

On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum

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Abstract. Consider a spectrally one-sided Lévy process X and reflect it at its past infimum I . Call this process Y . For spectrally positive X , Avram et al. [2] found an explicit expression for the law of the first time that $Y = X - I$ crosses a finite positive level a . Here we determine the Laplace transform of this crossing time for Y , if X is spectrally negative. Subsequently, we find an expression for the resolvent measure for Y killed upon leaving $[0, a]$. We determine the exponential decay parameter ϱ for the transition probabilities of Y killed upon leaving $[0, a]$, prove that this killed process is ϱ -positive and specify the ϱ -invariant function and measure. Restricting ourselves to the case where X has absolutely continuous transition probabilities, we also find the quasi-stationary distribution of this killed process. We construct then the process Y confined in $[0, a]$ and prove some properties of this process.

Keywords. reflected Lévy process, first passage, exponential decay, excursion theory, h -transform

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0 Introduction

A spectrally one-sided Lévy process is a real-valued stochastic process with stationary and independent increments which has jumps of one sign. In this paper we will study such a Lévy process reflected at its past infimum, that is, the Lévy process minus its past infimum. In applied probability, these reflected processes frequently occur, for example in the study of the water level in a dam, the work load in a queue or the stock level (See e.g. [1, 8, 20] and references therein.) Moreover, the reflected Lévy process occurs in relation with a problem associated with mathematical finance. See [17] and references therein.

The paper consists of three parts. In the first part, we study the level-crossing probabilities of the reflected Lévy process. For spectrally positive Lévy processes X , Avram et al. [2] found an explicit expression for the Laplace transform of the first exit-time of the reflected process from $[0, a]$. In Section 3.2 we complement this study by obtaining the Laplace transform of the exit time for the reflected process of the dual, a spectrally negative Lévy process. Subsequently, in Section 4, we solve for the resolvent measure of the transition probabilities of the reflected Lévy process killed upon leaving $[0, a]$.

From the study of [2] and Section 3.2, it appears that the Laplace transforms of the exit times of X and Y from finite and semi-finite intervals can be expressed in terms of $W^{(q)}$ and $Z^{(q)}$, where $Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy$ and $W^{(q)}$ is the scale function of a spectrally negative Lévy process X that is killed at an independent exponential time with parameter q . See forthcoming Section 1 for a precise definition. The function $Z^{(q)}$ first occurred, although implicitly, in [4, 6]. By analogy of the theory of diffusions, the function $W^{(q)}$ is called a q -scale function since $\{\exp(-q(\widehat{T} \wedge t))W^{(q)}(X_{\widehat{T} \wedge t}), t \geq 0\}$ is a martingale where \widehat{T} the first exit time of the positive real line. In Section 3.3 we show that the function $Z^{(q)}$ has an analogous property for the reflected process Y : $\{\exp(-qt)Z^{(q)}(Y_t), t \geq 0\}$ is a martingale. Therefore we call $Z^{(q)}$ the *adjoint* q -scale function for X .

Bertoin [6] investigated the exponential decay and ergodicity for completely asymmetric Lévy processes killed upon leaving a finite interval. The purpose of the second part is to extend Bertoin's study to reflected Lévy processes killed upon up-crossing a finite level. We determine the exponential decay parameter ϱ of the semigroup, prove that the process is ϱ -positive in the classification of Tuominen and Tweedie [28] and specify the ϱ -invariant function and measure. Restricting ourselves to Lévy processes whose one-dimensional distributions are absolutely continuous with respect to the Lebesgue measure, we also find the quasi-stationary distribution. Section 7 contains the main results in that direction. Section 6 contains a study of the transition probabilities of the reflected Lévy process with preparatory results.

Important elements in the proof of the ergodic properties and the exponential decay are the special form of the earlier computed resolvent measure together with special properties of fluctuation theory of completely asymmetric Lévy processes, elementary properties of analytic functions and the R -theory developed by Tuominen and Tweedie [28] for a general irreducible Markov process.

The third part, to be found in section 8, starts with the construction by h -transform of the reflected process conditioned to stay below the level a . We study then this process: we show

that it is a positively recurrent Markov process and determine its stationary measure. If the one-dimensional distributions of the Lévy process are absolutely continuous, we observe that, as a direct consequence of the results of the second part mentioned above, the conditional probabilities of the reflected Lévy process conditioned on the fact that it exits $[0, a]$ after t , converge as t tends to infinity. The process constructed this way coincides with the earlier mentioned h -transform. Finally, we use excursion theory to determine the rate of convergence of the supremum of the reflected process to a . The obtained results reveal a similar pattern as those achieved by Lambert [18] in his study of a completely asymmetric Lévy process confined in a finite interval.

1 Preliminaries

This section reviews standard results on spectrally negative Lévy processes. For more background we refer to [7] or [5], Chapter VII.

Let $X = \{X_t, t \geq 0\}$ be a Lévy process without positive jumps defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. For all x the measure \mathbb{P}_x will denote the translation of \mathbb{P} under which $X_0 = x$. To avoid trivialities, we exclude the case where X has monotone paths. Since X has no positive jumps, the moment generating function $\mathbb{E}[e^{\theta X_t}]$ exists for all $\theta \geq 0$ and is given by

$$\mathbb{E}[\exp(\theta X_t)] = \exp(t \psi(\theta))$$

for some function $\psi(\theta)$ which is well defined at least on the positive half axis, where it is convex with the property $\lim_{\theta \rightarrow \infty} \psi(\theta) = +\infty$. Let $\Phi(0)$ denote its largest root. On $[\Phi(0), \infty)$ the function ψ is strictly increasing and we denote its right-inverse function by $\Phi : [0, \infty) \rightarrow [\Phi(0), \infty)$. It is well known, that the asymptotic behaviour of X can be determined from the sign of $\psi'_+(0)$, the right-derivative of ψ at zero. Indeed, X drifts to $-\infty$, oscillates or drifts to $+\infty$ according to whether $\psi'_+(0)$ is negative, zero or positive.

We use the notations $c \vee d = \max\{c, d\}$ and $c \wedge d = \min\{c, d\}$. Denote by I and S the past infimum and supremum of X respectively, that is,

$$I_t = \inf_{0 \leq s \leq t} (X_t \wedge 0), \quad S_t = \sup_{0 \leq s \leq t} (X_t \vee 0)$$

and introduce the notations $Y = X - I$ and $\widehat{Y} = \widehat{X} - \widehat{I} = S - X$ for the Lévy process X reflected at its past infimum I and its dual, the process X reflected at its supremum. Denote by $\eta(q)$ an exponential random variable with parameter $q > 0$ which is independent of X . The Wiener-Hopf factorization of X implies that $Y_{\eta(q)}$ and $I_{\eta(q)}$ are independent, where $Y_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ and

$$\mathbb{E}[\exp(\theta I_{\eta(q)})] = \frac{q}{q - \psi(\theta)} \cdot \frac{\Phi(q) - \theta}{\Phi(q)}. \quad (1)$$

By time reversal one can show that the pairs $(Y_{\eta(q)}, -I_{\eta(q)})$ and $(S_{\eta(q)}, \widehat{Y}_{\eta(q)})$ have the same distribution.

2 Scale functions

As in e.g. [6, 2], a crucial role will be played by the function $W^{(q)}$, which is closely connected to the two-sided exit problem. To be precise we give a definition for $W^{(q)}$ and review some of its properties.

Definition 1 For $q \geq 0$ the q -scale function $W^{(q)} : (-\infty, \infty) \rightarrow [0, \infty)$ is the unique function whose restriction to $[0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = (\psi(\theta) - q)^{-1}, \quad \theta > \Phi(q),$$

and is defined to be identically zero for $x < 0$.

By taking $q = 0$ we get the 0-scale function which is usually called just “the scale function” in the literature [5]. For $q > 0$, $W^{(q)}$ can be regarded as “the scale function” of the original process X killed at an independent exponential time with parameter q . It is known that $W = W^{(0)}$ is increasing, when restricted to $(0, \infty)$. Moreover, the value of W at 0 and infinity is connected to certain global properties of X . Indeed, $W(0)$ is zero precisely if X has unbounded variation. Secondly, $W(\infty) = \lim_{x \rightarrow \infty} W(x)$ is finite, precisely if X drifts to ∞ , which follows from a Tauberian theorem in conjunction with the earlier mentioned fact that $\psi'_+(0) > 0$ if and only if X drifts to ∞ .

Inverting now the Laplace transform (1), we find that

$$\mathbb{P}(\widehat{Y}_{\eta(q)} \in dy) = \frac{q}{\Phi(q)} W^{(q)}(dy) - q W^{(q)}(y) dy, \quad y \geq 0, \quad (2)$$

where $W^{(q)}(dy)$ denotes the Stieltjes measure associated with $W^{(q)}$ with mass $W^{(q)}(0)$ at zero.

For every fixed x , we can extend the mapping $q \mapsto W^{(q)}(x)$ to the complex plane by the identity

$$W^{(q)}(x) = \sum_{k \geq 0} q^k W^{*k+1}(x), \quad (3)$$

where W^{*k} denoted the k -th convolution power of $W = W^{(0)}$. The convergence of this series is plain from the inequality

$$W^{*k+1}(x) \leq x^k W(x)^{k+1} / k! \quad x \geq 0, k \in \mathbb{N},$$

which follows from the monotonicity of W . From the expansion (3) and the properties of W , we see that the q -scale function is continuous except possibly at zero and that it is increasing on $(0, \infty)$ for each $q \geq 0$.

Closely related to $W^{(q)}$ is the function $Z^{(q)}$. We recall the definition given in [2].

Definition 2 The *adjoint q -scale function* $Z^{(q)}$ is defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z) dz. \quad (4)$$

Note that this function inherits some properties from $W^{(q)}(x)$. Specifically it is increasing, differentiable and strictly convex on $(0, \infty)$ and is equal to the constant 1 for $x \leq 0$. Moreover, if X has unbounded variation, $Z^{(q)}$ is C^2 on $(0, \infty)$.

Example. A stable Lévy process X with index $\alpha \in (1, 2]$ has as cumulant $\psi(\theta) = \theta^\alpha$; its scale function and adjoint are respectively computed in [4] as

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha) \quad Z^{(q)}(x) = E_\alpha(qx^\alpha),$$

where E_α is the Mittag-Leffler function with parameter α

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1 + \alpha n)}, \quad y \in \mathbb{R}. \quad (5)$$

In the case $\alpha = 2$, the process $X/\sqrt{2}$ is a Brownian motion and $W^{(q)}, Z^{(q)}$ reduce to

$$W^{(q)}(x) = q^{-\frac{1}{2}} \sinh(x\sqrt{q}) \quad Z^{(q)}(x) = \cosh(x\sqrt{q}). \quad (6)$$

Hence for a standard Brownian motion $W^{(q)}, Z^{(q)}$ are found by replacing (x, q) by $(2x, q/2)$ in (6).

For later reference, we give four lemmas with some properties of $W^{(q)}$ and $Z^{(q)}$ which we will need later on.

Lemma 1 *The function $x \mapsto W^{(q)}(x)$ is right- and left-differentiable on $(0, \infty)$. Moreover, if X has unbounded variation or its Lévy measure has no atoms, $W^{(q)}$ is continuously differentiable on $(0, \infty)$.*

By $W_{\pm}^{(q)'}(x)$, we will denote the right and left-derivative of $W^{(q)}$ in x , respectively.

Proof In the proof of theorem VII.8 in [5] Bertoin shows that W satisfies for some constant K , $W(x) = K \exp(-\int_x^\infty \widehat{n}(h > t) dt)$, where \widehat{n} is the Itô excursion measure of $\widehat{Y} = S - X$ and h are excursion heights of excursions of \widehat{Y} away from zero. From this representation, we deduce that

$$W_+'(x) = W(x)\widehat{n}(h > x) \quad W_-'(x) = W(x)\widehat{n}(h \geq x). \quad (7)$$

It can be shown that the distribution of h under excursion measure \widehat{n} has no atoms if X has unbounded variation (see [18]) or if X has bounded variation but its Lévy measure Λ has no atoms (one way to see this is to invoke equation (20) to show that $\widehat{n}(h = x) = \mathfrak{d}^{-1}\Lambda(\{-x\})W(0)/W(x)$). Hence, under these conditions, W restricted to $(0, \infty)$ is continuously differentiable. Using the expansion (3) and the monotonicity of W , it is not hard to prove that the properties of W carry over to $W^{(q)}$ (see [18, Prop. 5.1]). \square

The second lemma is immediate from (3) and the definitions of $Z^{(q)}$ and $W_{\pm}^{(q)'}$.

Lemma 2 *The mapping $(x, q) \mapsto Z^{(q)}(x)$ is continuous on $[0, \infty) \times \mathbb{R}$ and, for every $x \geq 0$, $q \mapsto Z^{(q)}(x)$ and $q \mapsto W_{\pm}^{(q)'}(x)$ are analytic functions.*

Using the expansion (3), one can check the following convolution identities to be true:

Lemma 3 For $q, r \in \mathbb{C}$ and $a > 0$ we have

$$\begin{aligned} W^{(q)} \star W^{(r)}(a) &= \frac{1}{r - q} (W^{(r)}(a) - W^{(q)}(a)), \\ W^{(q)} \star dW^{(r)}(a) &= \frac{1}{r - q} (W^{(r)'}(a) - W^{(q)'}(a)) \end{aligned}$$

where for $q = r$ the expression is to be understood in the limiting sense and where $W^{(q)} \star dW^{(r)}(a) = \int_0^a W^{(q)}(a - x)W^{(r)}(dx)$.

The following result concerns the asymptotic behaviour of $W^{(q)}$ and $Z^{(q)}$. We write $f \sim g$ if $\lim(f/g) = 1$.

