

Stochastic Schrödinger evolution and symmetric Kähler manifolds of low dimension

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Abstract

We consider the manifold-valued, stochastic extension of the Schrödinger equation introduced by Hughston [8] in a manifestly covariant, differential-geometric framework, and examine the resulting quantum evolution on some specific examples of Kähler manifolds with many symmetries. We find conditions on the curvature for the evolution to be a 'collapse process' in the sense of Brody and Hughston [2] or, more generally, a 'reduction process', and give examples that satisfy these conditions. For some of these examples, we show that the Lüders projection postulate admits a consistent interpretation and remains valid in the nonlinear regime.

1 Introduction

In reference [8], in the context of a geometrical description of quantum mechanics, the first author investigated a generalisation of the Schrödinger equation called the stochastic Schrödinger evolution. This evolution is governed by a stochastic differential equation (SDE) for a random process with values in the state-manifold \mathbf{CP}^n , the space of one-dimensional subspaces of the usual linear space of states \mathbf{C}^{n+1} . The evolution requires for its definition various specific differential-geometric features of the state-manifold including its standard Kähler metric, the Fubini-Study metric.

In reference [2] Brody and Hughston generalised this extended quantum dynamics by defining the stochastic Schrödinger evolution (hereafter the SSE) in the case where the state space of the quantum system is given by a general Kähler manifold \mathcal{M} . In this theory observables are defined by holomorphic Killing vectors on \mathcal{M} , that is to say vector fields that preserve the complex structure J as well as the metric g . Such a Killing vector determines a Hamiltonian function, up to an additive constant. One of these Hamiltonian functions is then taken to be the Hamiltonian H that determines the SSE.

The SSE is an SDE on \mathcal{M} with the property that its drift reduces the dispersion V of H and its volatility is in the direction of the gradient vector field $\nabla^a H$. In particular, the stochastic Schrödinger evolution is defined in such a way that the random process for H is a martingale. Then it turns out that V is a supermartingale, or equivalently that the evolution reduces the expectation of V , if a certain holomorphic sectional curvature K_H is positive. If $K_H > 0$, then the expectation of V is reduced to zero asymptotically in the limit of large time, and Brody and Hughston [2] call the corresponding evolution on the state manifold \mathcal{M} a ‘collapse process’ for the observable H . If F is another observable, in the sense of being the Hamiltonian for another holomorphic Killing vector, which has zero Poisson bracket with H then the evolution reduces the expectation of the dispersion V^F of F if a certain holomorphic bisectional curvature is positive (these terms will be defined below). Refining the terminology of [2], we shall call this a ‘reduction process’ for F which will be called a ‘collapse process’ if the expectation of V^F is reduced to zero asymptotically in time.

The Fubini-Study metric on \mathbf{CP}^n has positive holomorphic bisectional curvature. It follows that the SSE on \mathbf{CP}^n is a collapse process for any choice of holomorphic Killing vector, with Hamiltonian H say, and that this

is simultaneously a reduction process for any other observable commuting with H . It is natural to consider other Kähler manifolds, beginning with low dimensional or otherwise familiar cases, to seek further examples of state manifolds admitting collapse and reduction processes, and that is the purpose of this article. We also consider the Lüders postulate, described in the next paragraph, which is known to be a theorem for the SSE in \mathbf{CP}^n (see [1]), to see if it can be interpreted in these examples, and, if so, if it still holds.

The Lüders postulate in standard quantum mechanics covers the situation when a wave function is a linear combination of eigenstates of a Hamiltonian, at least one of which corresponds to a degenerate eigenvalue. Suppose for simplicity that we have a finite-dimensional state space and an initial wave-function $\psi(0) = a_1\psi_1 + a_2\psi_2$, where ψ_1 lies in a one-dimensional eigenspace, U say, which therefore defines a point of the state-manifold \mathbf{CP}^n , while ψ_2 lies in a two-dimensional subspace, W say, which therefore defines a complex projective line in the state manifold. The Lüders postulate ([10]) is that the measurement process collapses ψ to ψ_1 with probability $|a_1|^2$ or to ψ_2 with probability $|a_2|^2$ despite the fact that states orthogonal to ψ_2 have the same energy as it. Thus one could write $\psi_2 = (\psi_3 + \psi_4)/\sqrt{2}$ say where ψ_3 and ψ_4 were orthogonal vectors in W . Now ψ_3 appears in the orthogonal expansion of $\psi(0)$ but there is zero probability of collapse to it. In the SSE on \mathbf{CP}^n , the evolution is confined to the projective line through $\psi(0)$ and ψ_1 and so can only terminate at one of ψ_1 and ψ_2 ; thus the Lüders postulate is a theorem in this case. If this is to be true in other state-manifolds, then we shall need to identify some geometrical objects equivalent to these various structures, in particular the submanifold to which the evolution is confined.

The plan of the article is as follows. In Section 2, we shall review the relevant background material on stochastic reduction on Kähler manifolds that appeared in [8] and [2]. In Section 3, we consider one-dimensional Kähler manifolds, that is to say, manifolds of two real dimensions. Here the requirement that the SSE determines a collapse process is strong enough to restrict the geometry quite severely: the metric is determined by a single function of one variable satisfying a convexity condition. In Section 4, we consider the two-dimensional case. Here there is room for several observables, in that one can have several commuting Killing vectors. A particularly interesting case is that of metrics with $U(2)$ symmetry transitive on hypersurfaces, or of LRS Bianchi-type IX in the language of general relativity. This case includes some familiar Kähler metrics, for example the Eguchi-Hansen metric [3] and some metrics of Hitchin [6]. In Section 5, we generalise some of the examples

of Section 4 to find Kähler metrics with $U(N)$ symmetry admitting collapse processes with many observables.

