Inflation Derivatives

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Abstract. A general theory for the pricing and hedging of inflation-linked derivatives is outlined in a complete market setting with no arbitrage. The market consists of nominal discount bonds and real discount bonds, together with the consumer price index, which acts as a kind of exchange rate to determine the nominal payout of a real discount bond at maturity. An analogy with foreign exchange is suggested as a basis for the design of new products.

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1 Introduction

The purpose of this note is to outline briefly a general model of inflation and inflation-linked derivatives. The idea is to formulate an approach to the valuation of inflation derivatives that is as close as possible to the methodologies we use for valuing foreign currency and interest rate derivatives. The theory of inflation has aspects that relate to both interest rates and foreign exchange. In particular, a very useful way of thinking about inflation is to treat the consumer price index (CPI) as if it were the price of a foreign currency. Not only does this lead to the correct approach for pricing and hedging inflation-linked derivatives, but it also is suggestive of new financial products and structures that might naturally arise.

We begin by considering an economy consisting of discount bonds and index-linked discount bonds. The indexing of the index-linked discount bonds is with respect to the consumer price index, which at time a has the value C_a. We think of C_a as representing the value, in units of the domestic currency (henceforth, dollars), of a typical basket of goods and services at that time. An increase in C_a over an interval of time then indicates that there has been inflation over that period. We shall define an inflation linked discount bond to be a bond that pays out C_b at the maturity date b. In other words, the inflation-linked bond pays out enough in dollars to buy a unit of goods and services at that time. There is clearly some idealisation involved here, since actual index-linked bonds are usually coupon bonds, rather than discount bonds. Furthermore, not all maturities are represented. Nevertheless, it is useful in the first place to think of the market as including index-linked discount bonds of all maturities. Then we can worry about the difficult but important effects of illiquidity and market incompleteness later. Another complication is that the value of the consumer price index that is paid at the maturity of an index-linked bond is typically fixed some months before, e.g. eight months in the UK index-linked gilts market. This is annoying, but not a fundamental problem. One thinks of such an actual lagged bond as being a kind of derivative that takes note of the consumer price index on a certain set date b, then pays it out at a somewhat later date c. Equivalently, the effective payout at time b is given by $C_b P^N_{bc}$, where P^N_{bc} is the value of a nominal (i.e. dollar) discount bond at time b with maturity c. That means that in actuality the data we are given are the market prices of these 'derivatives' rather than the 'proper' inflation-linked bonds.

Our problem now is to formulate a general theory for the price processes of the consumer price index and the index-linked bonds, and tie this in with the HJM theory of interest rate derivatives (Heath, Jarrow & Morton, 1992) in order to obtain a coherent modelling framework for the valuation of inflation-linked derivatives.

Indexation of debt is not a new idea. Deacon and Derry (1998) note an early example that occurs in 1742 when Massachusetts issued bills linked to the price of silver on the London Exchange. The risks in indexation to a single commodity became apparent a few years later when the price of silver rose in excess over general prices. As a consequence a law was passed in Massachusetts requiring a wider base of commodities for indexation.

In 1780 notes were issued again, indexed this time with the intention of preserving the value of notes issued as wages to soldiers in the American Revolution. In this case both the principal and the interest of the notes were indexed to the combined market values of five bushels of corn, sixty-eight and four-sevenths pounds of beef, ten pounds of sheep wool, and sixteen pounds of sole leather.

On the other hand, the inflation-linked derivatives market is only in its infancy. Let us begin by looking at some typical structures.

2 Inflation-linked payout structures

Suppose we denote by P^N_{ab} the value of a nominal discount bond at time a with maturity at time b. At maturity the nominal discount bond pays one dollar. Then a typical inflation-linked derivative has a payout or payouts given by functions of nominal discount bonds (at various times and of various maturities) and the consumer price index (at various times). Some examples are as follows.

(a) *Inflation cap.* This pays out if inflation (as measured by percentage appreciation in the consumer price index) exceeds a certain threshold K over a given period. Thus if the period in question is the interval (a, b), then the payout H_b at time b is given by:

$$H_b = X \max \left[\left(\frac{C_b}{C_a} - 1 \right) - K, 0 \right], \tag{1}$$

where X is some dollar notional. In practice the payout would have to delayed to some still later date c (to allow for official publication of the relevant CPI figure), so the effective payout is then

$$H_b = X \, P^N_{bc} \max \left[\left(\frac{C_b}{C_a} - 1 \right) - K, 0 \right]. \tag{2}$$

(b) *Inflation swap.* For a succession of intervals (a_i, b_i) $(i = 1, ..., n)$ we receive the inflation rate

$$I_{ab} = \frac{C_b}{C_a} - 1 \tag{3}$$

for that interval (with payment delayed to some slightly later time c_i), and pay a fixed rate, all on a fixed dollar notional.

