\infty \alpha_s^2 \, \mathrm{d}s \right) \left(\int_t^\infty \alpha_s^2 \, \mathrm{d}s \right)^{-1} \tag{8.3}$$

is also deterministic. We observe in particular that there exists a many-to-one map from the first Wiener chaos to the space of admissible yield curves (see figure 2). We note that $|\alpha_s|$, when suitably normalized, has the interpretation of the square root of the term-structure density. For instance, if we set

$$\alpha_s = \sqrt{R} \mathrm{e}^{-Rs/2} \tag{8.4}$$

in the case of a one-factor model, where R is a positive constant, then we find that $P_{tT} = \mathrm{e}^{-R(T-t)}$. More generally, in the case of a one-factor model, if we take

$$\alpha_s = \sqrt{\frac{1}{\Gamma(\kappa)}} R^{\kappa/2} s^{(\kappa-1)/2} \mathrm{e}^{-Rs/2}, \tag{8.5}$$

the square root of the gamma distribution, then a short calculation shows that

$$P_{tT} = \frac{\Gamma(\kappa, RT)}{\Gamma(\kappa, Rt)}, \tag{8.6}$$

where the symbol

$$\Gamma(a, z) = \int_z^\infty t^{a-1} \mathrm{e}^{-t} \, \mathrm{d}t \tag{8.7}$$

denotes the standard incomplete gamma function.

Example 8.2 (factorizable second chaos models). The special second chaos interest-rate models for which the random variable X_∞ is of the factorizable form

$$X_\infty = \int_0^\infty \alpha_{s_1} \, \mathrm{d}W_{s_1} + \int_0^\infty \beta_{s_1} \left(\int_0^{s_1} \gamma_{s_2} \, \mathrm{d}W_{s_2} \right) \mathrm{d}W_{s_1}, \tag{8.8}$$

where α_s, β_s and γ_s are deterministic functions of one variable, are considered in Hughston & Rafailidis (2002). In particular, we note that this model has the useful property that *analytic formulae for European-style bond options and swaptions can be obtained explicitly.* The state-price density in this case is given by an expression of the form

$$V_t = f_t + g_t R_t + h_t (R_t^2 - Q_t), \tag{8.9}$$

and the corresponding expression for the discount-bond system is

$$P_{tT} = \frac{f_T + g_T R_t + h_T (R_t^2 - Q_t)}{f_t + g_t R_t + h_t (R_t^2 - Q_t)}. \tag{8.10}$$

Here, the deterministic functions f_t, g_t and h_t are defined respectively by the expressions

$$f_t = \int_t^\infty (\alpha_s^2 + \beta_s^2 Q_s) \, \mathrm{d}s, \qquad g_t = 2 \int_t^\infty \alpha_s \beta_s \, \mathrm{d}s, \qquad h_t = \int_t^\infty \beta_s^2 \, \mathrm{d}s.$$

The Gaussian martingale

$$R_t = \int_0^t \gamma_s \, \mathrm{d}W_s$$

is the sole state variable, and the deterministic process

$$Q_t = \int_0^t \gamma_s^2 \, \mathrm{d}s$$

is its quadratic variation. The initial discount function in this model is given by $P_{0T} = f_T/f_0$.

9. Coherent interest-rate models

If each coefficient $\phi_{s_1 s_2 \cdots s_n}$ in the chaos expansion factorizes into a product of n functions of the variables $s_1, s_2, \ldots, s_n$, then the resulting term-structure dynamics simplifies, and in some cases we can create models that are analytically tractable. The factorizable second chaos model considered above provides a good example of this. In this section we examine the simplest factorizable model that involves *all* terms in the chaos expansion. For simplicity we consider the case when $\mathcal{F}_t$ is generated by a single Brownian motion, and the resulting interest-rate models are thus one-factor models.