2 Stochastic reduction on Kähler manifolds

In this section, we shall review the ideas of references [2] and [8]. As we have said, the stochastic Schrödinger evolution (or SSE) defined in these references is an SDE for a random process with values in a Kähler manifold \mathcal{M} . The idea that the state space of quantum theory can be generalised to a Kähler manifold with symmetries was introduced by Kibble [9], and has since been developed further by a number of authors (see, e.g., the references cited in [2] and [7]). Suppose that the manifold \mathcal{M} has metric g , complex structure J and Kähler form ω . The evolution requires for its definition a holomorphic Killing vector T . The Killing property of T is

$$\mathcal{L}_T g = 0$$

and the holomorphic property is

$$\mathcal{L}_T \omega = 0. \tag{1}$$

It follows from (2) that

$$i_T \omega = dH \tag{2}$$

for some function H , which is by definition the Hamiltonian function associated with the Killing vector T . In the quantum mechanical interpretation, H is an observable and its dispersion V is defined as

$$V = g^{ab} \nabla_a H \nabla_b H. \tag{3}$$

In the case of the Fubini-Study manifold it can be shown that V is the familiar squared uncertainty of H in the state corresponding to the given point of \mathcal{M} . The SSE of [2] and [8] is given by

$$dx_t^a = (2\omega^{ab} \nabla_b H - \frac{1}{4} \sigma^2 \nabla^a V) dt + \sigma \nabla^a H dW_t, \tag{4}$$

where x_t^a represents a random variable labelled by t and taking values in \mathcal{M} , dx_t^a is a covariant Ito differential and σ is a constant. The SDE (4) is

defined with reference to a fixed probability space and filtration, with respect to which W_t is a standard Brownian motion. We note in particular that (4) is covariant; in fact, the indices are abstract [8], and it reduces to the usual Schrödinger equation when σ vanishes. If σ does not vanish then the SDE (4) has the property that it reduces the expected value of the dispersion V , as we shall see. The argument is as follows. First, an application of Ito's lemma shows that the random process $H_t = H(x_t)$ is a martingale:

$$dH_t = \sigma V_t dW_t.$$

where $V_t = V(x_t)$. From (4), Hughston and Brody [2] derive an equation for V_t given by

$$dV_t = -\sigma^2 K_H V_t^2 dt + \sigma \nabla^a H \nabla_a V dW_t, \quad (5)$$

where K_H is a particular holomorphic sectional curvature, namely

$$K_H = \frac{1}{V^2} R_{apcq} J_b^p J_d^q \nabla^a H \nabla^b H \nabla^c H \nabla^d H. \quad (6)$$

Now if K_H is strictly positive, then the SSE (4) leads via (5) to a supermartingale condition on V_t , so that the expectation of V_t decreases. In this case it can be shown that the evolution reduces the expectation of V_t to zero (in the limit as $t \rightarrow \infty$) and halts at a critical point of H or equivalently at a fixed point of T . Following Brody and Hughston [2] we shall therefore call this a collapse process.

Next suppose that F denotes another observable. Suppose, in other words, that there is another holomorphic Killing vector X with Hamiltonian function F . We say that F and H commute iff their Poisson bracket vanishes, that is to say if and only if

$$\omega^{ab} \nabla_a F \nabla_b H = 0,$$

from which it follows that the corresponding Killing vectors X and T commute ([2]). In this case, one can ask whether a collapse process for H necessarily also reduces or even collapses F . In particular, suppose we now write

$$V^F = g^{ab} \nabla_a F \nabla_b F$$

for the dispersion of F , and V^H for the dispersion of H given in (3). Then for the process V^F one finds the SDE

$$dV_t^F = -\sigma^2 K_{FH} V_t^F V_t^H dt + \sigma \nabla^a H \nabla_a V^F dW_t, \quad (7)$$

where K_{FH} is the biholomorphic sectional curvature given by

$$K_{FH} = \frac{1}{V^F V^H} R_{apcq} J_b^p J_d^q \nabla^a H \nabla^b H \nabla^c F \nabla^d F. \quad (8)$$

If this biholomorphic sectional curvature is positive then the SSE, via (7), necessarily reduces the expectation of the dispersion of F . We shall call this a *reduction* process for F . If we have a collapse process for H that is simultaneously a reduction process for F but is also such that the expectation of V^F tends to zero asymptotically, we shall call this a *collapse* process for the observable F . In the terminology of random processes, we have assumed that V^H is a supermartingale, and we have deduced that it is then necessarily a *potential* ([1], [11]). To have simultaneously a reduction process for F , V^F must be a supermartingale, while to have a collapse process for F , V^F must also be a potential.

A process that simultaneously collapses H and F must terminate at a point that is a critical point of both H and F . A process that collapses H but merely reduces F must terminate at a degenerate critical point of H . It is possible, even in the case of \mathbf{CP}^2 with the Fubini-Study metric, to have a process that collapses H and reduces but does not collapse F . To see this one can take an observable H with a one-dimensional eigenspace, which defines a point, say U_1 , in \mathbf{CP}^2 , and a two-dimensional eigenspace, which comes from a degenerate eigenvalue and defines a complex projective line, say W , in \mathbf{CP}^2 . Now suppose F is an observable that commutes with H but has nondegenerate eigenvalues. Recall that an eigenvector of F thought of as a $GL(3, \mathbf{C})$ matrix that corresponds to a nondegenerate eigenvalue is necessarily an eigenvector of H (since F and H commute by assumption). Thus the eigenspaces of F will define points U_2 and U_3 on W , together with U_1 . Now the SSE will evolve an initial state that is not on either of the complex projective lines joining U_1 to U_2 and U_1 to U_3 either to U_1 or to a point in W that does not correspond to an eigenvector of F and so is not a critical point of F . This is a collapse of H and a reduction but not a collapse of F . We shall see in Case 1 of Section 4 that this phenomenon can continue to occur in the new examples of state-manifolds explored below.