(c) *Zero-strike floors on inflation.* Here the idea is to protect the receiver of the inflation leg in an inflation swap against a deflation scenario. Thus instead of simply receiving I_{ab}, which can go negative (deflation), one receives $\max(I_{ab}, 0)$.

(d) *Inflation swaption.* This confers the right to enter into an inflation swap (e.g., as a payer of the fixed rate) at some specified future time, with a given 'strike' fixed rate.

(e) *Inflation protected annuity.* This pays a fixed 'real' annuity on the future dates a_i:

$$H_{a_i} = \frac{fNC_{a_i}}{C_0} \tag{4}$$

Here f is the nominal annuity rate (e.g., 5%), N is the notional. The effect of the CPI is to inflate the actual payment appropriately. In actuality there will also be a lag effect.

(f) *Inflation protected annuity, also insulated against deflation.* In this case the annuity payment remains at the original nominal rate (e.g., 5%) if the consumer price index falls below its original value (deflation scenario):

$$H_{a_i} = fN\left[1 + \max\left(\frac{C_{a_i}}{C_0} - 1, 0\right)\right]. \tag{5}$$

The effect here is to incorporate a zero-strike floor on the total inflation up to each relevant annuity payment.

(g) *Knockout options.* A typical structure, for example, might pay if the total inflation exceeds a certain threshold K at time T. Knockout would occur if the total inflation drops below a certain specified critical level K' between time t and T:

$$H_T = N \max\left[\left(\frac{C_T}{C_a} - 1\right) - K', 0\right] \tag{6}$$

unless

$$\left(\frac{C_T}{C_a} - 1\right) - K' \leq 0 \tag{7}$$

at some time a in the interval $t \leq a \leq T$, in which case $H_T = 0$.

There are many variations possible on this kind of structure. The basic idea is to make the option premium cheaper by having the contract specify a cancelling of the structure in the event of certain circumstances.

(h) *Cap on 'real' interest rates.* This might, for example, pay off

$$H_b = X \max(L^R_{ab} - K, 0). \tag{8}$$

Actually, the real rates are not necessarily readily available as a basis for contract specification. Instead we can use a proxy.

(i) *Proxy cap on 'real' interest rates.* This instead would pay

$$H_b = X \max(L^N_{ab} - I_{ab} - K, 0), \tag{9}$$

where L^N_{ab} is the relevant per-period Libor rate. Then if the Libor rate exceeds the inflation rate over the given interval by more than a specified amount, there is a payoff. Here we have used the difference between the Libor rate and the inflation rate as a convenient (albeit crude) measure of the 'real' interest rate over the given interval.

Options can have knockout and American style features, etc., as well, but in practice the constraints imposed by the lag in the computation of the index have to be respected. Asian-type structures can be incorporated to deal with seasonality effects, if necessary. Clearly, more 'exotic' structures can also easily be represented. Analogues both from the foreign exchange world (treating C_a as a foreign exchange rate), and the interest rate world (treating I_{ab} as a kind of 'rate') can be formulated.

3 General theory of index-linked securities

We have three ingredients: the 'nominal' discount bonds P^N_{ab}, the 'real' discount bonds P^R_{ab}, and the consumer price index C_a. The real discount bonds are defined as follows. By P^R_{ab} we mean, intuitively, the price at time a, in units of goods and services, for one unit of goods and services to be delivered at time b. Thus P^R_{ab} is the discount function that characterises 'real' interest rates. If we lived in a pure barter economy, with no money, then P^R_{ab} would define the term structure of interest rates.

For example, if the price of bread happened to be a good proxy for goods and services in general, as it might in a suitably simple economy, then our 'unit' of goods and services could be represented by one hundred loaves of bread. The real term structure of interest rates would then supply information like how many loaves of bread you should in principle be willing to part with today in exchange for a sure delivery of one hundred loaves one year from now. The answer might be, say, ninety seven loaves, and that enables us to define the one-year real interest rate.

