

QUANTUM STATES AND SPACE-TIME CAUSALITY

DORJE C. BRODY AND LANE P. HUGHSTON

ABSTRACT. Space-time symmetries and internal quantum symmetries can be placed on equal footing in a hyperspin geometry. Four-dimensional classical space-time emerges as a result of a decoherence that disentangles the quantum and the space-time degrees of freedom. A map from the quantum space-time to classical space-time that preserves the causality relations of space-time events is necessarily a density matrix.

1. INTRODUCTION

This article presents a programme for the unification of space-time and internal quantum symmetries. An important role is played in this theory by certain higher-dimensional analogues of spinors. In four-dimensional space-time there is a local isomorphism between the Lorentz group $SO(1, 3)$ and the spin transformation group $SL(2, \mathbb{C})$. In higher dimensions, however, this relation breaks down and we are left with two concepts of spinors—one for the groups $SO(N, \mathbb{C})$, and one for the groups $SL(r, \mathbb{C})$. The spinors associated with $SO(N, \mathbb{C})$ are the so-called Cartan spinors. The study of Cartan spinors has a long history, and there is a beautiful geometry associated with these spinors. The spinors associated with $SL(r, \mathbb{C})$, called ‘hyperspinors’, have the advantage of being more directly linked with quantum mechanics. In fact, a relativistic model for hyperspin arises when one considers ‘multiplets’ of two-component spinors, i.e. expressions of the form $\xi^{\mathbf{A}i}$ and $\eta_i^{\mathbf{A}'}$, where \mathbf{A}, \mathbf{A}' are standard spinor indices and $i = 1, 2, \dots, n$ is an ‘internal’ index. In the general case ($n = \infty$) we can think of $\xi^{\mathbf{A}i}$ as an element of the tensor product space $\mathbb{S}^{\mathbf{A}i} = \mathbb{S}^{\mathbf{A}} \otimes \mathbb{H}^i$, where $\mathbb{S}^{\mathbf{A}}$ is the space of two-component spinors, and \mathbb{H}^i is an infinite-dimensional complex Hilbert space.

The theory of hyperspin constitutes a natural starting place for building up a theory of quantum geometry or, as we shall call it here, *quantum space-time*. The hyperspinor route has the virtue that the resulting higher-dimensional space-time has a rich causal structure associated with it, and as a consequence is well-positioned to form the geometrical basis of a physical theory.

2. RELATIVISTIC CAUSALITY

Let us review briefly the role of two-component spinors in the description of four-dimensional Minkowski space. We use bold upright Roman letters to denote two-component spinor indices, and we adopt the standard conventions for the algebra of two-component spinors [10]. Then we have the following correspondence between two-by-two Hermitian matrices $x^{\mathbf{A}\mathbf{A}'}$ ($\mathbf{A}, \mathbf{A}' = 1, 2$) and the positions x^a ($a = 0, 1, 2, 3$) of space-time points relative to some origin. More explicitly, in a standard basis this correspondence is given by

$$(1) \quad \frac{1}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \longleftrightarrow (t, x, y, z).$$

We thus obtain the fundamental relation $2 \det(x^{\mathbf{A}\mathbf{A}'}) = t^2 - x^2 - y^2 - z^2$. It follows that two-component spinors are connected both with quantum mechanics and with

the causal structure of space-time. It is a peculiar aspect of relativistic physics that there is a link of this nature between the spin degrees of freedom of spin one-half particles, and the causal geometry of four-dimensional space-time.

For the interval between a pair of points $x^{\mathbf{AA}'}$ and $y^{\mathbf{AA}'}$ in space-time we write $r^{\mathbf{AA}'} = x^{\mathbf{AA}'} - y^{\mathbf{AA}'}$. It follows that $2 \det(r^{\mathbf{AA}'}) = \epsilon_{\mathbf{AB}} \epsilon_{\mathbf{A}'\mathbf{B}'} r^{\mathbf{AA}'} r^{\mathbf{BB}'}$, where $\epsilon_{\mathbf{AB}}$ is the antisymmetric spinor. If we adopt the ‘index clumping’ convention and write $\mathbf{a} = \mathbf{AA}'$, $\mathbf{b} = \mathbf{BB}'$, and so on, whereby a pair of spinor indices, one primed and the other unprimed, corresponds to a space-time vector index, then we can write $\epsilon_{\mathbf{AB}} \epsilon_{\mathbf{A}'\mathbf{B}'} r^{\mathbf{AA}'} r^{\mathbf{BB}'}$ = $g_{\mathbf{ab}} r^{\mathbf{a}} r^{\mathbf{b}}$ for the corresponding squared space-time interval. Thus we identify $g_{\mathbf{ab}} = \epsilon_{\mathbf{AB}} \epsilon_{\mathbf{A}'\mathbf{B}'}$ as the metric of Minkowski space.