When the Wiener chaos coefficients for the random variable X_∞ take the special form

$$\phi_{s_1 s_2 \cdots s_n} = \phi_{s_1} \phi_{s_2} \cdots \phi_{s_n} \tag{9.1}$$

for each n, where ϕ_s is a deterministic square-integrable function of one variable, we call the resulting interest-rate system a *coherent term-structure model*. The term 'coherent' is used here by analogy with a well-known construction in laser physics, where similar mathematical ideas are used. In the case of a coherent term structure, we can take advantage of a formula due to Itô (1951) stating that

$$\int_0^s \int_0^{s_1} \cdots \int_0^{s_{n-1}} \phi_{s_1} \phi_{s_2} \cdots \phi_{s_n} \, \mathrm{d}W_{s_n} \cdots \mathrm{d}W_{s_2} \, \mathrm{d}W_{s_1} = \|\phi\|_s^n H_n\left(\frac{\Phi_s}{\|\phi\|_s}\right), \tag{9.2}$$

where

$$\Phi_s = \int_0^s \phi_u \, \mathrm{d}W_u, \qquad \|\phi\|_s^2 = \int_0^s \phi_u^2 \, \mathrm{d}u,$$

and $H_n(x)$ denotes the nth Hermite polynomial, which we define by

$$H_n(x) = \frac{1}{n!}(-1)^n \mathrm{e}^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2/2}. \tag{9.3}$$

Using this identity and the orthogonality relation

$$\mathbb{E}[H_n(X) H_m(Y)] = \frac{1}{n!} \delta_{nm} (\mathbb{E}[XY])^n \tag{9.4}$$

satisfied by the Hermite polynomials for any pair of $N(0,1)$ Gaussian random variables X and Y, one can then proceed to derive a closed form expression for the state-price density.

Alternatively, we can take a shortcut with the following observation. Let the $\mathcal{F}_T$-measurable random variable X_T be given by

$$X_T = \exp\left(\int_0^T f(s)\,\mathrm{d}W_s\right) \tag{9.5}$$

for some fixed choice of T, possibly infinite, where $f(s)$ is a deterministic square-integrable function. We would like to determine the chaos expansion of this random variable. First we recall that the generating function for the Hermite polynomials is given by

$$\exp(tx - \tfrac{1}{2}t^2) = \sum_{n=0}^{\infty} t^n H_n(x), \tag{9.6}$$

which implies that

$$\exp\left(\int_0^T f(s)\,\mathrm{d}W_s - \tfrac{1}{2}\|f\|_T^2\right) = \sum_{n=0}^{\infty} \|f\|_T^n H_n\left(\frac{F_T}{\|f\|_T}\right), \tag{9.7}$$

where

$$F_T = \int_0^T f(s)\,\mathrm{d}W_s \quad \text{and} \quad \|f\|_T^2 = \int_0^T f^2(s)\,\mathrm{d}s.$$

As a consequence, by use of (9.2) we see that the nth coefficient in the chaos expansion of X_T in (9.5) is given by

$$\phi_{s_1 s_2 \cdots s_n} = \exp(\tfrac{1}{2}\|f\|_T^2) f(s_1) f(s_2) \cdots f(s_n). \tag{9.8}$$

This result leads to the observation that the generator of a coherent term-structure model is given by an expression of the form

$$X_\infty = \exp\left(\int_0^\infty \phi_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^\infty \phi_s^2\,\mathrm{d}s\right). \tag{9.9}$$

It is then a straightforward exercise in stochastic calculus to calculate the conditional variance of X_∞, and thus to verify that the state-price density in the case of a coherent term-structure model is given by

$$V_t = \left(\exp\left(\int_0^\infty \phi_s^2\,\mathrm{d}s\right) - \exp\left(\int_0^t \phi_s^2\,\mathrm{d}s\right)\right) \exp\left(2\int_0^t \phi_s\,\mathrm{d}W_s - 2\int_0^t \phi_s^2\,\mathrm{d}s\right). \tag{9.10}$$

It should be evident that V_t factorizes into the product of two processes: an exponential martingale and a deterministic decreasing process. Thus, the associated bond-price system is deterministic:

$$P_{tT} = \left[1 - \exp\left(-\int_T^\infty \phi_s^2\,\mathrm{d}s\right)\right]\left[1 - \exp\left(-\int_t^\infty \phi_s^2\,\mathrm{d}s\right)\right]^{-1}. \tag{9.11}$$

Likewise, the short rate is also deterministic, and is given in this case by

$$r_t = \phi_t^2\left(\exp\left(\int_t^\infty \phi_s^2\,\mathrm{d}s\right) - 1\right)^{-1}. \tag{9.12}$$

Despite the fact that the interest-rate system is deterministic, the valuation of an option still requires taking an expectation, because the state-price density V_t itself is not deterministic in this model. For example, let us consider a European-style bond option with pay-off $H_t = (P_{tT} - K)^+$ at time t. For $0 \leqslant s \leqslant t$ we then have the valuation formula

$$C_s = V_s^{-1}\mathbb{E}_s[V_t(P_{tT} - K)^+], \tag{9.13}$$

from which we deduce in this case that $C_s = P_{st}(P_{tT} - K)^+$. This is the correct expression for the pay-off to be received at time t, discounted back to time s, where P_{tT} is given in (9.11).