In the following sections, we shall be interested in Kähler manifolds of low dimension with enough holomorphic Killing vectors to give interesting sets of observables, and with conditions of positivity on the curvature sufficient to lead to collapse and reduction processes.

3 Reduction processes on one-dimensional state manifolds

Any Riemannian manifold of two real dimensions defines a Kähler manifold of one complex dimension. Here there is only one independent component of curvature, namely the Gauss curvature. To have a collapse process this must be positive. It is a theorem of Cohn-Vossen (for references see [5]) that a complete Riemann surface with positive Gauss curvature is diffeomorphic to a sphere or a plane. If we assume, as seems reasonable, that a state manifold must be complete, then the state manifold for a collapse process is necessarily one of these two.

To define the SSE we need a holomorphic Killing vector, which in the case of a one-dimensional state manifold is any Killing vector. This could have open or closed trajectories on the plane but on the sphere must have closed trajectories. In either case the metric can be written in the form

$$ds^2 = d\theta^2 + (S(\theta))^2 d\phi^2 \quad (9)$$

in terms of a function $S(\theta)$, where the Killing vector is $T = \partial/\partial\phi$. For this metric we calculate the Gauss curvature and obtain

$$K = -\frac{1}{S} \frac{d^2 S}{d\theta^2}. \quad (10)$$

For a collapse process this needs to be positive, and it then follows that S must have a zero. For completeness of the state manifold at a zero of S the trajectories must be closed which makes ϕ periodic, and for definiteness we assume the period to be 2π . Also for completeness we need to avoid having a conical singularity at any zero of S , which requires $dS/d\theta = \pm 1$ there. This is because, for an infinitesimal circle around the zero, we want the ratio of circumference to radius in the limit of vanishing radius to be 2π . For a metric on the plane, S has a single zero; while for a metric on the sphere, S has two zeroes. For definiteness, we suppose that the necessary zero of S is at $\theta = 0$.

The Kähler form associated with the metric (9) is given by

$$\omega = S(\theta) d\theta \wedge d\phi.$$

Then as a consequence of (2) the Hamiltonian $H(\theta)$ satisfies

$$\frac{dH}{d\theta} = -S. \quad (11)$$

It then follows from (3) that the dispersion V , is given by

$$V = S^2. \tag{12}$$

The zero or zeroes of S are fixed points of the Killing vector T and critical points of the Hamiltonian H . The SSE will evolve towards such a zero.

Looking ahead to Section 5, we introduce a holomorphic coordinate tied to the symmetry and taking the form $z = re^{i\phi}$. The Kähler metric (9) necessarily has a Kähler potential $\Sigma(u)$ where $u = r^2$. In terms of Σ the metric can be written

$$ds^2 = 2\Sigma_{,z\bar{z}}dzd\bar{z},$$

and thus

$$ds^2 = 2(\dot{\Sigma} + u\ddot{\Sigma})(dr^2 + r^2d\phi^2),$$

where the dot denotes d/du . Comparing (3) with (9) and (12) we can make the identification

$$V = 2u(\dot{\Sigma} + u\ddot{\Sigma}). \tag{13}$$

Then as a consequence of (11) and (12) we deduce that $H = -u\dot{\Sigma}$ and $V = -2u\dot{H}$.

The zero of S at $\theta = 0$ we can suppose to correspond to a zero of V at $r = 0$. It is convenient to introduce a new radial coordinate $\chi = \log r$ so that the metric takes the form

$$ds^2 = V(d\chi^2 + d\phi^2).$$

With these coordinates, the SSE (4) becomes

$$d\chi_t = -\frac{1}{2}\sigma^2\frac{d\log V}{d\chi}dt + \sigma dW_t, \quad d\phi = 2dt. \tag{14}$$

Note that the evolution for ϕ is deterministic: this will be important later, in Section 5. In terms of the holomorphic coordinate z , we find

$$dz = z[2i + \frac{1}{2}\sigma^2(1 - \frac{d\log V}{d\chi})]dt + z\sigma dW_t. \tag{15}$$

This is an equation that will be generalised in Section 5. Finally the Gauss curvature (10) is given by

$$K = -\frac{1}{2V}\frac{d^2\log V}{d\chi^2}. \tag{16}$$

Comparing (14) and (16) we see that if the curvature K is positive then the drift in the SSE for χ is monotonic and increasing in χ .

By assumption there is a fixed point at $r = 0$ or $\chi = -\infty$. Here $S = 0$ and $dS/d\theta = 1$, and one can calculate the limit of the drift in χ in (14) as $-\sigma^2$. The drift is therefore towards the fixed point. Now consider increasing χ . If $dS/d\theta$ ever has a zero then the positivity of K forces S to have another zero and completeness then requires $dS/d\theta = -1$ at this zero. It is easy to see that this zero occurs at $\chi = \infty$. The manifold is now necessarily the sphere, and the limit of the drift near this fixed point is σ^2 , so again it is towards the fixed point. The drift changes sign at the maximum of V . If $dS/d\theta$ never has a zero then the drift never changes sign and for all χ is towards the fixed point at $\chi = -\infty$.

By calculating specific examples of surfaces of revolution in Euclidean three-space one finds that for large positive χ on an asymptotically hyperbolic surface the drift tends to $-k\sigma^2$ where k is a positive constant less than unity, while for an asymptotically parabolic surface the drift tends to zero as $O(\chi^{-1})$.