There are three different situations that can arise for the interval $r^{\mathbf{a}}$. The first case is $g_{\mathbf{ab}} r^{\mathbf{b}} = 0$; the second case is $g_{\mathbf{ab}} r^{\mathbf{b}} \neq 0$ and $g_{\mathbf{ab}} r^{\mathbf{a}} r^{\mathbf{b}} = 0$; and the third case is $g_{\mathbf{ab}} r^{\mathbf{a}} r^{\mathbf{b}} \neq 0$. Each of these cases gives rise to a canonical form for the interval $r^{\mathbf{AA}'}$, with various sub-cases, which can be summarised as follows: (i) $g_{\mathbf{ab}} r^{\mathbf{b}} = 0$ implies $r^{\mathbf{AA}'} = 0$ (zero separation); (ii) $g_{\mathbf{ab}} r^{\mathbf{a}} r^{\mathbf{b}} = 0$ implies $r^{\mathbf{AA}'} = \alpha^{\mathbf{A}} \bar{\alpha}^{\mathbf{A}'}$ (future-pointing null separation) or $r^{\mathbf{AA}'} = -\alpha^{\mathbf{A}} \bar{\alpha}^{\mathbf{A}'}$ (past-pointing null separation); (iii) $g_{\mathbf{ab}} r^{\mathbf{a}} r^{\mathbf{b}} \neq 0$ implies $r^{\mathbf{AA}'} = \alpha^{\mathbf{A}} \bar{\alpha}^{\mathbf{A}'} + \beta^{\mathbf{A}} \bar{\beta}^{\mathbf{A}'}$ (future-pointing time-like separation), $r^{\mathbf{AA}'} = \alpha^{\mathbf{A}} \bar{\alpha}^{\mathbf{A}'} - \beta^{\mathbf{A}} \bar{\beta}^{\mathbf{A}'}$ (space-like separation), or $r^{\mathbf{AA}'} = -\alpha^{\mathbf{A}} \bar{\alpha}^{\mathbf{A}'} - \beta^{\mathbf{A}} \bar{\beta}^{\mathbf{A}'}$ (past-pointing time-like separation). Once the canonical form for $r^{\mathbf{AA}'}$ is specified, so is the causal relationship that it determines.

3. HYPERSPIN SPACES AND QUANTUM SPACE-TIME

In the case of hyperspinors (introduced by Finkelstein [4], cf. also [5, 7, 8]) we replace the two-component spinors of four-dimensional space-time with r -component spinors. We can regard hyperspin space as the vector space \mathbb{C}^r with some extra structure. In particular, in addition to the primary hyperspin space we have three other vector spaces—the dual hyperspin space, the complex-conjugate hyperspin space, and the dual complex-conjugate hyperspin space.

Let us write \mathbb{S}^A and $\mathbb{S}^{A'}$ for the complex r -dimensional vector spaces of unprimed and primed hyperspinors. For hyperspinors we use italic indices to distinguish them from the boldface indices used for two-component spinors. We assume that \mathbb{S}^A and $\mathbb{S}^{A'}$ are related by an anti-linear isomorphism under complex conjugation. Thus if $\alpha^A \in \mathbb{S}^A$, then $\alpha^A \rightarrow \bar{\alpha}^{A'}$ under complex conjugation, where $\bar{\alpha}^{A'} \in \mathbb{S}^{A'}$. The dual spaces associated with \mathbb{S}^A and $\mathbb{S}^{A'}$ are denoted \mathbb{S}_A and $\mathbb{S}_{A'}$. If $\alpha^A \in \mathbb{S}^A$ and $\beta_A \in \mathbb{S}_A$, then their inner product is denoted $\alpha^A \beta_A$; if $\gamma^{A'} \in \mathbb{S}^{A'}$ and $\delta_{A'} \in \mathbb{S}_{A'}$ then their inner product is $\gamma^{A'} \delta_{A'}$. We also introduce the totally antisymmetric hyperspinors of rank r associated with \mathbb{S}^A , \mathbb{S}_A , $\mathbb{S}^{A'}$, and $\mathbb{S}_{A'}$. These are denoted $\epsilon^{AB\dots C}$, $\epsilon_{AB\dots C}$, $\epsilon^{A'B'\dots C'}$, and $\epsilon_{A'B'\dots C'}$, and satisfy the relations $\epsilon^{AB\dots C} \epsilon_{AB\dots C} = r!$, $\epsilon^{A'B'\dots C'} \epsilon_{A'B'\dots C'} = r!$, and $\epsilon_{A'B'\dots C'} = \bar{\epsilon}^{A'B'\dots C'}$.

Next we introduce the complex matrix space $\mathbb{C}^{AA'} = \mathbb{S}^A \otimes \mathbb{S}^{A'}$. An element $x^{AA'} \in \mathbb{C}^{AA'}$ is said to be *real* if it satisfies the (weak) Hermitian property $x^{AA'} = \bar{x}^{A'A}$. We shall have more to say about weak versus strong Hermiticity in connection with the idea of symmetry breaking. We denote the vector space of real elements of $\mathbb{C}^{AA'}$ by $\mathbb{R}^{AA'}$. The points of $\mathbb{R}^{AA'}$ constitute what we call the quantum space-time \mathcal{H}^{r^2} of dimension r^2 . We regard $\mathcal{CH}^{r^2} = \mathbb{C}^{AA'}$ as the complexification of \mathcal{H}^{r^2} . Many problems in \mathcal{H}^{r^2} are best first approached as problems in \mathcal{CH}^{r^2} .