Lemma 4 (i) For $q > 0$, we have as $x \rightarrow \infty$

$$W^{(q)}(x) \sim e^{\Phi(q)x} / \psi'(\Phi(q)), \quad Z^{(q)}(x) \sim qe^{\Phi(q)x} / (\Phi(q)\psi'(\Phi(q))).$$

(ii) As $x \downarrow 0$ the ratio $(W(x) - W(0))/x$ converges to a positive constant or to $+\infty$.

Proof (i) We can straightforwardly check that the Laplace-Stieltjes transforms of the functions $U(x) := e^{-\Phi(q)x}W^{(q)}(x)$ and $\tilde{U}(x) := e^{-\Phi(q)x}(Z^{(q)}(x) - 1)$ are given by

$$\begin{aligned} \int_0^\infty e^{-\lambda x} U(dx) &= \frac{\lambda}{\psi(\lambda + \Phi(q)) - q} = \frac{\lambda}{\psi(\lambda + \Phi(q)) - \psi(\Phi(q))}; \\ \int_0^\infty e^{-\lambda x} \tilde{U}(dx) &= \frac{q\lambda}{(\Phi(q) + \lambda)(\psi(\lambda + \Phi(q)) - q)}, \end{aligned}$$

where $dU, d\tilde{U}$ denote the Stieltjes measure associated to U, \tilde{U} respectively, which respectively assign masses $W^{(q)}(0)$ and 0 to zero. Since $\psi'(\Phi(q)) > 0$, the statements follow using a Tauberian theorem (e.g. [5, p.10]).

(ii) Recall that the Laplace-Stieltjes transform of W is given by $\lambda/\psi(\lambda)$. If the Brownian coefficient $s := \lim_{\lambda \rightarrow \infty} \psi(\lambda)/\lambda^2$ is positive, the same Tauberian theorem implies that $W(x) \sim x/s$ for x tending to infinity. Set $\mathfrak{d} := \lim_{\lambda \rightarrow \infty} \lambda/\psi(\lambda)$. If $s = \mathfrak{d}^{-1} = 0$, that is X has no Brownian component and unbounded variation, we find, again using the Tauberian theorem, that $W(x)/x$ tends to infinity for $x \downarrow 0$. Similarly, if the Lévy measure of X has finite mass m , we can check $W(x) - W(0) \sim mx/\mathfrak{d}$, whereas for X with bounded variation but infinite mass of the Lévy measure we can verify that $(W(x) - W(0))/x$ tends to infinity as $x \downarrow 0$.

□

3 Exit problems

3.1 Two-sided exit

We now turn our attention to the two-sided exit problem and review the main results. Denote the passage times \widehat{T}_a, T_a for X and $-X$ above and below the level a by

$$T_a = \inf\{t \geq 0 : X_t > a\} \quad \widehat{T}_a = \inf\{t \geq 0 : -X_t > a\}.$$

The following result, the origins of which go back to Takács [26], expresses the (discounted) probabilities of exiting the interval $[0, a]$ above and below in terms of $W^{(q)}$ and $Z^{(q)}$.

Proposition 1 *For $q \geq 0$, the Laplace transforms of the two-sided exit time $\widehat{T}_0 \wedge T_a$ on the part of the probability space where X starts at $x \in [0, a]$ and exits the interval $[0, a]$ above and below are respectively given by*

$$\mathbb{E}_x \left[e^{-qT_a} I_{(\widehat{T}_0 > T_a)} \right] = W^{(q)}(x)/W^{(q)}(a); \quad (8)$$

$$\mathbb{E}_x \left[e^{-q\widehat{T}_0} I_{(\widehat{T}_0 < T_a)} \right] = Z^{(q)}(x) - W^{(q)}(x)Z^{(q)}(a)/W^{(q)}(a). \quad (9)$$

Proof For $x \in (0, a)$, this result can be extracted directly out of existing literature. See for example [5, Thm. VII.8] for a proof of (8) using excursion theory. Combining this with [6, Cor 1], we find equation (9). Note by a small typographic mistake in [6] $\int_0^x W^{(q)}(x)dx$ is used instead of $q \int_0^x W^{(q)}(y)dy = Z^{(q)}(x) - 1$. Since 0 is regular for $(0, \infty)$ for X , the identities hold for $x = a$. Similarly, they hold for $x = 0$ if X has unbounded variation. If X has bounded variation, 0 is irregular for $(-\infty, 0)$ and hence $\widehat{T}_0 > 0$ almost surely. Since $T_\epsilon \downarrow 0$ almost surely if $\epsilon \downarrow 0$, the strong Markov property implies that

$$\begin{aligned} \mathbb{E}_0 \left[e^{-qT_a} I_{(\widehat{T}_0 > T_a)} \right] &= \lim_{\epsilon \downarrow 0} \mathbb{E}_0 \left[e^{-qT_\epsilon} I_{(\widehat{T}_0 > T_\epsilon)} W^{(q)}(\epsilon)/W^{(q)}(a) \right] \\ &= W^{(q)}(0)/W^{(q)}(a), \end{aligned}$$

whence (8) is valid for $x = 0$ as well. Analogously (9) is shown to hold for $x = 0$. \square

Remark. Let n be the Itô-excursion measure associated to the excursions of Y away from zero and let h, ζ denote the height and lifetime of the generic excursion respectively. In [5, Prop. VII.15] Bertoin related n and the scale function W as follows

$$n(h > a) = W(a)^{-1}.$$

The expansion (3) implies that $W^{(q)}(0) = W(0)$ and that $W_+^{(q)'}(0) = W_+'(0)$ if X has unbounded variation. Using Propositions VII.14 and VII.15 in [5] combined with Proposition 1, we find then the following links between n , $W^{(q)}$ and $Z^{(q)}$:

$$\begin{aligned} n(e^{-qT_a}, h > a) &= \lim_{x \downarrow 0} \frac{\mathbb{E}_x(e^{-qT_a} I_{(\widehat{T}_0 > T_a)})}{W(x)} = W^{(q)}(a)^{-1}, \\ n(e^{-q\zeta}, h > a) &= \frac{\mathbb{E}_a[e^{-q\widehat{T}_0}]}{W^{(q)}(a)} = \frac{Z^{(q)}(a)}{W^{(q)}(a)} - \frac{q}{\Phi(q)}, \end{aligned}$$

where, in the second line, we used the strong Markov property and Lemma 4(i). Letting $q \downarrow 0$ in the last equation yields that $n(h > a, \zeta < \infty) = n(h > a) - 1/W(\infty)$, which agrees with the fact that in case Y is not recurrent the excursions of Y away from zero form a Poisson point process stopped at rate $\psi'_+(0) = 1/W(\infty)$.

3.2 Mixed exit

As a next step, we study exit problems of $[0, a]$ for the reflected Lévy processes Y and \widehat{Y} . The first passage time of a positive level $a > 0$ will be denoted by

$$\tau_a = \inf\{t \geq 0 : Y_t > a\} \quad \text{and} \quad \widehat{\tau}_a = \inf\{t \geq 0 : \widehat{Y}_t > a\},$$

where we will use τ_0 and $\widehat{\tau}_0$, respectively, to denote the first time that Y and \widehat{Y} hit zero. The following result expresses the Laplace transforms of the exit times τ_a and $\widehat{\tau}_a$ in terms of the scale functions $W^{(q)}$ and $Z^{(q)}$. Note that $X_0 = x$ and hence $Y_0 = x$ under \mathbb{P}_x . Similarly, we see that \widehat{Y} starts from x under the measure \mathbb{P}_{-x} .

Proposition 2 *Let $x \in [0, a]$ and $q \geq 0$. Then we have*

- (i) $\mathbb{E}_x[e^{-q\tau_a}] = Z^{(q)}(x)/Z^{(q)}(a)$.
- (ii) $\mathbb{E}_{-x}[e^{-q\widehat{\tau}_a}] = Z^{(q)}(a-x) - qW^{(q)}(a-x)W^{(q)}(a)/W_+^{(q)}(a)$.

By analyticity in q (Lemma 2) and monotone convergence, we find from Proposition 2 the following expressions for the expectations of the stopping times τ_a and $\widehat{\tau}_a$ for $x \in [0, a]$:

$$\mathbb{E}_x[\tau_a] = \overline{W}(a) - \overline{W}(x), \quad \mathbb{E}_{-x}[\widehat{\tau}_a] = W(a-x)\frac{W(a)}{W_+'(a)} - \overline{W}(a-x), \quad (10)$$

where $\overline{W}(x) = \int_0^x W(y)dy$. If X is a standard Brownian motion, we recall the form of the q -scale function given in the example in Section 2 and we find back the following well known identities:

$$\mathbb{E}_x[e^{-q\tau_a}] = \cosh(x\sqrt{2q})/\cosh(a\sqrt{2q}), \quad \mathbb{E}_x[\tau_a] = (a^2 - x^2)/2,$$

for $q \geq 0$ and $x \in [0, a]$.

Proof (ii) Denote by \mathcal{D} the subset of $a \in (0, \infty)$ where $W_+^{(q)'}(a) > 0$. Since $x \mapsto W^{(q)}(x)$ is increasing, the complement of \mathcal{D} in $(0, \infty)$ is a closed set with empty interior. For any $a \in \mathcal{D}$ the Laplace transform in (ii) can be directly inferred from Theorem 1 in [2]. However, for any positive $a \notin \mathcal{D}$, we would immediately reach a contradiction in view of the form of the Laplace transform of $\widehat{\tau}_a$ and the fact that $\widehat{\tau}_a$ is increasing in a . Hence (ii) holds for all $a > 0$ and $\mathcal{D} = (0, \infty)$.

(i) To prove the form of the first Laplace transform, we use ideas developed in [2]. From the two-sided exit probability (8) we can extract that

$$M_t = \exp(-q(t \wedge \widehat{T}_0 \wedge T_a))W^{(q)}(X(t \wedge \widehat{T}_0 \wedge T_a)) \quad t \geq 0,$$

is a martingale. Indeed, combining (8) and the fact that $W^{(q)}(X_{\widehat{T}_0 \wedge T_a})/W^{(q)}(a)$ is almost surely equal to the indicator of $\{\widehat{T}_0 > T_a\}$, we find for $x \in \mathbb{R}$

$$\mathbb{E}_x[e^{-q(\widehat{T}_0 \wedge T_a)} W^{(q)}(X_{\widehat{T}_0 \wedge T_a})] = W^{(q)}(x).$$

Combined with the Markov property of X we then see that

$$\begin{aligned} \mathbb{E}_x(e^{-q(\widehat{T}_0 \wedge T_a)} W^{(q)}(X_{\widehat{T}_0 \wedge T_a}) | \mathcal{F}_t) &= e^{-qt} W^{(q)}(X_t) \mathbf{1}_{\{t < \widehat{T}_0 \wedge T_a\}} \\ &\quad + e^{-q(\widehat{T}_0 \wedge T_a)} W^{(q)}(X_{\widehat{T}_0 \wedge T_a}) \mathbf{1}_{\{t \geq \widehat{T}_0 \wedge T_a\}} \\ &= e^{-q(t \wedge \widehat{T}_0 \wedge T_a)} W^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a}), \end{aligned}$$

so that we have constance of expectation. Similarly, the martingale property follows. Exactly in the same vein, now using the exit probability in equation (9), we conclude that

$$e^{-q(t \wedge \widehat{T}_0 \wedge T_a)} \left(Z^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a}) - W^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a}) \frac{Z^{(q)}(a)}{W^{(q)}(a)} \right), \quad t \geq 0$$

is a martingale. By taking a linear combination, we see that $e^{-q(t \wedge \widehat{T}_0 \wedge T_a)} Z^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a})$ is a martingale. Recall that $Z^{(q)}(\cdot)$ is once (twice) continuously differentiable on $(0, \infty)$ if X has (un)bounded variation, respectively. Applying Itô's lemma to $e^{-qt} Z^{(q)}(X_t)$ (Theorems II.31(32) in [22] in the case where X has (un)bounded variation) on the set $\{t \leq \widehat{T}_0\}$, we find that

$$e^{-q(t \wedge \widehat{T}_0)} Z^{(q)}(X_{t \wedge \widehat{T}_0}) - \int_0^{t \wedge \widehat{T}_0} e^{-qs} (\Gamma - q) Z^{(q)}(X_{s-}) ds$$

is a (local) martingale, where Γ is the infinitesimal generator of X .

The martingale property of $e^{-q(t \wedge \widehat{T}_0 \wedge T_a)} Z^{(q)}(X_{t \wedge \widehat{T}_0 \wedge T_a})$ implies now that

$$(\Gamma - q) Z^{(q)}(x) = 0, \quad x \in (0, a). \quad (11)$$

Let I^c be the continuous part of I . By applying (the appropriate version of) Itô's lemma to $N_t = \exp(-qt) Z^{(q)}(Y_t)$ and using $Z^{(q)}(x) = 1$ for $x \leq 0$, one can verify that

$$N_t - \int_0^t e^{-qs} (\Gamma - q) Z^{(q)}(Y_{s-}) ds + q \int_0^t W^{(q)}(Y_{s-}) dI_s^c$$

is a local martingale. Note that the last term in the previous display is identically zero. Indeed, if X has bounded variation $I^c \equiv 0$; otherwise we see that if dI_s^c is negative then $Y_{s-} = 0$ and $W^{(q)}(Y_{s-}) = 0$ in this case. Noting that $N_{t \wedge \tau_a}$ is bounded by $Z^{(q)}(a)$ we deduce from equation (11) that $N_{t \wedge \tau_a}$ is a uniformly integrable martingale. Hence, as $t \rightarrow \infty$,

$$Z^{(q)}(x) = \mathbb{E}_x[N_{t \wedge \tau_a}] \rightarrow \mathbb{E}_x[N_{\tau_a}] = Z^{(q)}(a) \mathbb{E}_x[e^{-q\tau_a}] \quad x \in [0, a],$$

where we used that \mathbb{P}_x -almost surely $\tau_a < \infty$ and $Y_{\tau_a} = a$. □

3.3 Martingales

Another consequence of Proposition 2 is the following martingale property, which justifies the name *adjoint q -scale function* for $Z^{(q)}$.