It is interesting to note, incidentally, the parallelism between the coherent term structure studied above and the properties of coherent light in quantum optics. In optics a coherent state is defined to be a state for which all the correlation functions of the electric field factorize, and the associated field becomes deterministic (see, for example, Glauber 1963). In the case of a coherent term structure, the chaos expansion factorizes, and the associated discount-bond system becomes deterministic.

10. Chaos and coherence

We have observed here that the coherent term-structure models admit analytical tractability, and that the resulting interest-rate systems are deterministic. This implies that the coherent term-structure models are themselves insufficient to characterize the random dynamics of the interest-rate markets. Nevertheless, it is possible to develop an analysis based on coherent term structures that is sufficiently general to treat an essentially arbitrary arbitrage-free term-structure evolution.

To see this, let us return to the general Wiener–Itô expansion of square-integrable random variables. We consider the space $\mathfrak{F}$ spanned by square-integrable vectors of the form $(f, f_{s_1}, f_{s_1 s_2}, f_{s_1 s_2 s_3}, \dots)$, where the function $f_{s_1 s_2 s_3 \cdots s_n}$ belongs to the space of a tensor product of n copies of a Hilbert space of square-integrable functions restricted to the subdiagonal domain for which $s_1 \geqslant s_2 \geqslant s_3 \geqslant \cdots \geqslant s_n$. The linear inner product space $\mathfrak{F}$ is known as a *Fock space*, and the results of § 7 show that any admissible term-structure model can be represented as an element of $\mathfrak{F}$. The coherent elements of $\mathfrak{F}$, which are expressible in the exponentiated form

$$\boldsymbol{C}_\phi = (1, \phi_{s_1}, \phi_{s_1}\phi_{s_2}, \phi_{s_1}\phi_{s_2}\phi_{s_3}, \dots), \tag{10.1}$$

play a particularly important role in this connection, because a general element of $\mathfrak{F}$ can be expressed as a linear combination of coherent vectors (see figure 3). In other words, any term-structure model can be represented by a linear superposition of a finite or infinite, possibly uncountable number of coherent term-structure models. We note incidentally that the inner product of a pair of coherent vectors in $\mathfrak{F}$ is given by

$$\begin{aligned}(\boldsymbol{C}_\phi, \boldsymbol{C}_\psi) &= 1 + \int_0^\infty \phi_{s_1}\psi_{s_1}\,\mathrm{d}s_1 + \int_0^\infty \int_0^{s_1} \phi_{s_1}\phi_{s_2}\psi_{s_1}\psi_{s_2}\,\mathrm{d}s_1\,\mathrm{d}s_2 + \cdots \\ &= 1 + \int_0^\infty \phi_{s_1}\psi_{s_1}\,\mathrm{d}s_1 + \frac{1}{2}\int_0^\infty \int_0^\infty \phi_{s_1}\phi_{s_2}\psi_{s_1}\psi_{s_2}\,\mathrm{d}s_1\,\mathrm{d}s_2 + \cdots \\ &= \exp\left(\int_0^\infty \phi_s\psi_s\,\mathrm{d}s\right), \end{aligned} \tag{10.2}$$

which shows that no pair of coherent vectors is ever orthogonal. Thus, the space $\mathcal{C}$ of coherent vectors forms an over complete basis for $\mathfrak{F}$. In particular, there exists a resolution of the identity operator on $\mathfrak{F}$ that allows an arbitrary element of $\mathfrak{F}$ to be represented as a superposition of elements of $\mathcal{C}$.