Let $x^{AA'}$ and $y^{AA'}$ be points in \mathcal{H}^{r^2} , and write $r^{AA'} = x^{AA'} - y^{AA'}$ for their separation vector, which is independent of the choice of origin. Using the index-clumping convention we set $x^{\mathbf{a}} = x^{AA'}$, $y^{\mathbf{a}} = y^{AA'}$, $r^{\mathbf{a}} = r^{AA'}$, and for the separation of $x^{\mathbf{a}}$ and $y^{\mathbf{a}}$ in \mathcal{H}^{r^2} we write $r^{\mathbf{a}} = x^{\mathbf{a}} - y^{\mathbf{a}}$. There is a natural causal structure induced on such intervals by Finkelstein’s *chronometric tensor* [4], defined by the

relation $g_{ab\dots c} = \varepsilon_{AB\dots C} \varepsilon_{A'B'\dots C'}$. The chronometric tensor is of rank r , is totally symmetric and is nondegenerate in the sense that $v^a g_{ab\dots c} \neq 0$ for any vector $v^a \neq 0$. We say that x^a and y^a in \mathcal{H}^{r^2} have a ‘degenerate’ separation if the chronometric form $\Delta(r) = g_{ab\dots c} r^a r^b \dots r^c$ vanishes for $r^a = x^a - y^a$. Degenerate separation is equivalent to the vanishing of the determinant of $r^{AA'}$.

If the hyperspin space has dimension $r = 2$, this condition reduces to the case where x^a and y^a are null-separated in Minkowski space. For $r > 2$ the situation is more complicated since various degrees of degeneracy can arise between two points of a quantum space-time. In the case $r = 3$, for example, the quantum space-time has dimension nine, and the chronometric form is $\Delta = g_{abc} r^a r^b r^c$. The different possibilities that can arise for the separation vector are as follows: (i) $g_{abc} r^c = 0$ implies $r^{AA'} = 0$ (zero separation); (ii) $g_{abc} r^b r^c = 0$ and $g_{abc} r^c \neq 0$ implies $r^{AA'} = \alpha^A \bar{\alpha}^{A'}$ (future-pointing null separation) or $r^{AA'} = -\alpha^A \bar{\alpha}^{A'}$ (past-pointing null separation); (iii) $\Delta = 0$ and $g_{abc} r^b r^c \neq 0$ implies $r^{AA'} = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'}$ (degenerate time-like future-pointing separation), $r^{AA'} = \alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'}$ (degenerate space-like separation), or $r^{AA'} = -\alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'}$ (degenerate time-like past-pointing separation); (iv) $\Delta \neq 0$ and $g_{ab} r^a r^b \neq 0$ implies $r^{AA'} = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'} + \gamma^A \bar{\gamma}^{A'}$ (future-pointing time-like separation), $r^{AA'} = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'} - \gamma^A \bar{\gamma}^{A'}$ (future semi-space-like separation), $r^{AA'} = \alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'} - \gamma^A \bar{\gamma}^{A'}$ (past semi-space-like separation), or $r^{AA'} = -\alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'} - \gamma^A \bar{\gamma}^{A'}$ (past-pointing time-like separation).

When the separation of two points of a quantum space-time is degenerate, we define the ‘degree’ of degeneracy by the rank of the matrix $r^{AA'}$. Null separation is the case for which the degeneracy is of the first degree, i.e. $r^{AA'}$ is of rank one, and thus satisfies a system of quadratic relations of the form $g_{ab\dots c} r^a r^b = 0$. This implies $r^{AA'} = \pm \alpha^A \bar{\alpha}^{A'}$ for some α^A . In the case of degeneracy of the second degree, $r^{AA'}$ is of rank two and satisfies a set of cubic relations given by $g_{abc\dots d} r^a r^b r^c = 0$. In this situation $r^{AA'}$ can be put into one of the following canonical forms: (a) $r^{AA'} = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'}$, (b) $r^{AA'} = \alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'}$, and (c) $r^{AA'} = -\alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'}$. In case (a), the point x^a lies to the future of the point y^a , and r^a can be thought of as a degenerate future-pointing time-like vector. In case (b), r^a is a degenerate space-like vector. In case (c), x^a lies to the past of y^a , and r^a is a degenerate past-pointing time-like vector. A similar analysis can be applied to degenerate separations of other intermediate degrees.

If the determinant of the r -by- r weakly Hermitian matrix $r^{AA'}$ is nonvanishing, and $r^{AA'}$ is thus of maximal rank, then the chronometric form is nonvanishing. In that case $r^{AA'}$ can be represented in the following canonical form:

$$(2) \quad r^{AA'} = \pm \alpha^A \bar{\alpha}^{A'} \pm \beta^A \bar{\beta}^{A'} \pm \dots \pm \gamma^A \bar{\gamma}^{A'},$$

with the presence of r nonvanishing terms, where the r hyperspinors $\alpha^A, \beta^A, \dots, \gamma^A$ are linearly independent. Let us write (p, q) for the numbers of plus and minus signs appearing in the canonical form for $r^{AA'}$. We call (p, q) the ‘signature’ of $r^{AA'}$. When the signature is $(r, 0)$ or $(0, r)$ we say that $r^{AA'}$ is future-pointing time-like or past-pointing time-like, respectively. Then we define the proper time interval between the events x^a and y^a by the formula $\|x - y\| = |\Delta|^{1/r}$. In the case $r = 2$ we recover the Minkowskian proper-time interval.

A remarkable feature of the causal structure of a quantum space-time is that many of the physical features of the causal structure of Minkowski space are preserved. In particular, the space of future-pointing time-like vectors forms a convex cone. The same is true for the structure of the associated momentum space, from which it follows that we can give a good definition of ‘positive energy’.