Proposition 3 For $q \geq 0$,

$$(e^{-q(t \wedge \widehat{T}_0)} W^{(q)}(X_{t \wedge \widehat{T}_0}), t \geq 0) \quad \text{and} \quad (e^{-qt} Z^{(q)}(Y_t), t \geq 0)$$

are martingales.

Proof The first assertion follows by applying Lemma VII.11 in [5] to a spectrally negative Lévy process that is killed at an independent exponential time $\eta(q)$.

As before we write $N_t = \exp(-qt) Z^{(q)}(Y_t)$. From the proof of Proposition 2, we know that $(N_{t \wedge \tau_a}, t \geq 0)$ is a martingale. We now claim that $N_{t \wedge \tau_a}$ converges in L^1 to N_t as a tends to infinity. Since for $s \leq t$

$$\mathbb{E}|\mathbb{E}(N_{t \wedge \tau_a} | \mathcal{F}_s) - \mathbb{E}(N_t | \mathcal{F}_s)| \leq \mathbb{E}|N_{t \wedge \tau_a} - N_t|,$$

the claim implies that N_t is a martingale. Let us now prove the claim. Write

$$N_{t \wedge \tau_a} = N_t \mathbf{1}_{\{t < \tau_a\}} + N_{\tau_a} \mathbf{1}_{\{t \geq \tau_a\}}.$$

Since $\tau_a < \infty$ a.s., monotone convergence implies that the first term on the right-hand side increases to N_t in L^1 if a tends to infinity. For the second term we note that by the Cauchy-Schwarz-inequality

$$\mathbb{E}_x[e^{-q\tau_a} \mathbf{1}_{\{t \geq \tau_a\}}] \leq (\mathbb{E}_x[e^{-2q\tau_a}])^{1/2} \mathbb{P}_x(t \geq \tau_a)^{1/2}. \quad (12)$$

Recall that $\eta(r)$ denotes an independent exponential random variable with parameter r . Since $\tau_a \rightarrow \infty$ if $a \rightarrow \infty$ and we can check that, for $r > 0$,

$$\begin{aligned} \mathbb{P}_x(\tau_a \leq \eta(r)) &= \mathbb{P}_x(\tau_a \leq t) + \mathbb{P}_x(\tau_a \in (t, \eta(r)], t < \eta(r)) \\ &\quad - \mathbb{P}_x(\tau_a \in (\eta(r), t], \eta(r) < t), \end{aligned}$$

there exists an a_r large enough such that $\mathbb{P}_x(\tau_a \leq t)$ is bounded by $\mathbb{P}_x(\tau_a \leq \eta(r))$ for all $a \geq a_r$. Combining this property with equations (12) and Proposition 2(i), we find that for a large enough

$$\mathbb{E}_x[N_{\tau_a} \mathbf{1}_{\{\tau_a \leq t\}}] \leq Z^{(2q)}(x) Z^{(q)}(a) / Z^{(2q)}(a).$$

By Lemma 4 (recalling that Φ is increasing), we conclude that the expectation in the previous display converges to zero, which finishes the proof. \square

4 Resolvent measure

The Lévy process killed when it exits from $[0, a]$ has the strong Markov property; denote its transition probabilities by $(P^t, t \geq 0)$, that is, for a Borel set $A \subseteq [0, a]$ we have

$$P^t(x, A) = \mathbb{P}_x(X_t \in A, t < T_a \wedge \widehat{T}_0) \quad \text{for } x \in [0, a].$$

and its q -resolvent kernel by

$$U^q(x, A) = \int_0^\infty P^t(x, A)e^{-qt} dt = \mathbb{E}_x \left(\int_0^{T_a \wedge \widehat{T}_0} e^{-qt} \mathbf{1}_{\{X_t \in A\}} dt \right), \quad q \geq 0.$$

Since the Lévy process has an absolute continuous resolvent kernel, it follows from the Radon-Nikodym theorem that $U^q(x, \cdot)$ has a density with respect to the Lebesgue measure, which will be denoted by $u^q(x, \cdot)$. Suprun [25] showed that, for $x, y \in [0, a]$,

$$u^q(x, y) = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \quad (13)$$

is a version of this density. Now we consider the Lévy processes Y and \widehat{Y} killed upon leaving $[0, a]$. These killed processes still have the strong Markov property and we write $(Q^t, t \geq 0)$ and $(\widehat{Q}^t, t \geq 0)$ respectively to denote their transition probabilities. To be more precise, for Borel-sets $A \subseteq [0, a]$, we denote the transition probabilities of Y and \widehat{Y} by

$$Q^t(x, A) = \mathbb{P}_x(Y_t \in A, t < \tau_a), \quad \widehat{Q}^t(x, A) = \mathbb{P}_{-x}(\widehat{Y}_t \in A, t < \widehat{\tau}_a).$$

and the corresponding q -resolvent kernels by $R^q(x, A)$ and $\widehat{R}^q(x, A)$, respectively. We state the following result:

Theorem 1 (i) *The measure $R^q(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure and a version of its density is given by*

$$r^q(x, y) = \frac{Z^{(q)}(x)}{Z^{(q)}(a)} W^{(q)}(a-y) - W^{(q)}(x-y), \quad x, y \in [0, a].$$

(ii) *Let $\widehat{r}^q(x, 0) = W^{(q)}(a-x)W^{(q)}(0)/W_+^{(q)'}(a)$ for $x \geq 0$ and set*

$$\widehat{r}^q(x, y) = W^{(q)}(a-x) \frac{W_+^{(q)'}(y)}{W_+^{(q)'}(a)} - W^{(q)}(y-x) \quad x, y \in [0, a], y \neq 0.$$

Then $\widehat{r}^q(x, 0)\delta_0(dy) + \widehat{r}^q(x, y)dy$ is a version of the measure $\widehat{R}^q(x, dy)$.

Example. If X is a standard Brownian motion, a famous result of Lévy states that $|X| = Y$, where the equality is in law. Let τ' be the first exit time of $|X|$ from $[0, a]$ and as before $\eta(q)$ is an independent exponential random variable with parameter $q > 0$. Recalling from

the example in Section 2 the form of the functions $W^{(q)}, Z^{(q)}$ for a Brownian motion and substituting in Theorem 1, we find, after some algebra,

$$\mathbb{P}_x(|X|_{\eta(q)} \in dy, \eta(q) < \tau')/dy = \frac{\sqrt{q}}{\sqrt{2}} \cdot \frac{\sinh((a - |y - x|)\sqrt{2q}) + \sinh((a - x - y)\sqrt{2q})}{\cosh(a\sqrt{2q})}$$

for $0 \leq x, y \leq a$. This formula is well known in the literature (e.g. [9, 3.1.1.6]).

Proof of part (i) Pick $x, y \in [0, a]$ arbitrary and let $q > 0$. By applying the strong Markov property of Y at the stopping time τ_x and using the lack of memory property of the exponential distribution, we find

$$\begin{aligned} \mathbb{P}_0(Y_{\eta(q)} \in dy, \eta(q) < \tau_a) &= \mathbb{P}_0(Y_{\eta(q)} \in dy, \eta(q) < \tau_x) \\ &+ \mathbb{E}_0[e^{-q\tau_x}] \mathbb{P}_x(Y_{\eta(q)} \in dy, \eta(q) < \tau_a) \end{aligned} \quad (14)$$

Analogously, the probability in (14) admits as second decomposition

$$\mathbb{P}_0(Y_{\eta(q)} \in dy, \eta(q) < \tau_a) = \mathbb{P}_0(Y_{\eta(q)} \in dy) - \mathbb{E}_0[e^{-q\tau_a}] \mathbb{P}_a(Y_{\eta(q)} \in dy). \quad (15)$$

Combining the two decompositions (14) and (15) we find

$$\mathbb{P}_x(Y_{\eta(q)} \in dy, \eta(q) < \tau_a) = \mathbb{P}_a(Y_{\eta(q)} \in dy) - \frac{\mathbb{E}_0[e^{-q\tau_a}]}{\mathbb{E}_0[e^{-q\tau_x}]} \mathbb{P}_x(Y_{\eta(q)} \in dy). \quad (16)$$

Our next step is to evaluate the probability $\mathbb{P}_x(Y_{\eta(q)} \in dy)$. Applying as before the strong Markov property at the stopping time τ_0 , we find the decomposition

$$\begin{aligned} \mathbb{P}_x(Y_{\eta(q)} \in dy) &= \mathbb{P}_x(Y_{\eta(q)} \in dy, \eta(q) < \tau_0) + \mathbb{E}_x[e^{-q\tau_0}] \mathbb{P}_0(Y_{\eta(q)} \in dy) \\ &= \mathbb{P}_x(X_{\eta(q)} \in dy, \eta(q) < \widehat{T}_0) + \mathbb{E}_x[e^{-q\widehat{T}_0}] \mathbb{P}_0(Y_{\eta(q)} \in dy) \end{aligned} \quad (17)$$

where in the second line we used that $(Y_t, t \leq \tau_0)$ has the same law as $(X_t, t \leq \widehat{T}_0)$. Suprun [25] showed that a version of the resolvent density of the process X killed upon entering the negative half-line is given by

$$q^{-1} \mathbb{P}_x(X_{\eta(q)} \in dy, \eta(q) < \widehat{T}_0)/dy = e^{-\Phi(q)y} W^{(q)}(x) - W^{(q)}(x - y). \quad (18)$$

By integrating this resolvent density over y (or letting $a \rightarrow \infty$ in equation (9)), we find the Laplace transform of \widehat{T}_0 to be equal to

$$\mathbb{E}_x[e^{-q\widehat{T}_0}] = Z^{(q)}(x) - q\Phi(q)^{-1} W^{(q)}(x). \quad (19)$$

Substituting (19) and (18) into (17) and recalling that $Y_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ we end up with

$$\mathbb{P}_x(Y_{\eta(q)} \in dy)/dy = Z^{(q)}(x)\Phi(q)e^{-\Phi(q)y} - qW^{(q)}(x - y).$$

Substituting this into equation (16) and recalling from Proposition 2 that $\mathbb{P}_0(\tau_x < \eta(q))$ is given by $Z^{(q)}(x)^{-1}$, we get the formula as stated in the Theorem for $q > 0$. For $q = 0$, the result follows by letting $q \downarrow 0$. \square

Proof of part (ii) Let $x, y \in [0, a]$ and let $q > 0$. Since $(\widehat{Y}_t; t < \widehat{\tau}_0)$ has the same law as $(-X_t; t < T_0)$, the strong Markov property of \widehat{Y} enables us to write

$$\begin{aligned} \mathbb{P}_{-x}(\widehat{Y}_{\eta(q)} \in dy, \eta(q) < \widehat{\tau}(a)) &= \mathbb{P}_{-x}(-X_{\eta(q)} \in dy, \eta(q) < T_0 \wedge \widehat{T}_a) \\ &+ \frac{W^{(q)}(a-x)}{W^{(q)}(a)} \mathbb{P}_0(\widehat{Y}_{\eta(q)} \in dy, \eta(q) < \widehat{\tau}(a)), \end{aligned}$$

where we used the two-sided exit probability (8). The first quantity on the right-hand side is seen to be equal to $qu^q(a-x, a-y)dy$, where u^q is given in (13). To evaluate the probability in the second term on the right-hand side we are going to make use of the Master formula of excursion theory (e.g. [5, Cor. IV.11]). We shall use standard notation (see Bertoin [5, Ch. IV]). To this end, we introduce the excursion process $\widehat{e} = (\widehat{e}_t, t \geq 0)$ of \widehat{Y} , which takes values in the space of excursions

$$\mathcal{E} = \{f \in D[0, \infty) : f \geq 0, \exists \zeta = \zeta(f) \text{ such that } f(\zeta) = 0\}.$$

of càdlàg functions f with a generic life time $\zeta = \zeta(f)$ and is given by

$$\widehat{e}_t = (\widehat{Y}_s, L^{-1}(t^-) \leq s < L^{-1}(t)) \quad \text{if } L^{-1}(t^-) < L^{-1}(t)$$

where L^{-1} is the right-inverse of a local time L of \widehat{Y} at zero; else $\widehat{e}_t = \partial$, some isolated point. We take the running supremum S to be this local time L (cf. [5, Ch. VII]). The space \mathcal{E} is endowed with the Itô excursion measure \widehat{n} . A famous theorem of Itô states that \widehat{e} is a Poisson point process with characteristic measure \widehat{n} , if \widehat{Y} is recurrent; otherwise $(\widehat{e}_t, t \leq L(\infty))$ is a Poisson point process stopped at the first excursion of infinite lifetime. For an excursion $\epsilon \in \mathcal{E}$ its supremum is denoted by $\bar{\epsilon}$. By $\epsilon_g = (\widehat{Y}_{g+t}, t \leq \zeta_g)$ we denote the excursion of \widehat{Y} with left-end point g , where ζ_g and $\bar{\epsilon}_g$ denote its lifetime and supremum respectively.