Let us consider the simplest non-trivial combination of coherent elements, given by $a\boldsymbol{C}_\phi + b\boldsymbol{C}_\psi$. Note that a linear combination of coherent vectors cannot itself be expressed in the coherent form (10.1). The result of § 9 thus implies that any realistic term structure is necessarily represented by an *incoherent* element of $\mathfrak{F}$. The linearity of the Fock space $\mathfrak{F}$ implies that, when the chaos expansion is given by $a\boldsymbol{C}_\phi + b\boldsymbol{C}_\psi$, the corresponding random variable takes the form

$$X_\infty = a\exp\biggl(\int_0^\infty \phi_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^\infty \phi_s^2\,\mathrm{d}s\biggr) + b\exp\biggl(\int_0^\infty \psi_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^\infty \psi_s^2\,\mathrm{d}s\biggr). \tag{10.3}$$

Now for any square-integrable function f_s let us define the associated exponential martingale M_t^f according to the scheme

$$M_t^f = \exp\biggl(\int_0^t f_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^t f_s^2\,\mathrm{d}s\biggr), \tag{10.4}$$

and a corresponding positive deterministic, decreasing function Δ_t^f by

$$\Delta_t^f = \exp\biggl(\int_0^\infty f_s^2\,\mathrm{d}s\biggr) - \exp\biggl(\int_0^t f_s^2\,\mathrm{d}s\biggr). \tag{10.5}$$

It is then a straightforward exercise to calculate the conditional variance of the expression for X_∞ given in (10.3), and thus to verify that

$$V_t = a^2\Delta_t^\phi M_t^{2\phi} + b^2\Delta_t^\psi M_t^{2\psi} + 2ab\Delta_t^{\sqrt{\phi\psi}} M_t^{\phi+\psi}. \tag{10.6}$$

The corresponding expression for the bond price is

$$P_{tT} = \frac{a^2\Delta_T^\phi M_t^{2\phi} + b^2\Delta_T^\psi M_t^{2\psi} + 2ab\Delta_T^{\sqrt{\phi\psi}} M_t^{\phi+\psi}}{a^2\Delta_t^\phi M_t^{2\phi} + b^2\Delta_t^\psi M_t^{2\psi} + 2ab\Delta_t^{\sqrt{\phi\psi}} M_t^{\phi+\psi}}. \tag{10.7}$$

We note that P_{tT} depends on the weighting factors a and b through the ratio a/b. Unlike the case of a coherent term structure, stochasticity is inherent in an incoherent term structure.

More generally, if the chaos expansion of a term-structure model is given by an incoherent element of the form

$$X_\infty = \sum_k c_k \boldsymbol{C}_{\phi_k}, \tag{10.8}$$

then the state-price density admits a simple representation of the form

$$V_t = \sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}. \tag{10.9}$$

From this expression, the corresponding bond prices

$$P_{tT} = \frac{\sum_{k,l} c_k c_l \Delta_T^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}{\sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}} \tag{10.10}$$

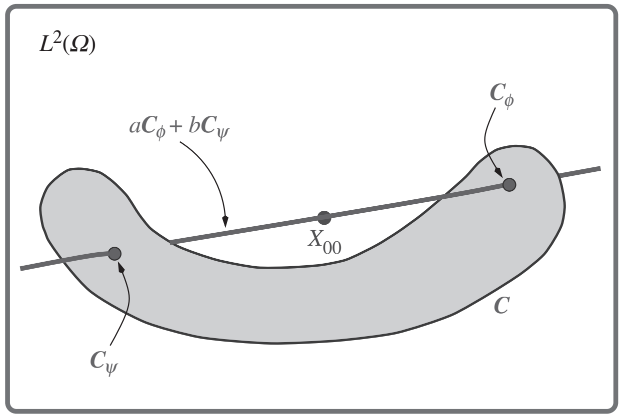

Figure 3. In the Fock space $\mathfrak{F} = L^2(\Omega)$ there is a submanifold $\mathcal{C}$ of coherent vectors corresponding to deterministic term-structure models. Given a pair $\boldsymbol{C}_\phi$, $\boldsymbol{C}_\psi$ of coherent vectors their linear span, expressible in the form $a\boldsymbol{C}_\phi + b\boldsymbol{C}_\psi$, is a line that joins $\boldsymbol{C}_\phi$ and $\boldsymbol{C}_\psi$. This line meets $\mathcal{C}$ at only these two points. An element of $\mathfrak{F}$ obtained by a linear combination of coherent vectors is necessarily incoherent, and thus generates a stochastic term-structure model. More generally, by use of the resolution of the identity operator it is possible to express any element of $\mathfrak{F}$ as a superposition of elements of $\mathcal{C}$.

and other related dynamical variables can be derived explicitly. For example, the short-rate process is given by

$$r_t = \left[\sum_{k,l} c_k c_l \phi_{kt}\phi_{lt} \exp\left(\int_0^t \phi_{ks}\phi_{ls}\,\mathrm{d}s\right) M_t^{\phi_k+\phi_l}\right]\left[\sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}\right]^{-1}, \tag{10.11}$$

whereas the risk premium vector can be expressed in the form

$$\lambda_t = -\frac{\sum_{k,l} c_k c_l (\phi_{kt} + \phi_{lt}) \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}{\sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}. \tag{10.12}$$