Associated with the system of real discount bonds we have a corresponding system of real interest rates. We denote a typical real rate with the notation L^R_{ab}. The index-linked discount bonds are related to the real discount bonds by the consumer price index, which acts as a kind of exchange rate. In other words, if we multiply the P^R_{ab} by C_a, that gives us the dollar value of the b-maturity real discount bond at time a. In the foreign exchange analogy, we think of the nominal (dollar) discount bonds as the 'domestic' bonds, the real discount bonds as 'foreign' discount bonds (say, DEM bonds), and the consumer price index plays the role of the exchange rate (price of one DEM in dollars). Note that the inflation rate I_{ab} for the period (a, b) is not strictly analogous to an interest rate in the usual sense; it is only known at time b (or later!). It is thus best thought of as an appreciation in an asset price.

But in that case what is the relation between 'real' rates, 'nominal' rates, and 'inflation' rates? Clearly care is required, and we must not confuse categories just because these are all loosely referred to as 'rates'. Part of the goal here is to gain some insight into the relation between these various 'rates'. Our approach will be motivated by derivatives pricing considerations.

4 Bond price processes

As usual in an HJM type framework (Heath, Jarrow and Morton 1992), we shall assume an economy where uncertainty in the future is modelled by a multi-dimensional Brownian motion defined with respect to the natural system of probabilities, which we denote by a probability measure $\mathbb{P}$. Then, assuming no arbitrage, and thus the existence of a universal market risk premium vector, we can write the stochastic equation for the price process of the nominal discount bonds in the form

$$\frac{dP^N_{ab}}{P^N_{ab}} = \left(r^N_a + \lambda^N_a \Omega^N_{ab}\right) da + \Omega^N_{ab}\, dW_a. \tag{10}$$

Here r^N_a is the nominal short rate, λ^N_a is the nominal risk premium vector, Ω^N_{ab} is the nominal vector volatility, and W_a is the Brownian motion vector. By analogy we can write

$$\frac{dP^R_{ab}}{P^R_{ab}} = \left(r^R_a + \lambda^R_a \Omega^R_{ab}\right) da + \Omega^R_{ab}\, dW_a \tag{11}$$

for the real discount bonds. It then follows from general considerations, by virtue of the foreign exchange analogy (see, e.g., Fisher & Gilles 1995, Flesaker & Hughston 1997a,b,c for details) that the price process for the consumer price index can be expressed in the form

$$\frac{dC_a}{C_a} = \left[r^N_a - r^N_a + \lambda^N_a \left(\lambda^N_a - \lambda^R_a\right)\right] da + \left(\lambda^N_a - \lambda^R_a\right) dW_a. \tag{12}$$

Here we note that the consumer price index volatility vector can be expressed as the difference between the nominal (dollars) and real (goods and services) risk premium vectors. Thus we can write

$$\frac{dC_a}{C_a} = \left[r_a^N - r_a^R + \lambda_a^N \nu_a\right] da + \nu_a dW_a, \tag{13}$$

where

$$\nu_a = \lambda_a^N - \lambda_a^R \tag{14}$$

is the CPI volatility vector.

The interpretation of this relation is as follows. In the absence of a risk premium, we see that the drift of the consumer price index is given by the difference between the nominal short rate and the real short rate. In reality, the drift of the consumer price index contains another term, given by the product of the nominal risk premium vector and the CPI volatility vector. Thus if by the rate of inflation I_a we mean the drift process for the consumer price index, we have:

$$I_a = r_a^N - r_a^R + \lambda_a^N \nu_a. \tag{15}$$

This is essentially a local expression of the so-called Fisher equation, which relates the inflation rate to the nominal interest rate minus the real interest rate plus a risk premium term.

5 Transfer to the risk-neutral measure

For the valuation of derivatives we want to introduce a change of measure such that the ratio of any of the assets P_{ab}^N to the dollar money market account is a martingale. Suppose we write B_a for the nominal money market account, which satisfies

$$dB_a = r_a^N B_a da. \tag{16}$$

Then we introduce a new probability measure $\mathbb{P}^N$ according to the scheme

$$\mathbb{E}_a^N[X_b] = \frac{\mathbb{E}_a[\rho_b X_b]}{\mathbb{E}_a[\rho_b]}, \tag{17}$$

where $\mathbb{E}_a^N$ denotes conditional expectation with respect to the measure $\mathbb{P}^N$ given the filtration up to time a, and where X_b is any random variable that is measurable at time b. We call $\mathbb{P}^N$ the nominal (or dollar) risk-neutral measure. Here the change-of-measure density process ρ_a is defined by

$$\rho_a = \exp\left[-\int_0^a \lambda_s^N dW_s - \tfrac{1}{2}\int_0^a \left(\lambda_s^N\right)^2 ds\right] \tag{18}$$