Letting $T_a(\epsilon) = \inf\{t \geq 0 : \epsilon(t) \geq a\}$ an application of the compensation formula yields for $y > 0$

$$\begin{aligned} &\mathbb{P}_0(Y_{\eta(q)} \in dy, \eta(q) < \widehat{\tau}(a)) \\ &= \mathbb{E} \left[\sum_g I \left(\epsilon_g(\eta(q)) \in dy, g < \eta(q) < g + \zeta_g, \eta(q) < \widehat{\tau}_a, \sup_{h < g} \bar{\epsilon}_h \leq a \right) \right] \\ &= \mathbb{E} \left[\int e^{-qs} I \left(\sup_{h < s} \bar{\epsilon}_h \leq a \right) dS_s \right] \widehat{n}(\epsilon(\eta(q)) \in dy, \eta(q) < \zeta \wedge T_a(\epsilon)). \end{aligned}$$

The first factor can be inferred from [2] to be equal to $W^{(q)}(a)/W_+^{(q)'}(a)$. For the second factor, we distinguish between the case that X has bounded or unbounded variation.

If X has bounded variation, it is well known (e.g. [23] or [19] for a more recent reference) that an excursion starts with a jump almost surely. Denote by \mathbf{d} and $\Lambda(dx)$ the drift and

Lévy measure of X , respectively. Note that in this case the time up to time t that the process \widehat{Y} has spent in zero is equal to local time S_t divided by the drift d . By the Markov property, under \widehat{n} , the excursion of \widehat{Y} , once in $(0, \infty)$, evolves as $-X$ killed at time T_0 . Furthermore, the entrance law of an excursion of \widehat{Y} under \widehat{n} is given by Λ/d . Indeed, letting $F : \mathcal{E} \rightarrow [0, \infty)$ be any bounded measurable functional on the space of excursions, we find that

$$\begin{aligned}
\int F(\epsilon) \widehat{n}(d\epsilon) &= \mathbb{E} \left[\sum_{0 \leq s \leq 1} F(e_s) \right] \\
&= \mathbb{E} \left[\sum_{0 \leq t < \infty} I(S_t \leq 1, X_{t-} = S_t, \Delta X_t < 0) F(\{-X_{s+t} + X_{t-}, s \leq \widehat{\tau}_0\}) \right] \\
&= \mathbb{E} \left[\int_0^\infty I(S_t \leq 1, X_{t-} = S_t) dt \int_{-\infty}^0 F(\{-X_s - x, s \leq T_{-x}\}) \Lambda(dx) \right] \\
&= \frac{1}{d} \int_{-\infty}^0 \mathbb{E}_x [F(\{-X_s, s \leq T_0\})] \Lambda(dx), \tag{20}
\end{aligned}$$

where on the first line we used as before the Master formula of excursion theory followed in the third line by an application of the compensation formula applied to the Poisson point process $(\Delta X_t, t \geq 0)$ with characteristic measure $\Lambda(dx)$ combined with the independent increments property of X . Applying this identity to $F(\epsilon) = I(\epsilon(t) \in dy, t < \zeta \wedge T_a(\epsilon))$, taking the Laplace transform in t and using (13), we find that

$$\begin{aligned}
\widehat{n}(\epsilon(\eta(q)) \in dy, \eta(q) < \zeta \wedge T_a) \\
&= \frac{q}{d} \int_{-\infty}^0 \left(\frac{W^{(q)}(a+x)W^{(q)}(y) - W^{(q)}(a)W^{(q)}(y+x)}{W^{(q)}(a)} \right) \Lambda(dx) dy.
\end{aligned}$$

We claim that the following identity holds true for all $a > 0$:

$$dW_+^{(q)'}(a) = \int_{-\infty}^0 (W^{(q)}(a) - W^{(q)}(a+x)) \Lambda(dx) + qW^{(q)}(a). \tag{21}$$

To see this, first note that the right and left-hand sides of (21) have the same Laplace transform in a . Moreover, equation (7) and the decomposition (3) imply that $W_+^{(q)'}$ is bounded on any compact interval in $(0, \infty)$. It follows that the right-hand side of (21) is right-continuous in $a > 0$, as is certainly the left-hand side of (21). This continuity combined with the almost sure unicity of the Laplace transform shows that the claim is true for $a > 0$.

After some algebra, we find that

$$\widehat{n}(\epsilon(\eta(q)) \in dy, \eta(q) < \zeta \wedge T_a) / dy = q \left(W_+^{(q)'}(y) - \frac{W_+^{(q)'}(a)}{W^{(q)}(a)} W^{(q)}(y) \right). \tag{22}$$

Substituting back the expression (22), we find the stated form of the density for $y > 0$. Noting that

$$1 - \mathbb{E}_{-x}[e^{-q\widehat{\tau}_a}] = \int_{0^+}^a q\widehat{r}^q(x, y) dy + \mathbb{P}_{-x}(\widehat{Y}_{\eta(q)} = 0, \eta(q) < \widehat{\tau}_a),$$

we can verify, by combining Proposition 2 with the just found density, that $\widehat{R}^q(x, 0) = W^{(a)}(a - x)W^{(a)}(0)/W_+^{(a)'}(a)$ which finishes the proof in the bounded variation case.

Suppose now X has unbounded variation. Let $g(\widehat{\tau}_a)$ and $d(\widehat{\tau}_a)$ be the last time before and first time after $\widehat{\tau}_a$ that \widehat{Y} visits zero. Consider now the excursion straddling $\widehat{\tau}_a$, $\{\widehat{Y}_t, t \in [g(\widehat{\tau}_a), d(\widehat{\tau}_a)]\}$, and denote its law by $Q^{(a)}$ and the completed right-continuous filtration generated by this process by $\{\mathcal{G}_t, t \geq 0\}$. Since we are in the case of unbounded variation, 0 is regular for \widehat{Y} for itself under \mathbb{P} . Then, in canonical notation, we have that X leaves continuously from zero $Q^{(a)}$ -almost surely and $T(x) = \inf\{t \geq 0 : X_t \geq x\}$ decreases to zero almost surely under $Q^{(a)}$ as $x \downarrow 0$. By right-continuity of the paths, the sequence of measures $(Q_x^{t,(a)}, x > 0)$ with

$$Q_x^{t,(a)}(A) := Q^{(a)}(X \circ \theta_{T(x)}(t) \in A, t < T(a)) \quad A \in \mathcal{F}_t$$

converges in finite distributions and weakly as a measure (by tightness) to $Q^{(a)}(\cdot, t < T(a))$. The strong Markov property implies that under $Q^{(a)}$ the shifted process $X \circ \theta_{T(x)}$ has the same law as $\widehat{X} = -X$ under \mathbb{P} starting at x and conditioned to exit $[0, a]$ at a . Using (13) and (8), we find for $A \in \mathcal{F}_t$

$$\begin{aligned} Q_x^{t,(a)}(A) &= \int \mathbb{P}_{-y}[-X_t \in A | \widehat{T}_a < T_0] Q^{(a)}(X_{T(x)} \in dy) \\ &= \int \int_A \frac{W(a) - W(a - z)}{W(a) - W(a - y)} \mathbb{P}_{-y}(-X_t \in dz, t < \widehat{T}_a \wedge T_0) Q^{(a)}(X_{T(x)} \in dy). \end{aligned}$$

From (13) it follows that

$$\lim_{x \downarrow 0} \mathbb{P}_{-x}(-X_{\eta(q)} \in dy, \eta(q) < \widehat{T}_a | \widehat{T}_a < T_0) = \frac{W(a) - W(a - y)}{W'(a)} f(y, a) dy$$

where the limit is in the sense of weak convergence and $f(y, a)$ is equal to the right-hand side of (22). Since $X_{T(x)}$ converges to zero $Q^{(a)}$ -a.e., we deduce by bounded convergence that

$$Q^{(a)}(X_{\eta(q)} \in dy, \eta(q) < T(a)) = \frac{W(a) - W(a - y)}{W'(a)} f(y, a) dy. \quad (23)$$

By a computation based on the compensation formula for excursion theory (cf. proof of Theorem 4 in [10]), one can verify that

$$\begin{aligned} &\mathbb{E} \left[\int_0^\infty I(S_t \leq x) q e^{-q(t-g_t)} I(\widehat{Y}_{t-g_t} \in dy, t - g_t < \widehat{\tau}_a) dt \right] \\ &= \mathbb{E} \left[\sum_{0 \leq g < \infty} I(S_g \leq x) \int_g^{g+\zeta_g} q e^{-q(t-g)} I(\widehat{Y}_{t-g} \in dy, t - g < \widehat{\tau}_a) dt \right] \\ &= x \widehat{n}(\epsilon(\eta(q)) \in dy, \eta(q) < \zeta \wedge T_a(\epsilon)), \end{aligned}$$

where $g_t = \sup\{s \leq t : \widehat{Y}_s = 0\}$. Thus, we find

$$\begin{aligned}
& \widehat{n}(\epsilon(\eta(q)) \in dy, \eta(q) < \zeta \wedge T_a(\epsilon)) \\
&= \lim_{x \downarrow 0} \frac{1}{x} \mathbb{E}_{-x} \left[\int_0^\infty q e^{-q(t-g_t)} I(\widehat{Y}_{t-g_t} \in dy, t-g_t < \widehat{\tau}_a \wedge \widehat{\tau}_0) \right] \\
&= \lim_{x \downarrow 0} \frac{W(a) - W(a-x)}{x} \times \\
&\times \lim_{x \downarrow 0} \frac{1}{W(a) - W(a-x)} \int_0^\infty q e^{-qt} \mathbb{E}_{-x} [I(X_{t-g_t} \in dy, t-g_t < \widehat{T}_a \wedge T_0) dt] \\
&= W'(a) Q^{(a)} \left[\frac{1}{W(a) - W(a - X_{\eta(q)})} I(X_{\eta(q)} \in dy, \eta(q) < T(a)) \right]. \tag{24}
\end{aligned}$$

Combining (24) and (23) we deduce that (22) is also valid in this case. \square

Remark. If X drifts to $-\infty$, we can relate the conditionings in the proof of the theorem to those in the literature on spectrally negative Lévy processes conditioned to stay in a half line. Recall that, since X drifts to $-\infty$, we have $\Phi(0) > 0$ and $\psi'(\Phi(0)) > 0$. We write $W(x) = e^{\Phi(0)x} W^\#(x)$ where $W^\#$ is the scale function of X under the measure $\mathbb{P}^\#$ which is for $A \in \mathcal{F}_t$ given by $\mathbb{P}^\#(A) = \mathbb{E}[\exp(\Phi(0)X_t) I_A]$. Since $\psi'_\#(0) = \psi'(\Phi(0)) > 0$, X drifts to $+\infty$ under $\mathbb{P}^\#$ and $W^\#$ is bounded. Then it follows from Proposition 1 that the probability $\mathbb{P}_{-x}(\widehat{T}_a < T_0)$ converges to $1 - \exp(-\Phi(0)x)$ as $a \rightarrow \infty$. By bounded convergence we then find for $A_t \in \mathcal{F}_t$

$$\begin{aligned}
\mathbb{P}_{-x}(A_t | \widehat{T}_a < T_0) &= (W(a) - W(a-x))^{-1} \mathbb{P}_{-x}((W(a) - W(a+X_t)) A_t) \\
&\rightarrow \mathbb{P}_{-x}^\perp(A_t) := \mathbb{E}_{-x} \left(\frac{1 - e^{-\Phi(0)X_t}}{1 - e^{-\Phi(0)x}} I(A_t) \right) \quad \text{as } a \rightarrow \infty.
\end{aligned}$$

Note that $\mathbb{P}_{-x}^\perp(A_t)$ is also equal to $\mathbb{P}_{-x}(A_t | S_\infty < 0)$. Hence the notation \mathbb{P}_{-x}^\perp is justified since under this measure the process always stay below zero with probability one. As $x \downarrow 0$ the measures \mathbb{P}_{-x}^\perp converge weakly (in the Skorohod topology) to a measure \mathbb{P}^\perp . For an analysis of this case, see [3].

5 Analytic continuation

In this subsection, we show that we can extend the resolvent measures $R^q(x, \cdot)$ and $\widehat{R}^q(x, \cdot)$ to some negative values of q . Let us define ϱ and $\widehat{\varrho}$ by

$$\varrho = \inf\{q \geq 0 : Z^{(-q)}(a) = 0\} \quad \widehat{\varrho} = \inf\{q \geq 0 : W_+^{(-q)'}(a) = 0\}. \tag{25}$$

Continuity of $q \mapsto Z^{(q)}(a)$ (Lemma 2) combined with the fact $Z^{(0)}(a) \equiv 1$ implies that ϱ is positive. Similarly, combining the continuity of $q \mapsto W_+^{(q)'}$ with the fact (from the proof of Proposition 2(ii)) that $W_+^{(q)'}$ is positive for all $a > 0$, we see that $\widehat{\varrho}$ is positive as well.

Proposition 4 Let $x \in [0, a]$ and A a Borel subset of $[0, a]$. We have for $q < \varrho$

$$\int_0^\infty e^{qt} Q^t(x, A) dt = \int_A \left\{ \frac{Z^{(-q)}(x)}{Z^{(-q)}(a)} W^{(-q)}(a - y) - W^{(-q)}(x - y) \right\} dy$$

and for $q < \widehat{\varrho}$

$$\int_0^\infty e^{qt} \widehat{Q}^t(x, A) dt = \int_A \frac{W^{(-q)}(a - x)}{W_+^{(-q)'}(a)} W^{(-q)}(dy) - \int_A W^{(-q)}(y - x) dy.$$

Proof For $q \leq 0$, the statement (i) rephrases Theorem 1. By Lemma 2 and the properties of $q \mapsto W^{(q)}(x)$ as listed in [6, Lemma 4], we can extend the right-hand side for $q < \varrho$. The coefficient c_n of q^n in the corresponding expansion as a power series at zero is given in terms of the left-derivative of the left-hand side,

$$c_n = \int_0^\infty t^n Q^t(x, A) dt / n!.$$

We know that the series $\sum_n c_n q^n$ converges for $|q| < \varrho$. The statement follows. The proof of (ii) is similar and left to the reader. \square

6 Irreducibility and Continuity

Let μ denote a σ -finite measure on $([0, a), \mathcal{B}_{[0,a)})$, the interval $[0, a)$ endowed with the Borel σ -algebra $\mathcal{B}_{[0,a)}$. Examples are the Lebesgue measure λ and the Dirac measure δ_x at $x \in [0, a)$. One says that transition probabilities $(P^t, t \geq 0)$ are μ -irreducible if, for every $A \in \mathcal{B}_{[0,a)}$ with $\mu(A) > 0$, their potential $U(x, A)$ of A is positive for every $x \in [0, a)$. Before we formulate the result, we set the condition (R) by

$$\left\{ \begin{array}{l} X \text{ has jumps of absolute size smaller than } a \\ \text{or the Brownian coefficient } s = \lim_{\lambda \rightarrow \infty} \lambda^{-2} \psi(\lambda) \text{ is positive} \end{array} \right\}. \quad (\text{R})$$

Proposition 5 Q^t is λ -irreducible and under condition (R) \widehat{Q}^t is $(\lambda + W(0)\delta_0)$ -irreducible.