The discount-bond volatility, on the other hand, is

$$\Omega_{tT} = \lambda_t + \frac{\sum_{k,l} c_k c_l (\phi_{kt} + \phi_{lt}) \Delta_T^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}{\sum_{k,l} c_k c_l \Delta_T^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}. \tag{10.13}$$

Taking this expression together with the expression (10.12), we see at once that the maturity condition $\Omega_{TT} = 0$ is satisfied. The forward short rates are

$$f_{tT} = \left[\sum_{k,l} c_k c_l \phi_{kT}\phi_{lT} \exp\left(\int_0^T \phi_{ks}\phi_{ls}\,\mathrm{d}s\right) M_t^{\phi_k+\phi_l}\right]\left[\sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}\right]^{-1}. \tag{10.14}$$

Analogously, we can obtain an expression for the simple forward rates L_{tab}, which are determined by the relation

$$\frac{P_{tb}}{P_{ta}} = \frac{1}{1 + (b-a)L_{tab}}. \tag{10.15}$$

From (10.10) it follows at once that

$$L_{tab} = \frac{1}{b-a}\left(\frac{\sum_{k,l} c_k c_l \Delta_a^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}{\sum_{k,l} c_k c_l \Delta_b^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}} - 1\right). \tag{10.16}$$

By taking the stochastic differential of this expression, we can check that the dynamics of the simple forward rates L_{tab} obey the well-known market model equation (Brace *et al*. 1997)

$$\frac{\mathrm{d}L_{tab}}{L_{tab}} = \frac{1+(b-a)L_{tab}}{(b-a)L_{tab}}(\Omega_{ta}-\Omega_{tb})(\mathrm{d}W_t - (\Omega_{tb}-\lambda_t)\,\mathrm{d}t), \tag{10.17}$$

by virtue of formula (10.13). The initial term structure determined by (10.10) is given by

$$P_{0T} = \left[\sum_{k,l} c_k c_l \Delta_T^{\sqrt{\phi_k\phi_l}}\right]\left[\sum_{k,l} c_k c_l \left(\exp\left(\int_0^\infty \phi_{ks}\phi_{ls}\,\mathrm{d}s\right)-1\right)\right]^{-1}, \tag{10.18}$$

and the associated initial short rate is

$$r_0 = \left(\sum_k c_k\phi_{k0}\right)^2\left[\sum_{k,l} c_k c_l \left(\exp\left(\int_0^\infty \phi_{ks}\phi_{ls}\,\mathrm{d}s\right)-1\right)\right]^{-1}. \tag{10.19}$$

The pricing of derivatives is also straightforward. For instance, the value at time $s \leqslant t$ of a bond option with pay-off $H_t = (P_{tT}-K)^+$ at time t is given by the valuation formula (9.13), which in this example takes the form

$$C_s = V_s^{-1}\mathbb{E}_s\left[\left(\sum_{k,l} c_k c_l (\Delta_T^{\sqrt{\phi_k\phi_l}} - K\Delta_t^{\sqrt{\phi_k\phi_l}})M_t^{\phi_k+\phi_l}\right)^+\right]. \tag{10.20}$$

This expression is similar to the valuation formula for a standard basket option, for which well-established numerical methods have been developed (see, for example, Turnbull & Wakeman 1991; Curran 1994).

Although linear combinations of coherent elements will exhaust all possible admissible term-structure models, we are not confined to adopt the coherent-state decomposition in order to generate interest-rate models. For instance, coherent elements can be combined with finite chaos elements, as the next example shows.