In the case when the manifold is a sphere, it is possible to calculate the probabilities π_+ and π_- , that the evolution terminates at $\chi = \infty$ or at $\chi = -\infty$ respectively, in terms of the initial value of χ . This is done by solving the backward Fokker-Planck equation associated with the SDE (14) with the relevant boundary conditions ([4]). The result for π_+ is

$$\pi_+ = \frac{H(\chi) - H(-\infty)}{H(\infty) - H(-\infty)}$$

with $\pi_- = 1 - \pi_+$.

For the asymptotically hyperbolic or parabolic (or even cylindrical) examples, the same calculation gives $\pi_- = 1$, as expected. In other words, the reduction proceeds towards the single ‘eigenstate’, ultimately resulting in collapse.

4 Reduction processes on two-dimensional state manifolds

In two dimensions there are many more curvature components, and as a consequence the restriction that just one holomorphic sectional curvature

should be positive is not a strong one. Simple examples of Kähler manifolds with this property can be constructed by taking products of a Riemann surface with an example from the previous section. More interesting examples arise if we assume that there are more observables and look for processes that collapse or reduce several of these observables at once. With this possibility in view, we consider metrics with a $U(2)$ action transitive on three-surfaces, equivalently stated as an action with three-dimensional principal orbits. That is to say, we have holomorphic Killing vectors T and X_i , for $i = 1, 2, 3$, with the commutators

$$[X_i, X_j] = -\epsilon_{ij}{}^k X_k; \quad [X_i, T] = 0. \quad (17)$$

We shall thus be led to examine explicit examples of nonlinear state manifolds with observables H and S_i associated with the symmetries T and X_i respectively, which we can think of as *energy* and *spin* respectively. We shall consider the evolution defined by T and seek manifolds that give rise to collapse processes for the energy. It will turn out that some of these simultaneously reduce the spin.

We begin by introducing a set of Euler angles (θ, ϕ, ψ) , in terms of which we shall take the Killing vectors to be given by the following expressions:

$$\begin{aligned} X_1 + iX_2 &= e^{i\phi} \left(i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} + \csc \theta \frac{\partial}{\partial \psi} \right), \\ X_3 &= \frac{\partial}{\partial \phi}, \\ T &= \frac{\partial}{\partial \psi}. \end{aligned} \quad (18)$$

We shall work with the usual basis of invariant one-forms σ_i , which are given in these coordinates by

$$\begin{aligned} \sigma_1 + i\sigma_2 &= e^{i\psi} (d\theta - i \sin \theta d\phi), \\ \sigma_3 &= d\psi + \cos \theta d\psi, \end{aligned} \quad (19)$$

so that $d\sigma_1 = \sigma_2 \wedge \sigma_3$, $d\sigma_2 = \sigma_3 \wedge \sigma_1$ and $d\sigma_3 = \sigma_1 \wedge \sigma_2$. The most general metric with the desired symmetry is now

$$ds^2 = dt^2 + a^2(t)(\sigma_1^2 + \sigma_2^2) + c^2(t)\sigma_3^2. \quad (20)$$

The metric depends on a pair of functions $a(t)$ and $c(t)$, where t , which we can think of as either a ‘time’ or a ‘radial’ coordinate, labels the surfaces

of homogeneity. The generic surface of homogeneity, or equivalently the principal orbit of the symmetry group $U(2)$, can be a three-sphere or a Lens space $L(3, n)$ which is a quotient of the three-sphere by a cyclic group of order n , and the orbits degenerate at zeroes of a and/or c . Given the topology of the principal orbit, the topology of the underlying manifold \mathcal{M} is determined by the nature of the degenerate orbits and the (one-dimensional) manifold say \mathcal{T} which t ranges over. Clearly this must be one of four possibilities: a circle, an interval, the whole line or a half-line. The first two give compact state-manifolds and the last two give noncompact state-manifolds.

It will be convenient to work with a particular orthonormal frame for this metric, namely

$$\theta^0 = dt, \theta^1 = a\sigma_1, \theta^2 = a\sigma_2, \theta^3 = c\sigma_3. \quad (21)$$

In terms of this frame we define the complex structure J by

$$J\theta^0 = \theta^3, \quad J\theta^1 = \theta^2.$$

The corresponding two-form is then given by

$$\omega = cdt \wedge \sigma_3 + a^2\sigma_1 \wedge \sigma_2. \quad (22)$$

It can be checked that the complex structure is automatically integrable and preserved by the Killing vectors, so that these are all holomorphic. The Kähler condition is that the form ω given by (22) should be closed. This requires that

$$\frac{d(a^2)}{dt} = c. \quad (23)$$

Equation (23) immediately shows that \mathcal{T} cannot be a circle: if a were periodic in t then c would have zeroes at which the metric would degenerate. On the other hand, since $c^2 = g(T, T)$, we need at least one zero in c or there are no fixed points of the evolution. This shows that \mathcal{T} must be either an interval, in which case there will be two critical values of H , or a half-line, when there will be just one.

We next find Hamiltonian functions for the four Killing vectors. Let us denote these Hamiltonian functions as H for T and S_i for X_i . Then from (18), (19) and (22) these are easily found as

$$\begin{aligned} H &= -a^2, \\ S_1 + iS_2 &= -a^2 \sin \theta e^{i\phi}, \\ S_3 &= -a^2 \cos \theta. \end{aligned}$$

As a consequence, we note that

$$H^2 = S_1^2 + S_2^2 + S_3^2.$$

With ω as in (22) it is a simple matter to check that the Poisson brackets $\omega(T, X_i)$ all vanish, so that the observables S_i commute with the Hamiltonian H . The S_i have the usual $SU(2)$ commutators with each other, i.e. $\omega(S_1, S_2) = S_3$, $\omega(S_2, S_3) = S_1$ and $\omega(S_3, S_1) = S_2$.