Proof The first statement follows since, by Theorem 1,

$$r^0(x, y) = W(a - y) - W(x - y) > 0 \quad \text{for all } x, y \in [0, a),$$

as W is increasing. For the second statement, we note that $\widehat{Q}^t(x, dy) \geq P^t(a - x, d(a - y))$. Thus $\widehat{r}^0(x, y) \geq u^0(a - x, a - y)$, where under condition (R) $u^0(a - x, a - y) > 0$ for all $x, y \in (0, a)$ by Corollary 3 in [6]. If $x = 0$, we see from (7) that for $y < a$

$$\widehat{r}^0(0, y) / (W(a)W(y)) = \frac{W'_+(y)}{W(y)} - \frac{W'_+(a)}{W(a)} = \widehat{n}(h > y) - \widehat{n}(h > a) > 0.$$

Finally, note that for $x \in [0, a)$ the measure $\widehat{R}^q(x, dy)$ has an atom at zero if and only if X has bounded variation and thus if and only if $W(0) > 0$. \square

Proposition 5 implies the following property of $Z^{(q)}$.

Corollary 1 (i) For every $q < \varrho$ and $x \in [0, a)$, we have $Z^{(-q)}(x) > 0$.

(ii) Similarly, for every $q < \widehat{\varrho}$ and $x \in (0, a)$, $W_+^{(-q)'} > 0$.

Proof (i) We know from Lemma 2 that $Z^{(-q)}(x) > 0$ if x is sufficiently small. Let x_0 be the smallest zero of $Z^{(-q)}(x) = 0$. If we had $x_0 < a$, then we would have $\int_0^\infty e^{qt} Q^t(x_0, (x_0, a)) dt = 0$ by Proposition 4, which conflicts with the fact that Q^t is Lebesgue irreducible.

(ii) Suppose first there would exist an $x_0 \in (0, a)$ such that $W_+^{(-q)'}(x_0) < 0$. Then, by right-continuity and Proposition 4, we would find the contradictory statement $\int_0^\infty e^{qt} Q^t(x_0 + \epsilon, (x_0, x_0 + \epsilon)) dt < 0$ for $\epsilon > 0$ small enough. Thus $W_+^{(-q)'} \geq 0$ on $(0, a)$ for all $q \leq \widehat{\varrho}$. Next suppose that there exists an $x_1 \in (0, a)$ with $W_+^{(-q)'}(x_1) = 0$. This would imply that $\frac{\partial}{\partial q} W_+^{(q)'}(x_1) = 0$, which conflicts with the second identity of Lemma 3 for $q = r$ combined with Corollary 4 in [6] (which also holds without (AC) as follows from part (i)). \square

In order to be able to prove continuity in space and time of the transition probabilities $(Q^t, t \geq 0)$ and $(\widehat{Q}^t, t \geq 0)$, we restrict ourselves to Lévy processes X whose one-dimensional distributions are absolutely continuous with respect to the Lebesgue measure, that is,

$$\mathbb{P}_0(X_t \in dy) \ll dy \quad \text{for all } t > 0. \quad (\text{AC})$$

It is known that (AC) holds whenever the Brownian coefficient is positive or when the mass of the absolutely continuous part of the Lévy measure is infinite (see Tucker [27]). We use the standard notation $Q^t f(x) = \int_{[0, a)} f(y) Q^t(x, dy)$. Recall that the family $(Q^t, t \geq 0)$ has the strong Feller property if for every bounded Borel function f , $Q^t f(\cdot)$ is a continuous function on $[0, a]$ for all $t > 0$. If a family of probability measures has the Feller as well as the strong Feller property it is called doubly Feller.

Proposition 6 Assume (AC) is satisfied. Then the following hold true:

(i) For every $x \in [0, a]$ and Borel set $A \subseteq [0, a]$ the mappings $t \mapsto Q^t(x, A)$ and $t \mapsto \widehat{Q}^t(x, A)$ are continuous on $(0, \infty)$.

(ii) For every $t > 0$, Q^t and \widehat{Q}^t have the strong Feller property.

To prove Proposition 6 we need the following auxiliary results.

Lemma 5 Assume (AC) holds.

(i) The one-dimensional distributions of the reflected Lévy process Y are absolutely continuous, that is,

$$\mathbb{P}_x(Y_t \in dy) \ll dy \quad \text{for every } t > 0, x \geq 0.$$

(ii) For any $t > 0, x \geq 0$, the measure $\mathbb{P}_x(\widehat{Y}_t \in dy)$ is absolutely continuous on $(0, \infty)$. If X has (un)bounded variation, $\mathbb{P}_x(\widehat{Y}_t = 0) > (=) 0$.

Proof (ii) Let $N \subset (0, \infty)$ be an arbitrary Borel set of measure zero and fix $t > 0$. The form of the law of $\widehat{Y}_{\eta^{(q)}}$ given in (2) combined with the absolute continuity of $W^{(q)}(dx)$ for $x > 0$ implies that

$$\mathbb{P}_0(\widehat{Y}_t \in N) = 0 \quad \text{for Lebesgue-almost all } t > 0. \quad (26)$$

Next we note that $\mathbb{P}_x(\widehat{\tau}_0 \in dt)$ has no atoms for $x > 0$. Indeed, since the sample paths of a Lévy process are continuous at each fixed time a.s. we see that under (AC)

$$\mathbb{P}_x(\widehat{\tau}_0 = t) = \mathbb{P}_{-x}(T_0 = t) \leq \mathbb{P}_{-x}(X_t = 0) = 0 \quad x > 0. \quad (27)$$

Applying the Markov property at $\widehat{\tau}_0$ yields that

$$\mathbb{P}_x(\widehat{Y}_t \in N) = \mathbb{P}_x(\widehat{Y}_t \in N, t < \widehat{\tau}_0) + \int_0^t \mathbb{P}_0(\widehat{Y}_{t-s} \in N) \mathbb{P}_x(\widehat{\tau}_0 \in ds). \quad (28)$$

Noting that $\mathbb{P}_x(\widehat{Y}_t \in N, t < \widehat{\tau}_0)$ is dominated by $\mathbb{P}_{-x}(-X_t \in N)$ and invoking (26) and (27), we deduce from (28) that $\mathbb{P}_x(\widehat{Y}_t \in N)$ is zero under (AC) for all $t, x > 0$. By an application of the Markov property at time $s \notin N$, we can now remove the ‘‘almost’’ in (26) and the first assertion follows. Recalling that \widehat{Y}_t has the same law as $-I_t$ and using (2), we see that $\mathbb{P}_0(\widehat{Y}_t = 0)$ is zero for all $t > 0$ if and only if X has unbounded variation. The proof of (ii) is complete. The proof of (i) is similar to (ii) and is left to the reader. \square

Lemma 6 *For $a > 0$, the distribution of $\widehat{\tau}_a$ has no atom, that is,*

$$\mathbb{P}_x(\widehat{\tau}_a = t) = 0 \quad \text{for every } x \in [0, a) \text{ and } t \geq 0.$$

Under (AC), the same holds for the distribution of τ_a .

Proof Since a Lévy process (and also a reflected Lévy process) is almost surely continuous at time t , we have

$$\mathbb{P}_x(\tau_y = t) \leq \mathbb{P}_x(Y_t = y) \text{ and } \mathbb{P}_x(\widehat{\tau}_y = t) \leq \mathbb{P}_x(\widehat{Y}_t = y)$$

which are both zero under (AC) by the first and second part of the Lemma 5 respectively. Suppose now (AC) is not satisfied; X is then a drift minus pure jump process of bounded variation. Hence \widehat{Y} can cross the level $a > 0$ only by a jump. However, the probability is zero that the Poisson point process $(\Delta X_t, t \geq 0)$ jumps at time t . \square

Lemma 7 *Assume (AC) holds and let $A \subseteq \mathbb{R}$ be an arbitrary Borel set.*

(i) *For every $t > 0$, $\mathbb{P}_x(Y_t \in \cdot)$ and $\mathbb{P}_x(\widehat{Y}_t \in \cdot)$ have the strong Feller property.*

(ii) *For every $x \geq 0$, $t \mapsto \mathbb{P}_x(Y_t \in A)$ and $t \mapsto \mathbb{P}_x(\widehat{Y}_t \in A)$ are continuous on $(0, \infty)$.*

Proof (i) Let f be any bounded Borel function. Since Y_t under \mathbb{P}_x has the same law as $X_t - (I_t \wedge (-x))$ under \mathbb{P}_0 , we have

$$\mathbb{E}_x[f(Y_t)] = \mathbb{E}[f(X_t + x)\mathbf{1}_{\{I_t \geq -x\}}] + \mathbb{E}[f(X_t - I_t)\mathbf{1}_{\{I_t < -x\}}].$$

Recall that $\mathbb{P}(X_t \in dx)$ is absolutely continuous with respect to the Lebesgue measure. Since $C_c(\mathbb{R})$, the continuous functions with compact support, are dense in L^1 , it follows by dominated convergence that $x \mapsto \mathbb{E}_x[f(Y_t)]$ is continuous. The proof of the second statement is similar.

(ii) From the proof of Theorem 2.2 in [14] and Lemma 5, we can deduce, following an analogous line of reasoning, that there exists a version $(t, x, y) \mapsto q(t, x, y)$ of the density of the one-dimensional distributions of Y , such that for all Borel bounded f and for all $x \geq 0$

$$\mathbb{E}_x[f(Y_t)] = \int f(y)q_t(x, y)dy$$

and $\int q_t(\cdot, z)q_s(z, \cdot)dz = q_{t+s}(\cdot, \cdot)$ for all $s, t > 0$. Moreover, by part (i) $x \mapsto \mathbb{E}_x[f(Y_t)]$ is continuous. By the weak convergence of $q_\epsilon(x, z)dz$ to the Dirac point measure at x as $\epsilon \downarrow 0$ and almost sure sample path continuity of Y at time ϵ , left- and right-continuity follow (as in the proof of [6, Lemma 2]).

To prove the second statement of (ii), we repeat the proof of part (i) where everywhere the Lebesgue measure dy is replaced by the measure $dy + \delta_0(y)$, the Lebesgue measure dy with an atom of size one at zero. \square

Proof of Proposition 6 We only prove the statements for Y , the proofs for \widehat{Y} are similar.

(i) By the strong Markov property of Y applied at τ_a , we find that

$$\mathbb{P}_x(Y_t \in A) = Q^t(x, A) + \int_0^t \mathbb{P}_x(\tau_a \in ds)\mathbb{P}_a(Y_{t-s} \in A).$$

The left-hand side is continuous in t on $(0, \infty)$ by Lemma 7. The same holds for the integral on the right-hand side, as the distribution of τ_a has no atom. Hence $t \mapsto Q^t(x, A)$ is continuous.

(ii) Proposition VI.1 in [5] states that Y has the Feller property. Combining this with Lemma 7, we see that Y is doubly Feller. From Chung [11], we know that a doubly Feller process killed upon hitting an open set remains doubly Feller. \square

7 Ergodicity and exponential decay

Under the assumption (AC) Bertoin [6] identifies the decay parameter of the transition probabilities $(P^t, t \geq 0)$ of X killed upon leaving $[0, a]$ as $\rho = \rho(a)$ where

$$\rho(a) = \inf\{q \geq 0 : W^{(-q)}(a) = 0\}.$$

Recall from (25) that we defined $\varrho = \varrho(a)$ and $\widehat{\varrho} = \widehat{\varrho}(a)$ as

$$\varrho(a) = \inf\{q \geq 0 : Z^{(-q)}(a) = 0\} \quad \widehat{\varrho}(a) = \inf\{q \geq 0 : W_+^{(-q)'}(a) = 0\}.$$

The result below concerns the ergodic properties of the transition probabilities Q^t and \widehat{Q}^t and identifies their decay parameters as ϱ and $\widehat{\varrho}$ respectively. The proof uses the R -theory of irreducible Markov processes developed by Tuominen and Tweedie [28]. For the terminology of R -theory used in this section, we refer to [28].

Theorem 2 (A) *We have that $\varrho \in (0, \infty)$ and ϱ is a simple root of $q \mapsto Z^{(-q)}(a)$ and the following hold true:*

(i) Q^t is ϱ -recurrent and, more precisely, ϱ -positive.