Example 10.1 (mixture models). Consider a chaos expansion of X_∞ in terms of the combination of a coherent element $\boldsymbol{C}_\phi$ and a first chaos vector of $\mathfrak{F}$ of the form $(0, \alpha_s, 0, 0, \dots)$ in the case of a single-factor economy. It follows from the linearity of $\mathfrak{F}$ that

$$X_\infty = \int_0^\infty \alpha_s\,\mathrm{d}W_s + \exp\left(\int_0^\infty \phi_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^\infty \phi_s^2\,\mathrm{d}s\right). \tag{10.21}$$

In order to obtain the state-price density V_t we must calculate the conditional variance of X_∞. Now, the contributions arising from each term in the right-hand side of (10.21) can be determined straightforwardly. The contribution from the cross term, on the other hand, can be obtained by changing the probability measure. Alternatively, we can introduce an auxiliary variable χ and write

$$\mathbb{E}_t\bigg[\bigg(\int_0^\infty \alpha_s\,\mathrm{d}W_s\bigg)\exp\bigg\{\int_0^\infty \phi_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^\infty \phi_s^2\,\mathrm{d}s\bigg\}\bigg] = \frac{\partial}{\partial\chi}\mathbb{E}_t\bigg[\exp\bigg\{\int_0^\infty (\phi_s+\chi\alpha_s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^\infty \phi_s^2\,\mathrm{d}s\bigg\}\bigg]\bigg|_{\chi=0}. \quad (10.22)$$

Then we find at once that the bond price in this model is given by

$$P_{tT} = \bigg[\int_T^\infty \alpha_s^2\,\mathrm{d}s + \Delta_T^\phi M_t^{2\phi} + \bigg(\int_T^\infty \alpha_s\phi_s\,\mathrm{d}s\bigg)M_t^\phi\bigg] \times \bigg[\int_t^\infty \alpha_s^2\,\mathrm{d}s + \Delta_t^\phi M_t^{2\phi} + \bigg(\int_t^\infty \alpha_s\phi_s\,\mathrm{d}s\bigg)M_t^\phi\bigg]^{-1}, \quad (10.23)$$

where we make use of the definitions (10.4) and (10.5). In the special case for which the supports of the functions α_s and ϕ_s share no overlap, equation (10.23) takes the form (4.5), and we recover a rational lognormal model. We note that the method used in (10.22) is effective in general when we consider models that are generated by combinations of finite and coherent elements.

11. Foreign-exchange systems

The interest-rate framework described here generalizes in a natural way to the situation where we also include a foreign-exchange system with a family of discount bonds associated with each currency. In particular, a chaos representation exists for the stochastic dynamics of the entirety of such an international system of interest rates and foreign exchange.

We adopt here the conventions of Flesaker & Hughston (1997*b*), and write S_t^{ij} for the price of one unit of currency i in units of currency j, where $i, j = 0, 1, \dots, N$, and we may for convenience think of the label $i = 0$ as referring to the particular base currency with respect to which the axioms (i)–(iv) are formulated. There is no special significance to the choice of base currency, and the entire system is symmetrical in the ensemble of currencies. For each currency we assume that there exists a strictly increasing absolutely continuous money-market asset B_t^i, with a corresponding strictly positive short-rate process r_t^i such that

$$B_t^i = B_0^i \exp\bigg(\int_0^t r_s^i\,\mathrm{d}s\bigg). \quad (11.1)$$

We also require the existence of a floating-rate note in each currency: that is to say, for each value of i we assume the existence of an asset of constant value in units of currency i, paying a dividend at the rate r_t^i.

If we write S_t^{i0} for the value of one unit of currency i in units of the base currency, we see that the product $S_t^{i0}B_t^i$ represents the base-currency price of an asset that pays no dividend. Therefore, by axiom (ii) we deduce for each value of i that $V_tS_t^{i0}B_t^i$ is a martingale, from which it follows that $V_tS_t^{i0}$ is a supermartingale, since B_t^i is increasing. Thus, if we define $V_t^i = V_t S_t^{i0}$, then, from the cyclic property $S_t^{ij} S_t^{j0} = S_t^{i0}$ that holds in the case of a frictionless exchange-rate system, we deduce that

$$S_t^{ij} = \frac{V_t^i}{V_t^j}. \tag{11.2}$$

This gives us a general expression for the exchange-rate process as a ratio of supermartingales (Rogers 1997; Flesaker & Hughston 1997*b*). As a consequence we find that the dynamics of the exchange-rate matrix S_t^{ij} are given (see, for example, Lipton 2001) by the following system of SDEs:

$$\frac{\mathrm{d}S_t^{ij}}{S_t^{ij}} = [r_t^j - r_t^i + \lambda_t^j(\lambda_t^j - \lambda_t^i)]\,\mathrm{d}t + (\lambda_t^j - \lambda_t^i)\,\mathrm{d}W_t. \tag{11.3}$$

Here λ_t^i denotes the market price of risk process associated with assets that are denominated in currency i. Thus, we see that the axiomatic scheme, when extended to include a money-market account and a floating-rate note for each foreign currency, is sufficient to generate the general foreign-exchange dynamics between the currencies.