It is convenient to introduce a new radial coordinate $R = 2a$, which is allowed because $\dot{a} \neq 0$ in the interior of the coordinate range (otherwise c would have a zero, which can only happen at the end of the range). Then $c = \frac{1}{2}R\dot{R}$ and the metric (20) can be written in the form

$$ds^2 = \frac{1}{F}dR^2 + \frac{1}{4}R^2(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}R^2F\sigma_3^2, \quad (24)$$

where $F(R) = \dot{R}^2$. Written like this, the metric depends on one function F of R . There are various possibilities for the topology of the underlying manifold, determined by the range of R for which F is positive and the behaviour of F near its zeroes. There is some standard terminology too: see, for example, reference [14]. If $R = 0$ is in the allowed range then for completeness of the metric we need $F = 1$ there: this is the condition that the coordinate singularity, which is like the singularity at the origin of polar coordinates, can be removed and the manifold can be completed by inserting a point. A coordinate singularity of this kind is called a ‘nut’ in the literature.

For completeness at a zero of F , say at $R = R_0 \neq 0$, we need

$$R_0F'(R_0) = \pm 2n \quad (25)$$

for a positive integer n , where the principal orbit is $L(3, n)$, so $n = 1$ if the principal orbit is a three-sphere. In this case the coordinate singularity can be removed and the manifold completed by inserting a two-sphere. A coordinate singularity of this kind is called a ‘bolt’ in the literature. If there are two bolts then for completeness they must have the same n with one plus and one minus in (25). If there is a nut and a bolt then the principal orbits are three-spheres and $n = 1$ at the bolt. There cannot be two nuts because between them \dot{a} would have a zero whence so would c by (23) and there would be a bolt before the second nut. We shall see below how the possibilities are constrained further by conditions of positivity on the curvature.

We can calculate the dispersions V and V_i associated with the observables H and S_i respectively by use of (18) and (24). These turn out to be

$$\begin{aligned}
V &= \frac{1}{4}R^2 F, \\
V_1 &= \frac{1}{4}R^2(\sin^2 \theta + F \cos^2 \theta), \\
V_2 &= \frac{1}{4}R^2(\cos^2 \phi + \sin^2 \phi(\cos^2 \theta + F \sin^2 \theta)), \\
V_3 &= \frac{1}{4}R^2(\sin^2 \phi + \cos^2 \phi(\cos^2 \theta + F \sin^2 \theta)). \tag{26}
\end{aligned}$$

The fixed points of the Killing vectors, which are the zeroes of the respective dispersions, can be read off from (26). All four Killing vectors vanish at the nut, if there is one. Thus a nut is an isolated fixed point of T , and therefore a non-degenerate critical point of H , as well as an isolated fixed point of each X_i . The observables H and S_i all vanish there. If there is a bolt, then T vanishes at all points of it while each X_i vanishes at a pair of antipodal points, and for varying i the pairs are symmetrically arranged. Thus a bolt is a whole two-sphere, in fact a \mathbf{CP}^1 , of fixed points of T that are degenerate critical points of H , and each S_i has two critical values on the bolt. The Hamiltonian takes the value $H_0 = -R_0^2/4$ on a bolt at $R = R_0$ and the critical values of the S_i are $\pm H_0$.

We next need to calculate the Riemann tensor for the metric (24). This is readily done in the orthonormal basis of (21), with the following result:

$$\begin{aligned}
R_{0101} &= R_{0202} = -\frac{1}{2R}F', \\
R_{0123} &= R_{0231} = -\frac{1}{2}R_{0312} = \frac{1}{2R}F', \\
R_{0303} &= -\frac{1}{2}(F'' + \frac{3}{R}F'), \\
R_{1212} &= \frac{4}{R^2}(1 - F), \tag{27}
\end{aligned}$$

where the prime denotes d/dR . Once we have the Riemann tensor, we can calculate the biholomorphic sectional curvatures. Suppose that U and W are

arbitrary unit vectors, so that

$$\begin{aligned} U &= Ae_0 + Be_1 + Ce_2 + De_3, \\ W &= \alpha e_0 + \beta e_1 + \gamma e_2 + \delta e_3, \end{aligned}$$

where $A^2 + B^2 + C^2 + D^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$ and the e_i are the basis of vector fields dual to the θ^i . Then

$$\begin{aligned} JU &= -De_0 - Ce_1 + Be_2 + Ae_3, \\ JW &= -\delta e_0 - \gamma e_1 + \beta e_2 + \alpha e_3, \end{aligned}$$

and the corresponding biholomorphic sectional curvature can be written

$$\begin{aligned} R(U, JU, W, JW) &= R_{0303}(\alpha^2 + \delta^2)(A^2 + D^2) \\ &\quad + R_{0312}((\alpha B + \beta A - \gamma D - \delta C)^2 \\ &\quad + (\alpha C + \beta D + \gamma A + \delta B)^2) \\ &\quad + R_{1212}(\beta^2 + \gamma^2)(B^2 + C^2). \end{aligned} \quad (28)$$

For the Killing vector $T = ce_0$ that determines the evolution, the relevant holomorphic sectional curvature K_H from (6) and above is R_{0303} . The condition for a collapse process for H is therefore

$$R_{0303} = -\frac{1}{2R^3}(R^3 F')' > 0. \quad (29)$$

If we want the condition for this process to reduce, say, S_3 as well, we first need X_3 in the basis (e_i) . This is

$$X_3 = a \sin \theta (\sin \psi e_1 - \cos \psi e_2) + c \cos \theta e_3.$$

Now for the biholomorphic sectional curvature, as defined by analogy with (8) we find

$$K_{HS_3} = \frac{1}{\Delta}(R_{0303}F \cos^2 \theta + R_{0312} \sin^2 \theta),$$

where $\Delta = F \cos^2 \theta + \sin^2 \theta$.