(ii) $x \mapsto Z^{(-\varrho)}(x)$ is positive on $[0, a)$ and ϱ -invariant for Q^t ; that is,

$$Q^t Z^{(-\varrho)}(x) = e^{-\varrho t} Z^{(-\varrho)}(x) \quad \text{for all } x \in [0, a). \quad (29)$$

(iii) $x \mapsto W^{(-\varrho)}(a - x)$ is positive almost everywhere on $(0, a)$ and the measure $\Pi(dx) = W^{(-\varrho)}(a - x)dx$ on $[0, a]$ is ϱ -invariant for Q^t , that is,

$$\Pi Q^t = e^{-\varrho t} \Pi. \quad (30)$$

(iv) Assume (AC) is satisfied. Then for every $x \in [0, a]$ we have

$$\lim_{t \rightarrow \infty} e^{\varrho t} Q^t(x, \cdot) = c^{-1} Z^{(-\varrho)}(x) \Pi(\cdot) \quad (31)$$

in the sense of weak convergence where $c = \frac{d}{dq} Z^{(q)}(a)|_{q=-\varrho} > 0$.

(B) *Suppose X satisfies (R). Then $\widehat{\varrho} \in (0, \infty)$ and $\widehat{\varrho}$ is a simple root of $q \mapsto W_+^{(-q)'}(a)$ and the following hold:*

(i) \widehat{Q}^t is $\widehat{\varrho}$ -recurrent and, more precisely, $\widehat{\varrho}$ -positive;

(ii) $x \mapsto W^{(-\widehat{\varrho})}(a - x)$ is positive on $(0, a)$ and $\widehat{\varrho}$ -invariant for \widehat{Q}^t ;

(iii) $x \mapsto W_+^{(-\widehat{\varrho})'}(x)$ is a.s. positive on $(0, a)$ and the measure $\widehat{\Pi}(dx) = W^{(-\widehat{\varrho})}(dx)$ on $[0, a]$ is $\widehat{\varrho}$ -invariant for \widehat{Q}^t ;

(iv) Assume (AC) is satisfied. Then for every $x \in [0, a]$ we have

$$\lim_{t \rightarrow \infty} e^{\widehat{\varrho} t} \widehat{Q}^t(x, \cdot) = \widehat{c}^{-1} W^{(-\widehat{\varrho})}(a - x) \widehat{\Pi}(\cdot)$$

in the sense of weak convergence where $\widehat{c} = \frac{d}{dq} W_+^{(q)'}(a)|_{q=-\widehat{\varrho}} > 0$.

Remarks.

- (i) Specialising Theorem 2(A,iv) and (B,iv) we get the following asymptotic identities for $t \rightarrow \infty$ and $x, y \in [0, a]$

$$\begin{aligned}\mathbb{P}_x(\tau_a > t) &\sim c' Z^{(-\varrho)}(x) e^{-\varrho t}, & \text{for some constant } c' > 0; \\ \mathbb{P}_x(\widehat{\tau}_a > t) &\sim \tilde{c} W^{(-\widehat{\varrho})}(a-x) e^{-\widehat{\varrho} t}, & \text{for some constant } \tilde{c} > 0; \\ \mathbb{P}_x(Y_t \in A | \tau_a > t) &\sim \Pi(A) / \Pi([0, a]), & \text{for Borel sets } A \subseteq [0, a].\end{aligned}$$

- (ii) Take $\alpha \in (1, 2]$. In the case X is stable process of index α we recall from the example in Section 2 that $Z^{(q)}(x) = E_\alpha(qx^\alpha)$. The roots introduced in (25) are hence respectively given by $\varrho = a^{-\alpha}r(\alpha)$ where $-r(\alpha)$ is the first negative root of E_α and $\widehat{\varrho} = a^{-\alpha}\tilde{r}(\alpha)$ where $-\tilde{r}(\alpha)$ is the first negative root of

$$\sum_{n=1}^{\infty} \frac{y^n}{\Gamma(1 + \alpha n)}.$$

In the special case $\alpha = 2$, $X/\sqrt{2}$ is a standard Brownian motion and $E_2(-x) = \cos \sqrt{x}$ for $x > 0$. In particular, $r(2) = \pi^2/4$ and

$$\varrho = \pi^2/(4a^2) \quad Z^{(-\varrho)}(x) = \cos\left(\frac{\pi}{2a}x\right).$$

Since in this case $W^{(-q)'}(x) = Z^{(-q)}(x) = \cos(x\sqrt{q})$, we see that $\widehat{\varrho} = \varrho$, as it should be.

- (iii) By the decomposition (3) and Lemma 3 we find that

$$\begin{aligned}\frac{\partial}{\partial q} Z^{(q)}(a) |_{q=-\varrho} &= W^{(-\varrho)} \star Z^{(-\varrho)}(a) \\ \frac{\partial}{\partial q} W_+^{(q)'}(a) |_{q=-\widehat{\varrho}} &= \int_0^a W^{(-\widehat{\varrho})}(a-x) W^{(-\widehat{\varrho})}(dx).\end{aligned}$$

Hence the constants c, \widehat{c} in the Theorem make $\mu(dx) = c^{-1} Z^{(-\varrho)}(x) \Pi(dx)$ and $\widehat{\mu}(dx) = \widehat{c}^{-1} W^{(-\widehat{\varrho})}(a-x) \widehat{\Pi}(dx)$ into probability measures.

- (iv) We now consider equation (25) to define mappings $a \mapsto \varrho(a)$ and $a \mapsto \widehat{\varrho}(a)$ from $(0, \infty)$ to $(0, \infty)$. Note that, by Corollary 1(i) and Theorem 2(A,ii), $Z^{(-q)}(x) > 0$ for all $x \in [0, a]$ and $q \leq \varrho$. Much in the same vein as [18], it then follows from Theorem 2 and Lemma 2 that the mapping ϱ is decreasing and continuously differentiable on $(0, \infty)$ with derivative

$$\varrho'(a) = -\varrho(a) W^{(-\varrho(a))}(a) / \frac{\partial}{\partial q} Z^{(q)}(a) |_{q=-\varrho(a)}. \quad (32)$$

Similarly, if we assume that W (and hence $W^{(-\varrho)}$) is twice continuously differentiable, we can show that $a \mapsto \widehat{\varrho}(a)$ is decreasing and continuously differentiable with derivative

$$\widehat{\varrho}'(a) = \frac{\partial^2}{\partial x^2} W^{(-\widehat{\varrho})}(x) |_{x=a} / \frac{\partial}{\partial q} W^{(q)'}(a) |_{q=-\widehat{\varrho}(a)}. \quad (33)$$

Introduce the set $\mathcal{D}_\varrho = \{a > 0 : \varrho'(a) < 0\}$. Note that it is open and its complement has an empty interior, as ϱ is decreasing. We have now the following relation between ρ on the one hand and $\widehat{\varrho}$ and ϱ on the other hand.

Corollary 2 *Suppose X satisfies (R). Then the following hold.*

(i) $\widehat{\varrho}(a) < \rho(a)$;

(ii) $\varrho(a) < [=]\rho(a)$ if and only if $a \in [\notin]\mathcal{D}_\varrho$. Moreover, $W^{(-\varrho(a))}(x) > 0$ for $x \in (0, a)$.

Proof of Corollary 2 Following the same line of reasoning as in the proof of Theorem 2, one can show that Theorem 2 (i)-(iv) and Corollary 4 in [6] continue to hold if assumption (AC) is replaced by the assumption (R).

(i) Since $\widehat{Q}^t > P^t$, we see that $\widehat{\varrho}$ is bounded above by ρ . Since $W^{(-\widehat{\varrho})}(0) = W(0)$ is nonnegative and $W^{(-\varrho)}(dx)$ has no atoms in $(0, \infty)$, it follows from Theorem 2(B,iii) that $W^{(\widehat{\varrho}(a))}(a) > 0$. By definition of ρ and continuity of $q \mapsto W^{(q)}(a)$, this implies that $\widehat{\varrho} \neq \rho$ and hence $\widehat{\varrho} < \rho$.

(ii) Since $Q^t > P^t$ we see that ϱ is bounded above by ρ . The second statement follows directly from Corollary 4 and Theorem 2(iii) in [6]. Equation (32) combined with the positivity of $\frac{\partial}{\partial q} Z^{(q)}(a)|_{q=-\varrho(a)}$ implies that $a \in \mathcal{D}_\varrho$ if and only if $W^{(-\varrho(a))}(a) > 0$. As in (i) the first statement follows. \square

Proof of Theorem 2 By a close reading of the proofs of Theorems 2 and 3 in [28] one notes that these remain valid under the requirement of irreducibility (instead of simultaneous irreducibility). By Proposition 5, we can thus use Theorems 2 and 3 from [28]. Recall from Section 5 that ϱ is positive. Moreover, Proposition 4 implies that $\varrho < \infty$, since otherwise $\int_0^\infty e^{qt} Q^t(x, A) dt$ would be finite for all $x \in [0, a)$ and $q > 0$, which would not agree with Theorem 2 in [28]. We identify ϱ as the decay parameter and show Q^t is ϱ recurrent. From Lemma 2 combined with Lemma 5 in [6] we know we can find a $\delta \in (0, a/2)$ such that $Z^{(-\varrho)}(x) > 1/2$ and $W^{(-\varrho)}(x) > 0$ if $x \in (0, \delta)$; Since $q \mapsto Z^{(q)}(x)$ and $q \mapsto W^{(q)}(x)$ are continuous as stated in Lemma 2 and [6, Lemma 4(i)] respectively, we find that, for every $x < \delta, y \in (a - \delta, a)$,

$$\lim_{q \uparrow \varrho} \frac{Z^{(-q)}(x) W^{(-q)}(a - y)}{Z^{(-q)}(a)} = \infty. \quad (34)$$

Let $A \subseteq (a - \delta, a)$ be any Borel set with positive Lebesgue measure. By Proposition 4, monotone convergence and Fatou's Lemma we deduce that

$$\int_0^\infty e^{qt} Q^t(x, A) dt = \infty \quad \text{for every } x \in (0, \delta). \quad (35)$$

Theorem 2 in [28] now implies that ϱ coincides with the decay parameter and Q^t is ϱ -recurrent. In particular, it implies that (35) holds for all $x \in [0, a)$ and non-null Borel-sets $A \subseteq [0, a)$. Hence, we deduce that $Z^{(-\varrho)}(x)$ and $W^{(-\varrho)}(x)$ are positive for all respectively Lebesgue almost all $x \in (0, a)$. By remark (iii) after the Theorem, we now see that $\frac{\partial}{\partial q} Z^{(-\varrho)}(a) > 0$ thus ϱ is a simple root.

Using the identity of Lemma 3 combined with the observation $Z^{(q)}(x) - 1 = q(1 \star W^{(q)}(x))$ and the form of the resolvent given in Theorem 1, one finds, after some algebra, that $\int e^{-qt} Q^t Z^{(-\varrho)}(x) dt = Z^{(-\varrho)}(x)/(q + \varrho)$. By unicity of the Laplace transform, we find that there exists a null set N such that (29) holds for $t \notin N$. Since N is a Lebesgue null-set, for any $t \in N$, there exists an $s \notin N$ such that $(t - s) \notin N$. Applying the Markov property at s , we see

$$Q^t Z^{(-\varrho)}(x) = Q^{(t-s)}(Q^s Z^{(-\varrho)}(x)) = e^{-\varrho s} Q^{(t-s)} Z^{(-\varrho)}(x) = e^{-\varrho t} Z^{(-\varrho)}(x),$$

from which we see that (29) holds for all $t > 0$. Hence $Z^{(-\varrho)}$ is the ϱ -invariant function for Q^t (unicity from Theorem 3 in [28]). Analogously, one can prove that $W^{(-\varrho)}(a - x) dx$ is the ϱ -invariant measure for Q^t .

(iv) By Proposition 6 and Theorem 1 from [28], we are allowed to apply Theorem 5 and 7 in [28]. Noting that the ϱ -invariant measure Π has a finite mass and $Q^t 1(x)$ converges to zero for all x as t tends to ∞ , we find from Theorem 7 in [28] that for Π -almost every (and hence Lebesgue-almost every) $x \in [0, a)$

$$Q^t(x, A)/Q^t(x, [0, a)) \rightarrow \Pi(A)/\Pi([0, a)) \quad \text{as } t \rightarrow \infty.$$

Combining with Theorem 5(i) in [28] this proves (31) for almost every $x \in [0, a)$. The Markov property combined with the absolute continuity of Q^t under (AC) then implies that the last statement is valid for all $x \in [0, a]$. This completes the proof of part (A).