4. EQUATIONS OF MOTION

Let $\lambda \mapsto x^{AA'}(\lambda)$ define a smooth curve γ in \mathcal{H}^{r^2} for $\lambda \in [a, b] \subset \mathbb{R}$. Then γ is said to be time-like if the tangent vector $v^{AA'}(\lambda) = dx^{AA'}(\lambda)/d\lambda$ is time-like and future-pointing. In that case we define the proper time s elapsed along γ by

$$(3) \quad s = \int_a^b [g_{ab\dots c} v^a v^b \dots v^c]^{1/r} d\lambda.$$

In the case of a very small time interval, we can write this in the pseudo-Finslerian form $(ds)^r = g_{ab\dots c} dx^a dx^b \dots dx^c$. For $r = 2$ this reduces to the standard pseudo-Riemannian expression for the line element.

Now consider the condition γ must satisfy in order to be a geodesic in \mathcal{H}^{r^2} . In the case of a time-like curve, we can choose the proper time as the parameter along the curve. The equation of motion for a time-like geodesic is obtained by an application of the calculus of variations to formula (3). We assume the variation vanishes at the endpoints. Writing L for the integrand in (3), we can use a standard argument to show that $x^a(s)$ describes a geodesic only if the velocity vector v^a satisfies the Euler-Lagrange equation $d(\partial L/\partial v^a)/ds = 0$. It follows that if y^a and z^a are quantum space-time points such that $y^a - z^a$ is time-like and future-pointing, then the affinely parametrised geodesic γ connecting these points in \mathcal{H}^{r^2} is given by (cf. Busemann [3])

$$(4) \quad x^a(s) = z^a + \frac{y^a - z^a}{[\Delta(y, z)]^{1/r}} s,$$

for $s \in (-\infty, \infty)$, where $\Delta(y, z) = g_{ab\dots c} (y^a - z^a)(y^b - z^b) \dots (y^c - z^c)$.

5. CONSERVED QUANTITIES

The chronometric form for the separation between two points is invariant when the points of \mathcal{H}^{r^2} are subjected to transformations of the form

$$(5) \quad x^{AA'} \rightarrow \lambda_B^A \bar{\lambda}_{B'}^{A'} x^{BB'} + \beta^{AA'}.$$

Here $\beta^{AA'}$ is a translation in the quantum space-time, and λ_B^A is an element of $SL(r, \mathbb{C})$. The relation of this group of transformations to the Poincaré group in the case $r = 2$ is clear. Indeed, one of the attractive features of the extension of space-time geometry that we are presenting is that the hyper-Poincaré group admits such a description, which has a number of important physical consequences.

More generally, the proper hyper-Poincaré group preserves the signature of any space-time interval, whether or not the interval is degenerate, and hence preserves the causal relations between events. If $L_b^a = \lambda_B^A \bar{\lambda}_{B'}^{A'}$ for some $\lambda_B^A \in SL(r, \mathbb{C})$, we refer to a map of the form $r^a \rightarrow L_b^a r^b$ as a *hyper-Lorentz transformation*.

The real dimension of the hyper-Lorentz group is $2r^2 - 2$, and hence the real dimension of the hyper-Poincaré group is $3r^2 - 2$. The dimension of the hyper-Poincaré group thus grows linearly with the dimension of the quantum space-time. This can be contrasted with the dimension of the group arising if we endow \mathbb{R}^{r^2} with a Lorentzian metric with signature $(1, r^2 - 1)$. In that case the associated pseudo-orthogonal group has dimension $\frac{1}{2}r^2(r^2 - 1)$, which together with the translation group gives a total dimension of $\frac{1}{2}r^2(r^2 + 1)$. The comparatively low dimensionality of the hyper-Poincaré group arises from the fact that it preserves a more elaborate system of causal relations than what one has in the Lorentzian case.

In Minkowski space the symmetries of the Poincaré group are associated with a ten-parameter family of Killing vectors. Thus for $r = 2$ we have the Minkowski metric g_{ab} , and the Poincaré group is generated by vector fields ξ^a on \mathcal{H}^4 that satisfy $\mathcal{L}_\xi g_{ab} = 0$, where \mathcal{L}_ξ denotes the Lie derivative. For any vector field ξ^a and

any symmetric tensor field g_{ab} we have $\mathcal{L}_\xi g_{ab} = \xi^c \nabla_c g_{ab} + 2g_{c(a} \nabla_{b)} \xi^c$. If g_{ab} is the metric and ∇_a denotes the torsion-free covariant derivative satisfying $\nabla_a g_{bc} = 0$, we obtain the Killing equation $\nabla_{(a} \xi_{b)} = 0$, where $\xi_a = g_{ab} \xi^b$. The condition $\mathcal{L}_\xi g_{ab} = 0$ therefore implies that ξ^a is a Killing vector.