We shall have a reduction process for S_3 as well as for H if this is positive too, which requires

$$R_{0312} = -\frac{1}{2R}F' > 0 \quad (30)$$

in addition to (29). We would obtain the same condition by considering S_1 or S_2 instead of S_3 , so that (29) and (30) taken together are necessary and sufficient for the evolution defined by T to collapse H and simultaneously reduce any one of the S_i .

We obtain a range of examples by considering various cases, which we now proceed to summarise.

Case 1: one nut and one bolt

Suppose that F is positive at $R = 0$, has a zero at $R = R_0 > 0$, and is positive in between, so that the state manifold \mathcal{M} is compact. For completeness we need a nut at $R = 0$, so that $F(0) = 1$, $F'(0) = 0$, and the principal orbit is a three-sphere, and a bolt at $R = R_0$, so that $R_0 F'(R_0) = -2$. We can now identify \mathcal{M} as topologically (and therefore in fact biholomorphically) equivalent to \mathbf{CP}^2 .

For a collapse process for H we need (29), but since $F'(0) = 0$ this will ensure that $F'(R) < 0$, from which (30) will necessarily follow: by insisting on a collapse process for H we automatically obtain a reduction process for S_3 . Finally, F is decreasing with $F(0) = 1$ so that $F < 1$ and by (27) this ensures that $R_{1212} > 0$. Putting this together with (28) we now have enough to make every biholomorphic sectional curvature positive. It is a theorem of Siu and Yau ([15], see also [12]) that a compact Kähler manifold with positive biholomorphic sectional curvatures is necessarily \mathbf{CP}^n .

The reduction process for S_3 will not necessarily be a collapse process as we shall see from a discussion of the Lüders postulate in Section 5. However, since every biholomorphic sectional curvature is positive given (29), we could instead use S_3 as Hamiltonian. This new SSE would give a collapse process for S_3 which would also necessarily reduce H . Now the critical points of S_3 are nondegenerate and therefore are also critical points of H , as we can see in any case from (26). Thus the reduction process for H is in fact a collapse process for H and this process simultaneously collapses S_3 and H .

This example, with a metric on \mathbf{CP}^2 that is different from the Fubini-Study metric, is very similar to the example with a degenerate H discussed in the Section 1, and can be seen as a natural generalisation of this example.

Case 2: two bolts

The other compact case arises if F has zeroes at two nonvanishing values of R and is positive in between. Call these R_0 and R_1 with $R_0 \leq R_1$. Then we have two bolts and we require that

$$R_0 F'(R_0) = -R_1 F'(R_1) = 2n \quad (31)$$

for some positive integer n , so that the principal orbit is $L(3, n)$. Looking at (27) we see that R_{0312} must change sign, so that by (30) we cannot have a reduction process for S_3 even though, by (29), we can arrange to have a collapse process for H . Note also that the critical points of H lie on two distinct complex projective lines, the two bolts. This could not happen in \mathbf{CP}^2 where two complex projective lines necessarily meet.

There are some specific examples of this case due to Hitchin [6]. In our notation they are defined by

$$F(R) = \frac{(R^2 - 1)(sn + 1 - R^2)}{sR^2},$$

where s is a positive real constant and n is a positive integer greater than 1. There are bolts at the two positive zeroes of F and (31) is satisfied with the same n at both. The state manifold \mathcal{M} is a rational surface, specifically the projective space $\mathbf{P}(\mathcal{O}(n) + \mathcal{O}(0))$ of the vector bundle $\mathcal{O}(n) + \mathcal{O}(0)$ over \mathbf{CP}^1 .

From (29) we find $R_{0303} = 4/s$ which is positive, so that we do have a collapse process for H , but as was already noted, this process does not simultaneously reduce any of the S_i . The interest of this example for Hitchin was that for small enough s , in fact for $s < 1/n^2$, the holomorphic sectional curvatures are all positive (although the biholomorphic sectional curvatures are not, as can be seen from the theorem of Siu and Yau [15] already cited).

Putting these two cases together, we can assert that for a compact \mathcal{M} with the symmetry assumed here, if the stochastic Schrödinger evolution collapses H and reduces S_3 then the manifold must be \mathbf{CP}^2 with a metric of positive biholomorphic sectional curvature.

Case 3: semi-infinite with a nut

The simplest noncompact case has a nut at $R = 0$, for which therefore $F(0) = 1, F'(0) = 0$, and F is everywhere positive. If we have a collapse

process for H then, by (29), we must have $(R^3 F')' > 0$. With the boundary conditions from the nut this forces $F' < 0$ and so $0 < F < 1$ for all positive R . The state manifold in this case is \mathbf{C}^2 .

Now from (27) and (28) all biholomorphic sectional curvatures are positive. In particular the collapse process for H is simultaneously a reduction process for all the S_i . Further, this reduction process is also a collapse process because the nut is a nondegenerate critical point for all the observables. However this case is less interesting because the endpoint of the stochastic evolution is the origin, where all the observables are zero.

A simple example of this case is given by

$$F = \frac{1 + \lambda R^2}{1 + R^2},$$

where λ is a real constant with $0 \leq \lambda \leq 1$. The corresponding metric is the flat metric on \mathbf{C}^2 for $\lambda = 1$, is asymptotic to a deformed ('Berger') three-sphere at infinity if $0 < \lambda < 1$ and is ALF (or asymptotically locally flat) in the language of [14] if $\lambda = 0$.

Case 4: semi-infinite with a bolt

The final case has a bolt at say $R = R_0$ so that $F(R_0) = 0$, $R_0 F'(R_0) = 2n$, with $F > 0$ for $R > R_0$. We can arrange to satisfy (29) so that we have a collapse process for H , but we cannot satisfy (30) and simultaneously obtain a reduction process for S_i . A familiar example of this case is the Eguchi-Hanson metric ([3], [14]) which has $F = 1 - a^4/R^4$. This has a bolt with $n = 2$ at $R = a$ and is asymptotically locally Euclidean in the standard terminology (see, for example, [14]).