Part (B) follows along the same lines as part (A). By Proposition 5, we can again use Theorems 2 and 3 of [28], where the role of the measure m is now played by the Lebesgue measure with an atom of size one at zero. We invoke Lemma 5 in [6] to find a $\delta \in (0, a/2)$ such that $W^{(-\widehat{\varrho})}(y) > 0$ for $y \in (0, \delta)$. By the expansion (3) we see that $W_+^{(-\widehat{\varrho})'}(0) = W_+'(0) - \widehat{\varrho} W(0)^2$. Combined with Lemma 4 [and monotonicity of $\widehat{\varrho}(\cdot)$ in the compound Poisson case] this implies $W_+^{(-\widehat{\varrho})'}(0)$ is positive or infinite. By right-continuity (Lemma 2) of $x \mapsto W_+^{(-\widehat{\varrho})'}(x)$ we can then find a δ' such that $W^{(-\widehat{\varrho})'}(y) > 0$ for $y \in (0, \delta')$. Analogously as for part (A), we can then prove the $\widehat{\varrho}$ -recurrence of \widehat{Q}^t and the stated properties of $W_+^{(-\widehat{\varrho})'}(x), W^{(-\widehat{\varrho})}(x)$. To identify the $\widehat{\varrho}$ -invariant function and measure we follow an analogous line of reasoning using the second identity in Lemma 3. The proof of (iv) goes along the same lines as above. \square

8 The processes Y and \widehat{Y} conditioned to stay below a

We study the processes Y and \widehat{Y} conditioned to stay below a fixed level $a > 0$. We introduce the measures \mathbb{P}^\diamond and $\widehat{\mathbb{P}}^\diamond$ by

$$d\mathbb{P}_{x|\mathcal{F}_t}^\diamond = H_t d\mathbb{P}_{x|\mathcal{F}_t} \quad \text{and} \quad d\widehat{\mathbb{P}}_{x|\mathcal{F}_t}^\diamond = \widehat{H}_t d\mathbb{P}_{-x|\mathcal{F}_t}$$

where

$$H_t = e^{\varrho t} \mathbf{1}_{\{t < \tau_a\}} \frac{Z^{(-\varrho)}(Y_t)}{Z^{(-\varrho)}(x)} \quad \text{and} \quad \widehat{H}_t = e^{\widehat{\varrho} t} \mathbf{1}_{\{t < \widehat{\tau}_a\}} \frac{W^{(-\widehat{\varrho})}(a - \widehat{Y}_t)}{W^{(-\widehat{\varrho})}(a - x)}.$$

Theorem 2 implies that \mathbb{P}_x^\diamond and $\widehat{\mathbb{P}}_x^\diamond$ are h -transforms of \mathbb{P}_x and \mathbb{P}_{-x} respectively. Indeed, by the Markov property of Y under the probability measure \mathbb{P} :

$$\begin{aligned}\mathbb{E}_x(H_{t+s}|\mathcal{F}_t) &= \frac{e^{\varrho(t+s)}}{Z^{(-\varrho)}(x)} \mathbb{E}_x(\mathbf{1}_{\{t+s < \tau_a\}} Z^{(-\varrho)}(Y_{t+s})|\mathcal{F}_t) \\ &= \frac{e^{\varrho(t+s)}}{Z^{(-\varrho)}(x)} \mathbf{1}_{\{t < \tau_a\}} \mathbb{E}_{Y_t}(\mathbf{1}_{\{s < \tau_a\}} Z^{(-\varrho)}(Y_s))\end{aligned}$$

and the martingale property of H follows from Theorem 2(A,iii). Similarly, using Theorem 2(B,iii) we can verify that \widehat{H} is a martingale under \mathbb{P}_{-x} . The next result proves properties of the constructed processes and shows that, if (AC) holds, the h -transforms are equal to the limit as t tends to infinity of the conditional probabilities of Y (resp. \widehat{Y}) exiting $[0, a]$ after t . Recall the measures μ and $\widehat{\mu}$ given in note (iii) after Theorem 2.

Theorem 3 *Let $x \in [0, a)$. The following are true:*

(i) *Under \mathbb{P}^\diamond , Y has the strong Markov property and is positively recurrent with stationary probability measure μ . Moreover, we have in the sense of weak convergence*

$$\lim_{t \rightarrow \infty} \mathbb{P}_x^\diamond(Y_t \in \cdot) = \mu. \quad (36)$$

(ii) *If X satisfies (R), (i) continues to hold if we replace the triple $(Y, \mathbb{P}^\diamond, \mu)$ by $(\widehat{Y}, \widehat{\mathbb{P}}^\diamond, \widehat{\mu})$.*

(iii) *Suppose (AC) holds. Then the convergence in (i) and (ii) holds in total variation norm. Moreover, for any $s \geq 0$ and $A \in \mathcal{F}_s$, the conditional laws converge as $t \rightarrow \infty$*

$$\mathbb{P}_x(A|\tau_a > t) \rightarrow \mathbb{P}_x^\diamond(A) \quad \text{and} \quad \mathbb{P}_{-x}(A|\widehat{\tau}_a > t) \rightarrow \widehat{\mathbb{P}}_x^\diamond(A).$$

Example. Let X be a standard Brownian motion and define the space-time function h by

$$h(t, x, y) = e^{\frac{\pi^2}{4a^2}t} \cos\left(\frac{\pi}{2a}y\right) / \cos\left(\frac{\pi}{2a}x\right), \quad x, y \in [0, a).$$

Theorem 3 implies that the process Y conditioned to stay below a has infinitesimal generator L which acts on f in its domain $D = \{f \in C^2(0, a) : f'_+(0) = 0\}$ as

$$\begin{aligned}Lf(x) &= \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial y^2} \right) \Big|_{(t,y)=(0,x)} h(t, x, y) f(y) \\ &= \frac{1}{2} f''(x) - \frac{\pi}{2a} \tan\left(\frac{\pi}{2a}x\right) f'(x).\end{aligned} \quad (37)$$

By a famous theorem of Lévy, the process Y is in law equal to the process $|X|$. Hence, by symmetry and $\tan(x) = -\tan(-x)$, we find, as in [16], that the generator of Brownian motion conditioned to stay in $(-a, a)$, is given on $(-a, a)$ by (37) for all functions f in

$C^2(-a, a)$. This conditioned Brownian motion is called the Brownian Taboo process with taboo states $\{-a, a\}$ in the nomenclature of [16].

Proof We only prove the part of the theorem involving Y , leaving the rest to the reader.

(i) It is well known that under \mathbb{P}_x^\diamond and $\widehat{\mathbb{P}}_x^\diamond$ the strong Markov property is preserved [12, Thm. XVI.28 p. 329] and the process has as semigroup $P_t^\diamond(x, dy) = Q^t(x, dy)e^{ot} \frac{Z^{(-o)}(y)}{Z^{(-o)}(x)}$. The positive recurrence and invariance of μ for P_t^\diamond are immediate from Theorem 2(A;i,iii) combined with the form of the resolvents of the process under \mathbb{P}_x^\diamond , which follows now directly from Theorem 1. The form of the constant follows from note (iii) after Theorem 2. To prove the convergence, we will use the regenerative property of Y under \mathbb{P}^\diamond . To be more precise, under \mathbb{P}^\diamond , Y is a Markov process and hence a delayed regenerative process, where the delay is the time to reach zero and a cycle starts at zero and ends again at the first return to zero after a crossing of the level $a/2$. Denoting by T^* the cycle length, we see from forthcoming Proposition 7 that T^* has a finite mean. Note that T^* has the same distribution as $\widehat{\tau}_0 \circ \theta_{\tau_{\frac{a}{2}}}$ under \mathbb{P}_0^\diamond . From Lemma 6 we see that the distribution of $\widehat{\tau}_0 \circ \theta_{\tau_{\frac{a}{2}}}$ under \mathbb{P}_0^\diamond has no atoms. In particular, T^* is not concentrated on a lattice. Theorem V.1.2 from Asmussen [1] now implies the weak convergence (36).

(iii) Suppose now (AC) holds. As in the proof of Theorem 2(A; iv), we invoke Theorem 5(i) of [28] to find that (36) holds in total variation norm for Π -a.e. $x \in [0, a)$. Combining the Markov property with the absolute continuity of the transition probabilities P_t^\diamond of Y under \mathbb{P}^\diamond under (AC), we find

$$\|P_t^\diamond(x, \cdot) - \mu\| \leq P_s^\diamond(x, \|P_{t-s}^\diamond(Y_s, \cdot) - \mu\|)$$

where $\|\cdot\|$ denotes the total variation norm. By bounded convergence the right-hand side converges to zero as t tends to infinity.

To prove the convergence of the conditional laws, pick $s, t > 0$. From the notes after Theorem 2, we see that the random variables

$$H_{t,s} = \frac{\mathbb{P}_x(\tau_a > t + s | \mathcal{F}_t)}{\mathbb{P}_x(\tau_a > t + s)} = \mathbf{1}_{\{\tau_a > t\}} \frac{\mathbb{P}_{Y_t}(\tau_a > s)}{\mathbb{P}_x(\tau_a > t + s)}$$

converge to H_t a.s. as $s \rightarrow \infty$. Since $\mathbb{E}_x(H_{t,s}) = 1 = \mathbb{E}_x(H_t)$, it follows from Scheffe's lemma that the preceding convergence holds in L^1 . We deduce that $\mathbb{E}_x(AH_{t,s})$ converges to $\mathbb{E}_x(AH_t)$ for every $A \in L^\infty(\mathcal{F}_t)$. By the Markov property this means:

$$\lim_{s \rightarrow \infty} \mathbb{E}_x(A | \tau_a > t + s) = \mathbb{E}_x(AH_t) = \mathbb{E}_x^\diamond(A). \quad \square$$

In the sequel, we will frequently use the fact (from the optional stopping theorem) that for every finite stopping time S and $A \in L_+(\mathcal{F}_S)$

$$\mathbb{P}_x^\diamond(A) = \mathbb{E}_x(AH_S), \quad \widehat{\mathbb{P}}_x^\diamond(A) = \mathbb{E}_{-x}(A\widehat{H}_S).$$

We now collect some Laplace transforms of hitting times under \mathbb{P}^\diamond .

Proposition 7 For any $0 < b < x < c < a, q \geq 0$ the following hold:

(i) Two sided exit problem under \mathbb{P}^\diamond : if $T' = \inf\{t \geq 0 : Y_t \notin (b, c)\}$,

$$\mathbb{E}_x^\diamond(e^{-qT'} \mathbf{1}_{\{Y_{T'}=c\}}) = \frac{Z^{(-\varrho)}(c) W^{(q-\varrho)}(x-b)}{Z^{(-\varrho)}(x) W^{(q-\varrho)}(c-b)}.$$

(ii) Passage at an upper level:

$$\mathbb{E}_x^\diamond(\exp(-q\tau_c)) = \frac{Z^{(-\varrho)}(c) Z^{(q-\varrho)}(x)}{Z^{(-\varrho)}(x) Z^{(q-\varrho)}(c)}$$

(iii) Passage time below a lower level: if $T'' = \inf\{t \geq 0 : Y_t \notin (b, a]\}$,

$$\begin{aligned} \mathbb{E}_x^\diamond(\exp(-qT'') \mathbf{1}_{\{Y_{T''-} \in dy\}} \mathbf{1}_{\{\Delta Y_{T''} \in dz\}}) &= \frac{Z^{(-\varrho)}(y+z)}{Z^{(-\varrho)}(x)} \times \\ &\times \left(\frac{W^{(q-\varrho)}(x-b) W^{(q-\varrho)}(a-y)}{W^{(q-\varrho)}(a-b)} - W^{(q-\varrho)}(x-y) \right) dy \Lambda(dz). \end{aligned}$$

Similarly, we state some Laplace transforms of hitting times under $\widehat{\mathbb{P}}^\diamond$.

Proposition 8 For any $0 < b < x < c < a$ and $q \geq 0$ the following hold:

(i) Two sided exit problem under $\widehat{\mathbb{P}}^\diamond$: if $T' = \inf\{t \geq 0 : \widehat{Y}_t \notin (b, c)\}$,

$$\widehat{\mathbb{E}}_x^\diamond(e^{-qT'} \mathbf{1}_{\{X_{T'}=b\}}) = \frac{W^{(-\widehat{\varrho})}(a-b) W^{(q-\widehat{\varrho})}(c-x)}{W^{(-\widehat{\varrho})}(a-x) W^{(q-\widehat{\varrho})}(c-b)}.$$

(ii) Passage at an upper level: if $\widehat{\tau}_{\{c\}} = \inf\{t \geq 0 : \widehat{Y}_t = c\}$ and \widehat{Y} has unbounded variation,

$$\widehat{\mathbb{E}}_x^\diamond(\exp(-q\widehat{\tau}_{\{c\}})) = \frac{W^{(-\widehat{\varrho})}(a-c)}{W^{(-\widehat{\varrho})}(a-x)} \left[\frac{W^{(q-\widehat{\varrho})}(a-x)}{W^{(q-\widehat{\varrho})}(a-c)} - \frac{W^{(q-\widehat{\varrho})'}(a) W^{(q-\widehat{\varrho})}(c-x)}{W^{(q-\widehat{\varrho})'}(c) W^{(q-\widehat{\varrho})}(a-c)} \right]$$

(iii) Passage time above an upper level: if $T'' = \inf\{t \geq 0 : \widehat{Y}_t \notin [0, c]\}$,

$$\begin{aligned} \widehat{\mathbb{E}}_x^\diamond(\exp(-qT'') \mathbf{1}_{\{X_{T''-} \in dy\}} \mathbf{1}_{\{\Delta X_{T''} \in dz\}}) &= \frac{W^{(-\widehat{\varrho})}(a-y-z)}{W^{(-\widehat{\varrho})}(a-x)} \times \\ &\times \left(\frac{W^{(q-\widehat{\varrho})}(c-x) W^{(q-\widehat{\varrho})}(dy)}{W_+^{(q-\widehat{\varrho})'}(c)} - W^{(q-\widehat{\varrho})}(y-x) dy \right) \Lambda(dz). \end{aligned}$$

Proof of Propositions 7 and 8 Proposition 7 and statements (i) and (iii) of Proposition 8 can be proved adapting the line of reasoning followed in the proof of [18, Prop. 3.2].