For $r > 2$ the usual relations between symmetries and Killing vectors are lost. Instead, we obtain a system of higher-rank Killing tensors. More specifically, to generate a symmetry of the quantum space-time the vector field ξ^a has to satisfy $\mathcal{L}_\xi g_{ab\dots c} = 0$, where $g_{ab\dots c}$ is the chronometric tensor. For a vector field ξ^a and a symmetric tensor field $g_{ab\dots c}$ we have

$$(6) \quad \mathcal{L}_\xi g_{ab\dots c} = \xi^d \nabla_d g_{ab\dots c} + r g_{d(a\dots b} \nabla_{c)} \xi^d.$$

In the case of the quantum space-time \mathcal{H}^{r^2} we let ∇_a be the natural flat connection for which $\nabla_a g_{bc\dots d} = 0$. Then to generate a symmetry of the chronometric structure of \mathcal{H}^{r^2} the vector field ξ^a has to satisfy $g_{d(a\dots b} \nabla_{c)} \xi^d = 0$. This equation can be written in a suggestive form if we define a symmetric tensor $K_{ab\dots c}$ of rank $r - 1$ by setting $K_{ab\dots c} = g_{ab\dots cd} \xi^d$. Then it follows that $K_{ab\dots c}$ satisfies the conditions for a symmetric Killing tensor: $\nabla_{(a} K_{bc\dots d)} = 0$. We thus see that \mathcal{H}^{r^2} provides a symmetry group generated by a family of Killing tensors. The symmetries of the quantum space-time are generated by a system of $3r^2 - 2$ irreducible symmetric Killing tensors of rank $r - 1$. The significance of Killing tensors is that they are associated with conserved quantities. In particular, if the vector field v^a satisfies the geodesic equation, which on a quantum space-time of dimension r^2 is given by $g_{abc\dots d} (v^e \nabla_e v^b) v^c \dots v^d = 0$, and if $K_{ab\dots c}$ is the Killing tensor of rank $r - 1$, then we have the following conservation law: $v^e \nabla_e (K_{ab\dots c} v^a v^b \dots v^c) = 0$. In other words, $K_{ab\dots c} v^a v^b \dots v^c$ is a constant of the motion.

6. HYPER-RELATIVISTIC MECHANICS

In higher-dimensional quantum space-times the conservation laws and symmetry principles of relativistic physics remain intact. In particular, the conservation of hyper-relativistic momentum and angular momentum for a system of interacting particles can be formulated by use of principles similar to those of the Minkowskian case. For this purpose we introduce the idea of an ‘elementary system’ in hyper-relativistic mechanics. Such a system is defined by its momentum and angular momentum. The hyper-relativistic momentum of an elementary system is given by a momentum covector P_a . The associated mass m is given (cf. [4]) by: $m = (g^{ab\dots c} P_a P_b \dots P_c)^{1/r}$. The hyper-relativistic angular momentum is given by a tensor L_a^b of the form $L_a^b = l_A^B \delta_{A'}^{B'} + \bar{l}_{A'}^{B'} \delta_A^B$, where the hyperspinor l_A^B is trace-free: $l_A^A = 0$. The angular momentum is defined with respect to a choice of origin. Under a change of origin defined by a shift vector β^a we have $l_A^B \rightarrow l_A^B + P_{AC'} \beta^{BC'}$. In the case $r = 2$ these formulae reduce to the usual expressions for momentum and angular momentum in a Minkowskian setting. The real covector $S_{AA'} = im^{-1} (l_A^B P_{A'B} - \bar{l}_{A'}^{B'} P_{AB'})$ is invariant under a change of origin, and can be interpreted as the intrinsic spin of the elementary system. The magnitude of the spin is $S = |g^{ab\dots c} S_a S_b \dots S_c|^{1/r}$. In the case of a set of interacting hyper-relativistic systems we require that the total momentum and angular momentum should be conserved. This implies conservation of the total mass and spin. We thus see that the idea of relativistic mechanics carries through to the case of a general quantum space-time. We shall see later, once we introduce the idea of symmetry breaking, that hyper-momentum can be interpreted as the momentum operator for a relativistic quantum system. Conservation of hyper-momentum then can be thought of as conservation of four-momentum, in relativistic quantum mechanics, in the Heisenberg representation.

7. WEAK AND STRONG HERMITICITY

As a prelude to our discussion of symmetry breaking in a quantum space-time, we digress briefly to review the notions of weak and strong Hermiticity. This material is relevant to the origin of unitarity in quantum mechanics. Intuitively speaking, when the weak Hermiticity condition is imposed on a hyperspinor $x^{AA'}$, then $x^{AA'}$ belongs to the real subspace $\mathbb{R}^{AA'}$. The hyper-relativistic symmetry of a quantum space-time is not affected by the imposition of this condition. If, however, we break the symmetry by selecting a preferred time-like direction, then we can speak of a stronger reality condition whereby *an isomorphism is established between the primed and unprimed hyperspin spaces*.

We begin with the weak Hermitian property. Let \mathbb{S}^A denote, as before, an r -dimensional complex vector space. We also introduce the spaces \mathbb{S}_A , $\mathbb{S}^{A'}$, and $\mathbb{S}_{A'}$. In general, there is no natural isomorphism between $\mathbb{S}^{A'}$ and \mathbb{S}_A , and there is no natural matrix multiplication law or trace operation defined for elements of $\mathbb{S}^A \otimes \mathbb{S}^{A'}$. Certain matrix operations are well defined. For example, the determinant of a generic element $\mu^{AA'}$ is given by $r! \det(\mu) = \varepsilon_{AB\dots C} \varepsilon_{A'B'\dots C'} \mu^{AA'} \mu^{BB'} \dots \mu^{CC'}$. The weak Hermitian property is also well-defined: if $\bar{\mu}^{A'A}$ is the complex conjugate of $\mu^{AA'}$, then we say that $\mu^{AA'}$ is weakly Hermitian if $\mu^{AA'} = \bar{\mu}^{A'A}$.