Consider now the discount-bond system for foreign currency number i, and denote by P_{tT}^i the value at time t of a bond that pays one unit of currency i at time T. In this case $S_t^{i0} P_{tT}^i$ is the price in base currency of an asset that pays no dividend, and therefore $V_t S_t^{i0} P_{tT}^i$ is a martingale by axiom (ii). It follows that $V_t S_t^{i0} P_{tT}^i = \mathbb{E}_t[V_T S_T^{i0} P_{TT}^i]$. Because $V_t S_t^{i0} = V_t^i$ and $P_{TT}^i = 1$, we thus conclude that each foreign bond system admits a representation of the form

$$P_{tT}^i = \frac{\mathbb{E}_t[V_T^i]}{V_t^i}. \tag{11.4}$$

To proceed further, we make the assumption that $\lim_{T\to\infty} P_{0T}^i = 0$ for each currency. Then a conditional variance representation exists for the state-price density associated with each currency. In other words, there exists a system of random variables X_∞^i $(i = 0, 1, \dots, N)$, belonging to $L^2(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$V_t^i = \mathbb{E}_t[(X_\infty^i - \mathbb{E}_t[X_\infty^i])^2], \tag{11.5}$$

and each of these random variables admits a chaos representation. Thus, once the random variables X_∞^i have been specified for $i = 0, 1, \dots, N$, the international system of interest rates and foreign exchange is determined by (11.2), (11.4) and (11.5).

Let us now, as an illustration, examine the special case when each of the state-price densities V_t^i is generated by a coherent element of $\mathfrak{F}$. In this situation, each currency is characterized by a deterministic interest rate r_t^i and a deterministic bond-price system P_{tT}^i. However, the exchange-rate evolution is not deterministic, and is characterized rather by an extended geometric Brownian motion with a deterministic local volatility and drift. More specifically, let $\boldsymbol{C}_{\phi^i}$ generate the dynamics of the currency associated with the state-price density V_t^i. Then V_t^i takes the form of (9.10) for each i, and as a consequence we obtain

$$\mathrm{d}V_t^i = -\left(\phi_t^{i2}\left[\exp\left(\int_t^\infty \phi_s^{i2}\,\mathrm{d}s\right) - 1\right]^{-1}\right) V_t^i\,\mathrm{d}t + 2\phi_t^i V_t^i\,\mathrm{d}W_t \tag{11.6}$$

for the SDE satisfied by the state-price density associated with currency number i. Therefore, in the case of such an 'internationally' coherent exchange-rate system, we find that the exchange-rate volatility $\lambda_t^j - \lambda_t^i$ for the currency pair (i, j) is given by the simple expression

$$\lambda_t^j - \lambda_t^i = 2(\phi_t^i - \phi_t^j), \tag{11.7}$$

while the short rate for currency i is given by

$$r_t^i = \phi_t^{i2}\left[\exp\left(\int_t^\infty \phi_s^{i2}\,\mathrm{d}s\right) - 1\right]^{-1}. \tag{11.8}$$

Although equation (11.8) is not obviously invertible in closed form, it is nevertheless evident that there is a simple relationship in this model between the short rate and the exchange-rate volatility.

The single-factor coherent exchange model presented above is overly determined in the sense that there are not enough exogenously specifiable independent degrees of freedom. We can nevertheless improve the situation by considering the multi-factor extension of the above model, as indicated in the following example.

Example 11.1 (two-factor coherent exchange models). Consider a foreign-exchange system consisting of two currencies, say, sterling (GBP) represented by £ and the yen (JPY) represented by ¥, and assume that the term structure is given by a two-factor coherent model. We observe first that, even in the multi-factor context, the random variable X_∞ for the coherent term structure and the corresponding state-price density V_t admit representations (9.9) and (9.10), respectively, where the function ϕ_t and the Wiener process W_t are now vectorial quantities. In the present context, we take

$$\phi_t^{£} = (f_t, g_t) \quad \text{and} \quad \phi_t^{¥} = (h_t, 0), \tag{11.9}$$

where f_t, g_t and h_t are $L^2(\mathbb{R}_+)$-functions, and consider the exchange rate $S_t^{£¥}$. We have two factors for one of the currencies and one factor for the other, because we can absorb one of the degrees of freedom into the others without loss of generality. Now, in this market, there are four invariant scalar quantities, namely, the short rate for each currency, the risk premium for the exchange rate $S_t^{£¥}$, and the squared length of the volatility vector. It is a straightforward exercise to verify, for a two-factor coherent term-structure model, that we have