The second statement of Proposition 8 follows from potential theory of Markov processes. Recall that under $\widehat{\mathbb{P}}^\diamond$ the process \widehat{Y} is a Markov process with semigroup $\widehat{P}_t^\diamond(x, dy) = \widehat{Q}^t(x, dy)e^{\widehat{\partial}t}W^{(-\widehat{\partial})}(a-y)/W^{(-\widehat{\partial})}(a-x)$. If \widehat{Y} has unbounded variation, Theorem 1 implies that the process \widehat{Y} under $\widehat{\mathbb{P}}^\diamond$, has an absolutely continuous q -resolvent measure \widehat{U}_q^\diamond with a version of its density given by

$$\widehat{u}_q^\diamond(x, y) = \widehat{r}^{q-\widehat{\partial}}(x, y)W^{(-\widehat{\partial})}(a-y)/W^{(-\widehat{\partial})}(a-x).$$

Then one has the identity

$$\mathbb{E}_x^\diamond[e^{-q\widehat{\tau}_{\{c\}}}] = \frac{\widehat{u}_q^\diamond(x, c)}{\widehat{u}_q^\diamond(c, c)},$$

which follows by approximating the potential density as in Theorem II.19(ii) in [5]. \square

8.1 Excursion measure away from a point

Recall that a point $x \in [0, a)$ is said to be regular (for itself) under \mathbb{P}^\diamond if

$$\mathbb{P}_x^\diamond(\inf\{s > 0 : Y_s = x\}) = 1.$$

Obviously, $x > 0$ is regular under \mathbb{P}^\diamond if and only if $x > 0$ is regular under \mathbb{P} , hence if and only if X has unbounded variation under \mathbb{P} . We assume this throughout from now on. The local time at level x , denoted by L^x is defined as the occupation density

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|Y_s - x| < \epsilon\}} ds.$$

Let σ_s be its right-continuous inverse:

$$\sigma_s = \inf\{t > 0 : L_t^x > s\}, \quad s \geq 0.$$

Analogously to what we did in the proof of Theorem 1, we now consider the excursion process $e = (e_s, s \geq 0)$ of Y away from $\{x\}$ where

$$e_s = (Y_u, \sigma_{s-} \leq u < \sigma_s) \quad \text{if } \sigma_{s-} < \sigma_s$$

and else e_s takes the value ∂ where ∂ is an additional isolated point. A famous theorem of Itô states that e is a Poisson point process valued in the space \mathcal{E}'

$$\mathcal{E}' = \{f \in D[0, \infty) : \exists \zeta = \zeta(f) \text{ such that } f(0) = f(\zeta) = 0\}.$$

Its characteristic measure is denoted by n_x under \mathbb{P} (and n_x^\diamond under \mathbb{P}^\diamond) and is called the excursion measure away from $\{x\}$. In this section we present some useful formulas involving

the local time L^x and the excursion measure n_x^\diamond . For every excursion of Y away from $\{x\}$, we denote its height by $m = m(\epsilon)$:

$$m(\epsilon) = \sup_{u \leq \zeta(\epsilon)} (\epsilon_u - \epsilon_0) = \sup_{u \leq \zeta(\epsilon)} \epsilon_u - x.$$

Recall that σ stands for the inverse of the local time L^x . As well known, σ is a subordinator. Define Laplace exponent Φ_x^\diamond by

$$\mathbb{E}_x^\diamond(e^{-\lambda\sigma_t}) = \exp(-t\Phi_x^\diamond(\lambda)), \quad \lambda \geq 0.$$

Analogously, replacing everywhere in the above definitions the process (Y, \mathbb{P}^\diamond) by $(\widehat{Y}, \widehat{\mathbb{P}}^\diamond)$ we define the local time \widehat{L} , its inverse $\widehat{\sigma}$ with exponent $\widehat{\Phi}_x$ and the excursion process \widehat{e} with its heights \widehat{m} and characteristic measure \widehat{n}_x^\diamond .

Proposition 9 (i) For any nonnegative λ and any $\eta \in [0, a - x]$,

$$n_x^\diamond(1 - \mathbf{1}_{\{m < \eta\}} e^{-\lambda\zeta}) = \frac{Z^{(\lambda-\varrho)}(x + \eta)}{Z^{(\lambda-\varrho)}(x)W^{(\lambda-\varrho)}(\eta)}.$$

In particular, for any nonnegative λ and for any $\eta \in [0, a - x]$,

$$\Phi_x^\diamond(\lambda) = \frac{Z^{(\lambda-\varrho)}(a)}{Z^{(\lambda-\varrho)}(x)W^{(\lambda-\varrho)}(a-x)} \quad n_x^\diamond(m > \eta) = \frac{Z^{(-\varrho)}(x + \eta)}{Z^{(-\varrho)}(x)W^{(-\varrho)}(\eta)}.$$

(ii) For any nonnegative λ and any $\eta \in [0, a - x]$,

$$\widehat{n}_x^\diamond(1 - \mathbf{1}_{\{\widehat{m} < \eta\}} e^{-\lambda\widehat{\zeta}}) = \frac{W^{(\lambda-\widehat{\varrho})'}(x + \eta)}{W^{(\lambda-\widehat{\varrho})'}(x)W^{(\lambda-\widehat{\varrho})'}(\eta)}.$$

In particular, for any nonnegative λ and for any $\eta \in [0, a - x]$,

$$\widehat{\Phi}_x^\diamond(\lambda) = \frac{W^{(\lambda-\widehat{\varrho})'}(a)}{W^{(\lambda-\widehat{\varrho})'}(x)W^{(\lambda-\widehat{\varrho})'}(a-x)} \quad \widehat{n}_x^\diamond(m > \eta) = \frac{W^{(-\widehat{\varrho})'}(x + \eta)}{W^{(-\widehat{\varrho})'}(x)W^{(-\widehat{\varrho})'}(\eta)}.$$

Propositions 9 can be proved much in the same vein as Proposition 4.2 in [18] and therefore we only sketch their proofs. Note that in both propositions the last two assertions follow easily from the first (by taking $\eta = a - x$ and $\lambda = 0$ respectively). Let $\tau_{\{c\}}$ and $\widehat{\tau}_{\{c\}}$ denote the first hitting time of $\{c\}$ by Y and \widehat{Y} respectively:

$$\tau_{\{c\}} = \inf\{t \geq 0 : Y_t = c\} \quad \widehat{\tau}_{\{c\}} = \inf\{t \geq 0 : \widehat{Y}_t = c\}.$$

The proof of the Proposition 9(i) starts from the identity

$$n_x^\diamond(1 - \mathbf{1}_{\{m < \eta\}} e^{-\lambda\zeta}) = \left[\mathbb{E}_x^\diamond \left(\int_0^{\tau_{\{x+\eta\}}} e^{-\lambda t} dL_t^x \right) \right]^{-1},$$

which follows by an application of the exponential formula to the Poisson point processes $m(e)$. Next step consists in establishing

$$\mathbb{E}_y^\diamond \left(\int_0^\infty e^{-\lambda t} dL_t^x \right) = u_\lambda^\diamond(y, x)$$

following a line of reasoning similar to the first lines of Proposition V.2 in [5]. An application of the optional sampling theorem yields that

$$(n_x^\diamond (1 - \mathbf{1}_{\{m < \eta\}} e^{-\lambda \zeta}))^{-1} = u_\lambda^\diamond(x, x) - u_\lambda^\diamond(x + \eta, x) \mathbb{E}_x^\diamond [e^{-\lambda \tau_{\{x+\eta\}}}]$$

and after substituting the expressions for u_λ^\diamond and for the Laplace transform (from Proposition 7(ii)) we end up with the stated expression. The first assertion of Proposition 9(ii) follows analogously replacing everywhere Y by \widehat{Y} .

The previous propositions enable us to specify the asymptotic behaviour of the local time. Recall from the Theorem 3 that the stationary measure μ and $\widehat{\mu}$ of the conditioned processes are absolutely continuous with respective densities p and \widehat{p} , say.

Corollary 3 (i) *If $x \in (0, a)$ or $x = 0$ and $a \in \mathcal{D}_\varrho$, we have*

$$\lim_{t \rightarrow \infty} \frac{L_t^x}{t} = \frac{\mu(dx)}{dx} = p(x) \quad a.s. \quad (38)$$

(ii) *If $x \in (0, a)$ or $x = 0$ and $a \in \mathcal{D}_{\widehat{\varrho}}$, equation (38) remains valid if we replace the triple (L^x, μ, p) by $(\widehat{L}^x, \widehat{\mu}, \widehat{p})$.*

Proof We deduce from Proposition 9 that Φ_x^\diamond has a right-derivative at 0 equal to

$$\frac{C(a)}{Z^{(-\varrho)}(x) W^{(-\varrho)}(a-x)} = p(x)^{-1}.$$

Hence using $\mathbb{E}_x^\diamond(\sigma_t) = t/p(x)$ and that $\{\sigma_t : t \geq 0\}$ is a Lévy process, the strong law of large numbers entails that almost surely

$$\lim_{t \rightarrow \infty} \frac{L_t^x}{t} = \lim_{t \rightarrow \infty} \frac{t}{\sigma_t} = p(x).$$

The proof of the second statement is similar and omitted. □

8.2 Convergence of the supremum

The fact that the conditioned processes are recurrent implies that under the measures \mathbb{P}^\diamond and $\widehat{\mathbb{P}}^\diamond$ the suprema of Y and \widehat{Y} , $M_t = \sup\{Y_s; s \in [0, t]\}$ and $\widehat{M}_t = \sup\{\widehat{Y}_s; s \in [0, t]\}$ respectively, converge to a as t tends to infinity. Our purpose is to investigate the rate of convergence. We still assume that X has unbounded variation.

Let $f : [0, \infty) \rightarrow (0, \infty)$ be a decreasing function and write

$$l_f = \liminf_{t \rightarrow \infty} \frac{a - M_t}{f(t)}, \quad L_f = \limsup_{t \rightarrow \infty} \frac{a - M_t}{f(t)}$$

and $\widehat{l}_f, \widehat{L}_f$ for the corresponding quantities involving \widehat{M} . Recall that a real-valued function is said to be slowly varying at infinity if for any $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{g(\lambda t)}{g(t)} = 1.$$

Finally, we set the notations $\mathcal{D}_\varrho = \{a > 0 : \varrho'(a) < 0\}$ and $\mathcal{D}_{\widehat{\varrho}} = \{a > 0 : \widehat{\varrho}'(a) < 0\}$.

Theorem 4 *Assume $a \in \mathcal{D}_\varrho$ for the statements involving M_t . Assume that $W^{(-\widehat{\varrho})}(\cdot)$ is twice continuously differentiable and let $a \in \mathcal{D}_{\widehat{\varrho}}$ for the statements involving \widehat{M}_t . Then following three assertions hold.*

- (i) *The random variables $t(a - M_t)$ and $t(a - \widehat{M}_t)$ converge in distribution as $t \rightarrow \infty$ to exponential random variables with parameters $|\varrho'(a)|$ and $|\widehat{\varrho}'(a)|$ respectively.*
- (ii) *The random variables l_f and \widehat{l}_f are 0 or ∞ almost surely according to whether the integral $\int_1^\infty f(s)ds$ converges or diverges.*
- (iii) *Assume further that $t \mapsto tf(t)$ is increasing and slowly varying at infinity and let*

$$\gamma_f = \inf \left\{ \gamma > 0 : \int_1^\infty f(t)e^{-\gamma tf(t)} dt < \infty \right\},$$

with the convention $\inf \emptyset = +\infty$. Then $L_f = |\varrho'(a)|^{-1}\gamma_f$ and $\widehat{L}_f = |\widehat{\varrho}'(a)|^{-1}\gamma_f$ almost surely.

Remarks

- (i) Let $I_f(\gamma) = \int_1^\infty dt f(t)e^{-\gamma tf(t)}$. One easily sees that if $\gamma_f < \infty$, I_f is finite (and decreasing) on (γ_f, ∞) and that if $\gamma_f > 0$, $I_f = \infty$ on $[0, \gamma_f)$.
- (ii) If \log_k denotes the k -th iterate of the logarithm, then for

$$\begin{aligned} f(t) = t^{-1} \log t, & & L_f = 0, & & l_f = 0; \\ f(t) = t^{-1} \log_2 t, & & L_f = |\varrho'(a)|^{-1}, & & l_f = 0; \\ f(t) = t^{-1} \log_3 t, & & L_f = \infty, & & l_f = 0. \end{aligned}$$

- (iii) Note that (32) implies that $Z^{(-\varrho(a))'}(a) < 0$ for $a \in \mathcal{D}_\varrho$. Similarly, if W is twice continuously differentiable and $a \in \mathcal{D}_{\widehat{\varrho}}$, it follows from (33) that $W^{(-\widehat{\varrho}(a))''}(a) < 0$. These facts are crucial in the proof of the theorem.

- (iv) Recall that for $\alpha \in (1, 2]$, $-r(\alpha)$ denotes the first negative root of E_α , where E_α is the Mittag-Leffler function of parameter α . In the case X is a stable process of index $\alpha \in (1, 2]$,

$$\limsup_{t \rightarrow \infty} \frac{t(a - M_t)}{\log_2 t} = \frac{a^{\alpha+1}}{\alpha r(\alpha)} \quad \text{a.s.},$$

which yields in the case X is a Brownian motion

$$\limsup_{t \rightarrow \infty} \frac{t(a - M_t)}{\log_2 t} = \frac{2a^3}{\pi^2} \quad \text{a.s.}$$

Following the lead of Lambert [18], the idea is to exploit the Poisson point character of the excursions away from $\{x\}$ under \mathbb{P}^\diamond and $\widehat{\mathbb{P}}^\diamond$. First note that Corollary 3 allows one to translate statements involving asymptotics of M_{σ_t} and $\widehat{M}_{\widehat{\sigma}_t}$ for large t into statements on asymptotics of M_t and \widehat{M}_t for large t respectively. An elementary observation is that M_{σ_t} and $\widehat{M}_{\widehat{\sigma}_t}$ are the maximum of the excursion heights $(m(\epsilon_s), s \leq t)$ and $(\widehat{m}(\widehat{\epsilon}_s), s \leq t)$ respectively. The study of the asymptotics of M_t and \widehat{M}_t is thus reduced to that of the maxima of some Poisson point process. In Proposition 9 the distribution functions of the measures n_x^\diamond and \widehat{n}_x^\diamond were computed explicitly and this allows one to perform explicit computations on the excursion heights. The key step in the proof of the third assertion is to appeal to results of Robbins and Siegmund [21] on laws of the iterated logarithm for maxima and minima of uniform random variables. Since the arguments are quite close to those of [18], we leave the details to the reader.

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