Next we consider the strong Hermitian property. In some situations there may exist a natural map $\mathbb{S}^{A'} \rightarrow \mathbb{S}_A$ defined by the context of the problem. Such a map is called a Hermitian correlation. In this case, the complex conjugate of an element $\alpha^A \in \mathbb{S}^A$ determines an element $\bar{\alpha}_A \in \mathbb{S}_A$. For any element $\mu_B^A \in \mathbb{S}^A \otimes \mathbb{S}_B$ we define the operations of determinant, matrix multiplication, and trace in the usual manner. The determinant is $r! \det(\mu) = \varepsilon_{AB\dots C} \varepsilon^{PQ\dots R} \mu_P^A \mu_Q^B \dots \mu_R^C$, and the Hermitian conjugate of μ_B^A is $\bar{\mu}_A^B$. The Hermitian correlation is given by the choice of a preferred element $t_{AA'} \in \mathbb{S}_A \otimes \mathbb{S}_{A'}$. Then we write $\bar{\alpha}_A = t_{AA'} \bar{\alpha}^{A'}$, where $\bar{\alpha}_A$ is now called the complex conjugate of α^A . When there is a Hermitian correlation $\mathbb{S}^{A'} \leftrightarrow \mathbb{S}_A$, we call the condition $\mu_B^A = \bar{\mu}_B^A$ the strong Hermitian property.

8. SYMMETRY BREAKING AND QUANTUM ENTANGLEMENT

We proceed to introduce a mechanism for symmetry breaking in a quantum space-time. We shall make the point that the breaking of symmetry in a quantum space-time is intimately linked to the notion of quantum entanglement. According to this point of view, the introduction of symmetry-breaking in the early stages of the universe can be understood as a *sequence of phase transitions*, the ultimate consequence of which is an approximate disentanglement of a four-dimensional ‘classical’ space-time.

The breaking of symmetry is represented by an ‘index decomposition’. In particular, if the dimension r of the hyperspin space is not a prime number, then a natural method of breaking the symmetry arises by consideration of the decomposition of r into factors. The specific assumption we make is that the dimension of the hyperspin space \mathbb{S}^A is *even*. Then we write $r = 2n$, where $n = 1, 2, \dots$, and set $\mathbb{S}^A = \mathbb{S}^{\mathbf{A}i}$, where \mathbf{A} is a two-component spinor index, and i will be called an ‘internal’ index ($i = 1, 2, \dots, n$). Thus we can write $\mathbb{S}^{\mathbf{A}i} = \mathbb{S}^{\mathbf{A}} \otimes \mathbb{H}^i$, where $\mathbb{S}^{\mathbf{A}}$ is a standard spin space of dimension two, and \mathbb{H}^i is a complex vector space of dimension n . The breaking of symmetry amounts to the fact that we can identify the hyperspin space with the tensor product of these two spaces.

We shall assume that \mathbb{H}^i is endowed with a strong Hermitian structure, i.e. that there is a canonical anti-linear isomorphism between the complex conjugate of the internal space \mathbb{H}^i and the dual space \mathbb{H}_i . If $\psi^i \in \mathbb{H}^i$, then we write $\bar{\psi}_i$ for the complex conjugate of ψ^i , where $\bar{\psi}_i \in \mathbb{H}_i$. We see that \mathbb{H}^i is a complex Hilbert space—and

indeed, although here we consider mainly the case for which n is finite, one should have in mind also the infinite dimensional situation. For the other hyperspin spaces we write $\mathbb{S}_A = \mathbb{S}_{\mathbf{A}i}$, $\mathbb{S}^{A'} = \mathbb{S}^{\mathbf{A}'i}$, and $\mathbb{S}_{A'} = \mathbb{S}_{\mathbf{A}'i}$, respectively. These identifications preserve the duality between \mathbb{S}^A and \mathbb{S}_A , and between $\mathbb{S}^{A'}$ and $\mathbb{S}_{A'}$; and at the same time are consistent with the complex conjugation relations between \mathbb{S}^A and $\mathbb{S}^{A'}$, and between \mathbb{S}_A and $\mathbb{S}_{A'}$. Hence if $\alpha^{\mathbf{A}i} \in \mathbb{S}^{\mathbf{A}i}$ then under complex conjugation we have $\alpha^{\mathbf{A}i} \rightarrow \bar{\alpha}^{\mathbf{A}'i}$, and if $\beta_{\mathbf{A}i} \in \mathbb{S}_{\mathbf{A}i}$ then $\beta_{\mathbf{A}i} \rightarrow \bar{\beta}_{\mathbf{A}'i}$.

In the case of a quantum space-time vector $r^{AA'}$ we have a corresponding structure induced by the identification $r^{AA'} = r^{\mathbf{A}\mathbf{A}'i}$. When the quantum space-time vector is real, the weak Hermitian structure on $r^{AA'}$ is manifested in the form of a weak Hermitian structure on the two-component spinor index pair, together with a strong Hermitian structure on the internal index pair. In other words, the Hermitian condition on the space-time vector $r^{AA'}$ is given by $\bar{r}^{\mathbf{A}'\mathbf{A}i} = r^{\mathbf{A}\mathbf{A}'i}$.