$$r_t^{£} = (f_t^2 + g_t^2)\left(\exp\left(\int_t^\infty (f_s^2 + g_s^2)\,\mathrm{d}s\right) - 1\right)^{-1} \tag{11.10}$$

and

$$r_t^{¥} = h_t^2\left(\exp\left(\int_t^\infty h_s^2\,\mathrm{d}s\right) - 1\right)^{-1} \tag{11.11}$$

for the short rates,

$$\lambda_t^{¥}(\lambda_t^{¥} - \lambda_t^{£}) = 4(h_t - f_t)h_t \tag{11.12}$$

for the risk premium, and

$$(\lambda_t^{¥} - \lambda_t^{£})^2 = 4(f_t^2 + g_t^2 + h_t^2 - 2f_t h_t) \tag{11.13}$$

for the squared length of the exchange-rate volatility vector. Therefore, in this model the exchange rate evolves stochastically, and we have three exogenously specifiable inputs determining four scalar market quantities.

The fact that there are relationships between the interest rates of the various currencies and the corresponding exchange-rate volatilities is valid in a more general context, although the correspondences are less transparent when the term structure of each currency is characterized by an incoherent element of $\mathfrak{F}$. In the general case, the exchange-rate matrix process can be expressed in the form

$$S_t^{ij} = \frac{\sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k \phi_l}} M_t^{\phi_k+\phi_l}}{\sum_{k,l} d_k d_l \Delta_t^{\sqrt{\psi_k \psi_l}} M_t^{\psi_k+\psi_l}}. \tag{11.14}$$

It should be evident that the relevant short rates r_t^i can be expressed in the form of (10.11), and the risk premium processes λ_t^i in the form (10.12), from which an indirect relationship between the short rates and the exchange volatilities can then be established.

12. Discussion

We have demonstrated here how Wiener chaos expansion techniques can be applied to the interest rate and foreign exchange markets, an idea originally proposed by Hughston & Rafailidis (2002). We then introduced the succulent idea of the coherent-state method to demonstrate that even a highly stochastic and complex market structure will admit a simple analytical representation for the various associated market observables. We conclude by noting that the foregoing formulation can be further extended to characterize the price of any asset in terms of another, providing that these prices are always positive and that we interpret the associated short-rate systems as continuous dividend streams. As a consequence we infer that the 'generic' model for an asset price is a process of the form

$$S_t = \frac{\mathrm{var}_t[Y_\infty]}{\mathrm{var}_t[X_\infty]}. \tag{12.1}$$

In other words, the price S_t is given by a ratio of conditional variances, where X_∞ and Y_∞ are elements of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The simplest one-factor models leading to a non-trivial asset price stochasticity on this basis are those for which X_∞ admits a first chaos expansion with coefficient ψ_t, and Y_∞ admits a coherent chaos expansion with coefficient ϕ_t. For example, if we set $\phi_t = \frac{1}{2}\sigma_t$ for a square-integrable function σ_t, then we find that the asset price trajectory is given by an extended geometric Brownian motion of the form

$$S_t = S_0 \exp\left(\int_0^t \sigma_s \, \mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2 \, \mathrm{d}s + \int_0^t (r_s - \delta_s) \, \mathrm{d}s\right), \tag{12.2}$$

where

$$S_0 = \left[\exp\left(\frac{1}{4}\int_0^\infty \sigma_s^2 \, \mathrm{d}s\right) - 1\right]\left[\int_0^\infty \psi_s^2 \, \mathrm{d}s\right]^{-1} \tag{12.3}$$

is the initial asset price, and the two deterministic rates r_t and δ_t are given by

$$r_t = \psi_t^2 \left[\int_t^\infty \psi_s^2 \, \mathrm{d}s\right]^{-1} \quad \text{and} \quad \delta_t = \tfrac{1}{4}\sigma_t^2 \left[\exp\left(\frac{1}{4}\int_t^\infty \sigma_s^2 \, \mathrm{d}s\right) - 1\right]^{-1}, \tag{12.4}$$

respectively. Such an example is perhaps rather artificial; nevertheless it should be evident that there are milliards of analytically tractable models that one can formulate in the present framework. It would be interesting in that connection to pursue some econometric studies of these models.

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