With respect to $\mathbb{P}^N$ the process W_a^N defined by

$$dW_a^N = dW_a + \lambda_a^N da \tag{19}$$

is a Brownian motion. Then for the processes P_{ab}^N and P_{ab}^R we can write

$$\frac{dP_{ab}^N}{P_{ab}^N} = r_a^N da + \Omega_{ab}^N dW_a^N \tag{20}$$

and

$$\frac{dP_{ab}^R}{P_{ab}^R} = \left(r_a^R - \nu_a \Omega_{ab}^R\right) da + \Omega_{ab}^R dW_a^N. \tag{21}$$

We note that the process (21) for the real discount bonds picks up a ‘quanto’ term in the drift in the nominal risk neutral measure. This is appropriate since the real discount bonds are not denominated in dollars. The process for the consumer price index in the risk neutral measure is:

$$\frac{dC_a}{C_a} = \left[r_a^N - r_a^R\right] da + \nu_a dW_a. \tag{22}$$

Thus in the risk neutral measure the nominal risk premium term disappears, and we see that the drift on the consumer price index is given by the difference between the nominal and real interest rates. The process is quite like that of a foreign currency, and we can think of the real interest rate as playing the role of the ‘foreign’ interest rate. Normally we expect r_a^N to be positive and r_a^R usually to be positive, but the difference can in principle take on either sign, analogous to the situation with a foreign exchange interest rate differential drift. We note that by construction the ratio process P_{ab}^N/B_a is a martingale in the nominal risk neutral measure. So is $C_a P_{ab}^R/B_a$, where $C_a P_{ab}^R$ is the (dollar) value of an index-linked discount bond.

6 Valuation of inflation-linked derivatives

Now let H_T be a random variable corresponding to the payout of an inflation-linked derivative. We can think of H_T as depending in a general way of the values of nominal discount bonds, real discount bonds, and the consumer price index at times between the present and the maturity date T. There are many examples of inflation-linked derivatives for which the payout depends in a direct way only on the nominal discount bonds and the consumer price index, but not on the real discount bonds. These we shall call 'index-linked' derivatives, and it should be noted that these structures are at least in principle more straightforward to value and hedge than general inflation-linked derivatives, which might also involve real interest rates in an essential way. In any case the basic derivatives valuation formula is given in the risk neutral valuation scheme by

$$H_0 = \mathbb{E}^N \left[\frac{H_T}{B_T} \right]. \tag{23}$$

In particular, we can consider the case where H_T is the payout C_T of an index-linked discount bond, normalised by the value of today's consumer price index. Then we have

$$P^N_{0T} = \mathbb{E}^N \left[\frac{C_T}{C_0 B_T} \right], \tag{24}$$

which shows that today's market for index-linked bonds tells us the initial real discount function. In reality, we have to work a bit harder, on account of the lagging effect, and the fact that we generally have to work with coupon bonds.

Finally, by use of the foreign exchange analogy let us consider a simple Black-Scholes type model for the valuation of index derivatives. Let us assume deterministic interest rates (nominal and real), and a deterministic consumer price index volatility, with a prescribed local volatility function ν_t. Then for the CPI process we can write:

$$C_t = \frac{C_0 P^R_{0t}}{P^N_{0t}} \exp \left[\int_{s=0}^{t} \nu_s dW_s - \tfrac{1}{2} \int_{s=0}^{t} \nu_s^2 ds \right], \tag{25}$$

where the expression $C_0 P^R_{0t} / P^N_{0t}$ is the forward value for the consumer price index. In this case the situation is entirely analogous to the corresponding problem for foreign exchange, and by use of the Black-Scholes formula we can get a crude valuation for some products in this way, though of course care is required in the case of longer dated structures.

7 General principles for the design of inflation-linked derivatives

Inflation linkage has foreign exchange like aspects, and interest rate like aspects. Thus, products can be designed by treating the consumer price index as if it were a foreign currency price (the calculation lag in the index needs to be respected). Products can also be designed by treating I_{ab} as a 'rate' (but known only after time b), for example, a cap on an inflation rate. Real interest rates can also be brought into play, at least in situations where they are clearly indexed (e.g., options on index-linked bonds) or proxied. Many analogies can be pursued. For example, an option to enter into an inflation-protected annuity in exchange for a lump sum payment of some fixed nominal quantity, can be thought of as a kind of cross currency swaption. Products can also be structured in the form of options that take advantage of CPI differentials or ratios (the so-called 'real' exchange rates) between countries—e.g., as in Europe, after currency unification.

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