One consequence of these relations is that we can interpret each point in a quantum space-time as being a *space-time-point-valued operator*. The ordinary classical space-time then ‘sits’ inside the quantum space-time in a canonical manner—namely, as the locus of those points of quantum space-time that factorise into the product of a space-time point $x^{\mathbf{A}\mathbf{A}'}$ and the identity operator on the internal space: $x^{\mathbf{A}\mathbf{A}'i} = x^{\mathbf{A}\mathbf{A}'}\delta_j^i$. Thus, in situations where special relativity is a satisfactory theory, we regard the relevant physical events as taking place on or in the immediate neighbourhood of the embedding of Minkowski space in \mathcal{H}^{4n^2} .

This picture can be presented in more geometric terms as follows. We introduce the notion of a *hypertwistor* as a pair of hyperspinors $(\omega^A, \pi_{A'})$ with the pseudo-norm $\omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'}$. The projective hypertwistor space \mathbb{P}^{2r-1} in the case $r = 2n$ admits a Segré embedding of the form $\mathbb{P}^3 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{4n-1}$. Many such embeddings are possible, though they are all equivalent under the action of the symmetry group $U(2n, 2n)$. If the symmetry is broken and one such embedding is selected out, then we can introduce homogeneous coordinates and write $Z^{\alpha i}$ for the generic hypertwistor. Here the Greek letter α denotes an ‘ordinary’ twistor index ($\alpha = 0, 1, 2, 3$) and i denotes an internal index ($i = 1, 2, \dots, n$). These two indices, when clumped together, constitute a hypertwistor index. The Segré embedding consists of those points in \mathbb{P}^{4n-1} for which we have a product decomposition given by $Z^{\alpha i} = Z^\alpha \psi^i$. Once symmetry breaking takes place—and this may happen in stages, corresponding to a successive factorisation of the underlying hypertwistor space—then one can think of ordinary four-dimensional space-time as becoming more or less disentangled from the rest of the universe, and behaving to some extent autonomously. Nonetheless, we might expect its global dynamics, on a cosmological scale, to be affected by the distribution of mass and energy elsewhere in the quantum space-time; and thus we obtain a possible model for ‘dark energy’.

The idea of symmetry breaking being put forward here is related to the notion of *disentanglement* in quantum mechanics [1, 6]. That is to say, at the unified level the degrees of freedom associated with space-time symmetry are quantum mechanically entangled with the internal degrees of freedom associated with microscopic physics. The phenomena responsible for the breakdown of symmetry are thus analogous to the mechanisms of decoherence through which quantum entanglements are gradually diminished. There is also in this connection an interesting relation to the so-called *twistor internal symmetry groups* (see, e.g., [9]).

Let us now examine the implications of our symmetry breaking mechanism for fields defined on a quantum space-time. For example, let $\phi(x^{AA'})$ be a scalar field on a quantum space-time. After we break the symmetry by writing $x^{AA'} = x^{\mathbf{A}\mathbf{A}'i}$, we consider an expansion of the field around the embedded Minkowski space. More

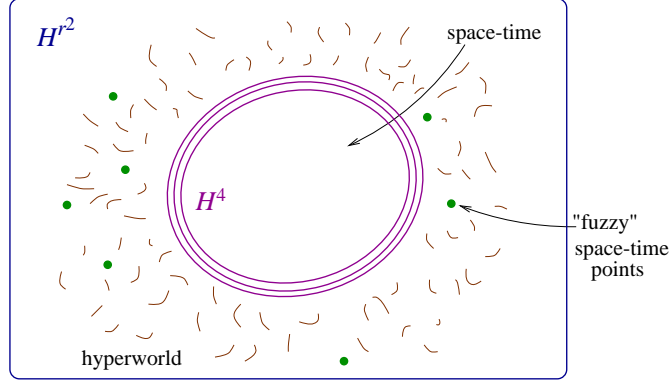


FIGURE 1. *Fuzzy space-time*. Once the symmetry is broken, a quantum space-time event can be interpreted as a space-time-point-valued operator, whose expectation determines a point in Minkowski space.

specifically, for such an expansion we have

$$(7) \quad \phi(x^{AA'}) = \phi^{(0)}(x^{AA'}) + \phi_{AA'j}^{(1)i}(x^{AA'}) \left(x^{AA'j} - x^{AA'} \delta_j^i \right) + \dots,$$

where $\phi^{(0)}(x^{AA'}) = \phi(x)|_{x=x^{AA'}\delta_j^i}$, and $\phi_{AA'j}^{(1)i}(x^{AA'}) = (\partial\phi(x)/\partial x)|_{x=x^{AA'}\delta_j^i}$. Therefore the order-zero term defines a classical field on Minkowski space, and the first-order term can be interpreted as a ‘multiplet’ of fields, transforming according to the adjoint representation of $U(n)$. Alternatively, if the internal space is infinite-dimensional, we can think of the first-order term as a field operator.

9. EMERGENCE OF QUANTUM STATES

The embedding of Minkowski space in the quantum space-time \mathcal{H}^{4n^2} implies a corresponding embedding of the Poincaré group in the hyper-Poincaré group. This can be seen as follows. The standard Poincaré group in \mathcal{H}^4 consists of transformations of the form $x^{AA'} \rightarrow \lambda_B^A \bar{\lambda}_{B'}^{A'} x^{BB'} + \beta^{AA'}$. This action lifts naturally to a corresponding action on \mathcal{H}^{4n^2} given by $x^{AA'i} \rightarrow \lambda_B^A \bar{\lambda}_{B'}^{A'} x^{BB'i} + \beta^{AA'} \delta_j^i$. On the other hand, the general hyper-Poincaré transformation in the broken symmetry phase can be expressed in the form

$$(8) \quad x^{AA'i} \rightarrow L_{Bk}^{Ai} \bar{L}_{B'j}^{A'l} x^{BB'k} + \beta^{AA'i}.$$

Thus the embedding of the Poincaré group as a subgroup of the hyper-Poincaré group is given by $L_{Bj}^{Ai} = \lambda_B^A \delta_j^i$ and $\beta^{AA'i} = \beta^{AA'} \delta_j^i$.