In summary, we have constructed a variety of two-dimensional quantum state manifolds admitting collapse processes for energy, and some of these simultaneously reduce the spin. The most interesting case is the first, which corresponds to a metric on \mathbf{CP}^2 with positive biholomorphic sectional curvatures but which is different from the Fubini-Study metric. In the next section, we consider a generalisation to metrics with N complex dimensions and $U(N)$ -symmetry.

5 Kähler metrics with $U(N)$ symmetry

We want to generalise the calculations of the last section to the consideration of Kähler metrics with $U(N)$ -symmetry transitive on hypersurfaces. To do this we mimic the calculation leading to (3). Suppose we have complex coordinates z^a , $a = 1, \dots, N$, where these are coordinate indices (not abstract indices), and a Kähler potential $\Sigma(u)$ where

$$u = r^2 = \delta_{a\bar{b}} z^a \bar{z}^b = \sum_{a=1}^N |z^a|^2. \quad (32)$$

The metric in these coordinates is $ds^2 = 2g_{a\bar{b}} dz^a d\bar{z}^b$ with $g_{a\bar{b}} = \partial^2 \Sigma / \partial z^a \partial \bar{z}^b$ so that

$$g_{a\bar{b}} = \dot{\Sigma} \delta_{a\bar{b}} + \ddot{\Sigma} z_{\bar{a}} \bar{z}_b, \quad (33)$$

where as before the dot means d/du , and we have introduced the notation

$$z_{\bar{a}} = \delta_{b\bar{a}} z^b, \quad \bar{z}_b = \delta_{b\bar{a}} \bar{z}^a.$$

The metric (33) is positive definite provided the following conditions hold:

$$\dot{\Sigma} > 0, \quad \dot{\Sigma} + u\ddot{\Sigma} > 0.$$

The metric degenerates where either of these quantities vanishes; but, as we shall see, these degeneracies may correspond to removable coordinate singularities. The inverse metric is easily found to be given by

$$g^{a\bar{b}} = \frac{1}{\dot{\Sigma}} (\delta^{a\bar{b}} - Q z^a \bar{z}^b), \quad (34)$$

where $Q = \ddot{\Sigma} / (\dot{\Sigma} + u\ddot{\Sigma})$. In these coordinates we can find the following holomorphic Killing vectors:

$$\begin{aligned} T &= i \left(z^a \frac{\partial}{\partial z^a} - \bar{z}^b \frac{\partial}{\partial \bar{z}^b} \right), \\ X &= i \left(z^a H_a^c \frac{\partial}{\partial z^c} - \bar{z}^b H_{\bar{b}}^{\bar{d}} \frac{\partial}{\partial \bar{z}^{\bar{d}}} \right), \end{aligned} \quad (35)$$

where $H_a^b = \delta^{b\bar{a}} H_{a\bar{a}}$ and $H_{\bar{b}}^{\bar{d}} = \delta^{b\bar{d}} H_{b\bar{b}}$ for an arbitrary trace-free Hermitian matrix $H_{a\bar{a}}$. It is easy to see that these are holomorphic and they are Killing vectors since they preserve the function u of (32). The Killing vectors of

the form X generate the Lie algebra of $SU(N)$ as $H_{a\bar{a}}$ runs over trace-free Hermitian matrices. The Hamiltonians for these Killing vectors turn out to be

$$H = -u\dot{\Sigma}, \quad (36)$$

and

$$S = -H_{a\bar{b}}z^a\bar{z}^b\dot{\Sigma}. \quad (37)$$

We note that, interestingly, (36) is formally just as in the one-dimensional case. With the aid of (34) the dispersion V of H is found to be

$$V = 2g^{a\bar{b}}\partial_a H \partial_{\bar{b}} H = -2u\dot{H},$$

which is also formally the same as in the one-dimensional case (13). The critical points of H are at the zeroes of V , that is to say at $u = 0$ or at $\dot{\Sigma} + u\ddot{\Sigma} = 0$, and the second of these is where the metric degenerates.

The dispersion V^S of S can be written in either of the following forms:

$$V^S = 2g_{a\bar{b}}H_{\bar{c}}^{\bar{b}}\bar{z}^c H_d^a z^d \quad (38)$$

or

$$V^S = 2(\dot{\Sigma}H_{\bar{c}}^{\bar{b}}H_{\bar{b}d}\bar{z}^c z^d + \ddot{\Sigma}(H_{a\bar{b}}z^a\bar{z}^b)^2). \quad (39)$$

From (38) one can see that the fixed points of X , which are the critical points of S and the zeroes of V^S , necessarily occur either at $u = 0$, where every z^a is zero, or on the surface where the metric degenerates. The latter critical points, by (39), occur for z^a satisfying

$$(\delta_{a\bar{b}}z^a\bar{z}^b)(H_{\bar{c}}^{\bar{e}}H_{\bar{e}d}z^d\bar{z}^c) = (H_{a\bar{b}}z^a\bar{z}^b)^2,$$

which holds when z^a is (proportional to) an eigenvector of H_a^b . We can assume that the eigenvalues of H_a^b are distinct (since we can choose a basis of trace-free Hermitian matrices all elements of which have distinct eigenvalues) and then the critical points of S will be nondegenerate.

We want to write down the SSE (4) in these coordinates. For this we note with the aid of (34) that

$$\nabla^a H = z^a$$

and

$$\nabla^a V = \frac{-2u\dot{V}}{V}z^a.$$

Now (4) becomes just

$$dz^a = z^a \left[2i + \frac{\sigma^2}{2} \left(1 - \frac{2u\dot{V}}{V} \right) \right] dt + z^a \sigma dW_t, \quad (40)$$

which should be compared with equation (15). Recall that the indices in (40) are coordinate indices. To give something like (3), we introduce a form of polar coordinates

$$z^a = r\zeta^a \quad (41)$$

where $\delta_{\bar{a}b}\zeta^a\bar{\zeta}^b = 1$. The metric then becomes

$$\begin{aligned} ds^2 &= 2g_{\bar{a}b}dz^a d\bar{z}^b \\ &= 2(\dot{\Sigma} + u\ddot{\Sigma})dr^2 + 2u\dot{\Sigma}\delta_{\bar{a}b}d\zeta^a d\bar{\zeta}^b + \frac{1}{4}u^2\ddot{\Sigma}\Theta^2, \end{aligned} \quad (42)$$

where Θ is defined by

$$\Theta = 2i\delta_{\bar{a}b}\zeta^a d\bar{\zeta}^b. \quad (43)$$

Equation (42) is the counterpart of (3). If we use the polar decomposition (41) in (40) we find

$$d\zeta^a = 2i\zeta^a dt, \quad (44)$$

which is again deterministic, just as the equation for ϕ in (14) was. This is the significant result for the Lüders postulate discussed in the Introduction. Suppose the SSE starts from an initial point with coordinates z_0^a . By a linear change of coordinates, which is a symmetry, we can arrange that $z_0^a = 0$ for $a = 2, \dots, N$, and then the evolution (44) will ensure that the state remains on this one-dimensional complex submanifold.

The other part of (40) is conveniently written in terms of $\chi = \frac{1}{2} \log u$ and then another application of Ito's lemma leads to

$$d\chi_t = -\frac{1}{2}\sigma^2 \frac{d \log V}{d\chi} dt + \sigma dW_t, \quad (45)$$

which is precisely the same as in the one-dimensional case (14).

To make the metric (42) more like (24) in appearance, we introduce the radial coordinate R by $R^2 = 2u\dot{\Sigma}$ to find

$$ds^2 = \frac{1}{F}dR^2 + \frac{1}{4}R^2 g_{FS} + \frac{1}{4}R^2 F\Theta^2, \quad (46)$$

where

$$F = \frac{\dot{\Sigma} + u\ddot{\Sigma}}{\dot{\Sigma}}, \quad g_{FS} = \delta_{\bar{a}b}d\zeta^a d\bar{\zeta}^b - \frac{1}{4}\Theta^2.$$

Here g_{FS} is the standard Fubini-Study metric on \mathbf{CP}^{N-1} , with constant holomorphic sectional curvature equal to one, and Θ from (43) is real because of the normalisation of ζ^a . By transforming the metric to look like (24), we make it easier to discuss the underlying topology. As before, the range of the ‘time’-coordinate is an interval for a compact state-manifold or a half-line for a noncompact state-manifold.

The next step is to find the connection and curvature, and for this we may use formulae from Kodaira and Morrow [13]:

$$\Gamma_{bc}^a = g^{a\bar{a}}\partial_b g_{c\bar{a}}, \quad R_{\bar{c}d}^a = \partial_{\bar{c}}\Gamma_{bd}^a.$$

These can be found explicitly in terms of Σ . For our purposes we need K_H as in (6) with the Killing vector T of (36). This turns out to be

$$K_H = -\frac{1}{2V} \frac{d^2 \log V}{d\chi^2}, \quad (47)$$

which is the same expression as (16). The positivity of \dot{H} is needed for the metric (24) to be positive definite so again the positivity of K_H forces the drift in (45) to be monotonic. If we use the metric form (46) then the conditions of positivity turn out to be precisely what they were in Section 4. Thus positivity of K_H is again (29) and for positivity of the holomorphic bisectional curvatures K_{HS} we need as well (30). Finally, all holomorphic bisectional curvatures are positive if as well R_{1212} as given in (27) is positive.

Following through the analysis of nuts and bolts, we have the four cases as before:

Case 1: one nut and one bolt is a metric on \mathbf{CP}^N with positive holomorphic bisectional curvature, so that the SSE gives a collapse process for the Hamiltonian that is simultaneously a reduction process for any one of the $SU(N)$ observables. The Hamiltonian has two critical values: one (non-degenerate) occurs at a single point (the nut), and one (degenerate) on a \mathbf{CP}^{N-1} (the bolt). Any one of the $SU(N)$ variables has N nondegenerate critical points on the bolt and one at the nut. The SSE confines the evolution to a submanifold that is actually a complex projective line through

the nut and the initial state. Thus the Lüders postulate holds here. This analysis includes the case $N = 2$ which we did not do separately in Section 4.

Case 2: two bolts is a metric on the projective space $\mathbf{P}(\mathcal{O}(n)+\mathcal{O}(0))$ of the vector bundle $\mathcal{O}(n)+\mathcal{O}(0)$ over \mathbf{CP}^{N-1} . The Hamiltonian has two degenerate critical values, at the two bolts respectively, and the Lüders postulate still holds but we do not have a reduction process for the $SU(N)$ -observables.

In particular, we observe that, as in Section 4, a compact state-manifold with the symmetry considered here gives a collapse process for H and simultaneously a reduction process for any one of the spins S if and only if it is biholomorphic to \mathbf{CP}^N with a metric of positive holomorphic bisectional curvature.

Case 3: semi-infinite with a nut and **Case 4: semi-infinite with a bolt** have a single critical value for the Hamiltonian, respectively nondegenerate and degenerate, and **Case 3** has positive bisectional holomorphic curvature while **Case 4** does not. The SSE confines the evolution to a linear submanifold, but we cannot speak of the Lüders postulate in these cases since there is only one eigenvalue.

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