Bearing this in mind, we construct a class of maps from the quantum space-time \mathcal{H}^{4n^2} to Minkowski space \mathcal{H}^4 . Under rather general physical assumptions, such maps are necessarily of the form

$$(9) \quad x^{AA'i} \rightarrow x^{AA'} = \rho_i^j x^{AA'j},$$

where ρ_i^j is a *density matrix*. By a density matrix we mean, as usual, a positive semi-definite strongly Hermitian matrix with unit trace. The maps thus arising here can be regarded as quantum expectations. In particular, let $\rho: \mathcal{H}^{4n^2} \rightarrow \mathcal{H}^4$ satisfy the following conditions: (i) ρ is linear and maps the origin of \mathcal{H}^{4n^2} to the origin of \mathcal{H}^4 ; (ii) ρ is Poincaré invariant; and (iii) ρ preserves causal relations. Then ρ is given by a density matrix on the internal space (we refer the reader to [2] for a proof).

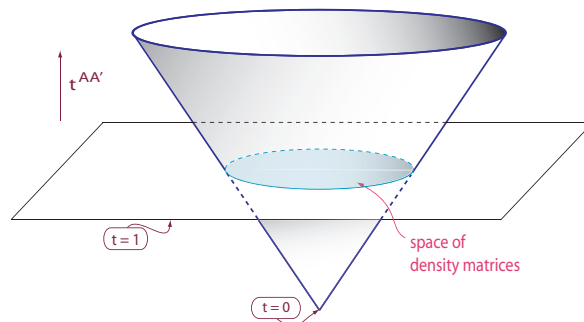


FIGURE 2. *Space of density matrices.* The intersection of the cone of positive Hermitian matrices with a space-like hypersurface is the space of density matrices.

This result shows how the causal structure of quantum space-time is linked with the probabilistic structure of quantum mechanics. The concept of a quantum state emerges when we ask for consistent ways of ‘averaging’ over the geometry of quantum space-time in order to obtain a reduced description of physical phenomena in terms of the geometry of Minkowski space. We see that a probabilistic interpretation of the map from a general quantum space-time to Minkowski space arises as a consequence of elementary causality requirements. We can thus view the space-time events in \mathcal{H}^{4n^2} as representing space-time-point-valued quantum observables, the totality of which constitute a ‘fuzzy space-time’ (see Figure 1), and the expectations of these fuzzy space-time points correspond to points of \mathcal{H}^4 .

Finally, we remark that the space of density matrices itself is endowed with a natural Finslerian metric induced from the ambient pseudo-Finslerian structure, as illustrated in Figure 2 (cf. [11]).

REFERENCES

- [1] D. C. Brody and L. P. Hughston *Geometric quantum mechanics*. J. Geom. Phys. **38**, 19-53 (2001).
- [2] D. C. Brody and L. P. Hughston *Theory of quantum space-time*. Proc. R. Soc. London A**461**, 2679-2699 (2005).
- [3] H. Busemann *Timelike spaces*. Dissertationes Math. Rozprawy Mat. **53**, 52pp. (1967).
- [4] D. Finkelstein *Hyperspin and hyperspace*. Phys. Rev. Lett. **56**, 1532-1533 (1986).
- [5] S. R. Finkelstein *Gravity in hyperspin manifolds*. Int. J. Theor. Phys. **27**, 251-272 (1988).
- [6] G. W. Gibbons *Typical states and density matrices*. J. Geom. Phys. **8**, 147-162 (1992).
- [7] C. Holm *The hyperspin structure of unitary groups*. J. Math. Phys. **29**, 978-986 (1988), Erratum *ibid.* **30**, 2451 (1989).
- [8] D. C. Honeycutt *Bergmannian relativity and bracket spaces*. Int. J. Theor. Phys. **30**, 1613-1644 (1991).
- [9] L. P. Hughston *Some new contour integral formulae*. In *Complex manifold techniques in theoretical physics*, D. Lerner and P. D. Sommers, eds., San Francisco: Pitman (1979).
- [10] R. Penrose *Structure of space-time*. In *Battelle Rencontres*, C. M. DeWitt and J. A. Wheeler, eds., New York: W. A. Benjamin (1968).
- [11] C. Liverani and M. P. Wojtkowski *Generalization of the Hilbert metric to the space of positive definite matrices*. Pacific J. Math. **166**, 339-355 (1994).

BLACKETT LABORATORY, IMPERIAL COLLEGE, LONDON SW7 2BZ, UK

E-mail address: dorje@imperial.ac.uk

URL: <http://www.imperial.ac.uk/people/d.brody>

DEPARTMENT OF MATHEMATICS, KING’S COLLEGE LONDON, LONDON WC2R 2LS, UK

E-mail address: lane.hughston@kcl.ac.uk

URL: <http://www.mth.kcl.ac.uk/staff/l.hughston.html>