Optimal Control of Itô Diffusions with an Ergodic Criterion and External Parameters

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Abstract

We address the problem of confining a general one-dimensional Itô diffusion within an externally specified region $]A, B[$ by means of an impulse control process. We minimise a long-term average criterion that penalises deviations of the state process from an unspecified nominal point within this region and the use of impulsive control effort. Whenever impulse control is exercised to reposition the state of the system, a fixed cost and a cost proportional to the size of the impulse are incurred. We solve the resulting optimisation problem and we provide an explicit optimal control strategy under general assumptions. Optimal central bank intervention to control an exchange rate or an inflation rate and optimal contribution policy in defined benefit pension plans are examples of potential applications of the model that we study.

1 Introduction

The dynamics of the stochastic system we study are described by the controlled, one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x \in \mathbb{R},$$

where W is a standard, one-dimensional Brownian motion. As long as the state process is inside the given interval $]A, B[$, the controller takes no action. Impulses are applied when

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the state process X reaches the bounds of the interval $]A, B[$. If the state process X assumes the value $x = B$, then control should be exercised to "push" it instantaneously to the level $\beta < B$. Similarly, whenever the state process X assumes the value $x = A$, control action should be used to reposition it at $\alpha > A$. The controlled process Z is assumed to be piecewise constant: the jumps of this process occur at the times when the system's controller intervenes to reposition the system's state by an amount equal to the associated jump sizes. The control problem is to determine the correct size of these jumps. The objective of the optimisation is to minimise the long-term average *pathwise* performance criterion defined by

$$J^{\mathrm{P}}(\mathbb{C}_x) = \limsup_{T\to\infty} \frac{1}{T}\left[\int_0^T h(X_t)\, dt + \sum_{t\in\,[0,T]} \left(K^+\Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t>0\}} \right.$$
$$\left. + \sum_{t\in\,[0,T]} \left(-K^-\Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t<0\}}\right],$$

as well as the long term average *expected* criterion

$$J^{\mathrm{E}}(\mathbb{C}_x) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_x\left[\int_0^T h(X_t)\, dt + \sum_{t\in\,[0,T]} \left(K^+\Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t>0\}} \right.$$
$$\left. + \sum_{t\in\,[0,T]} \left(-K^-\Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t<0\}}\right].$$

Here, h is a given function that penalises deviations of the state process X from an unspecified nominal operating point. The positive constants c^+ and K^+ (resp., c^- and K^-) provide a fixed and a proportional cost each time the controller experiences a jump of the system's state in the positive (resp., negative) direction.

There are several potential applications of the control problem that we solve. One of them arises in the context of intervening to control an exchange rate that must be maintained within a specific trading range. This trading range may be externally imposed, as was the case for European currencies that were subject to the Exchange Rate Mechanism (ERM) during the late 1980's and early 1990's, or even self-imposed, as is the case for several Asian currencies that are pegged to the US dollar at present. In this context, the state process represents the targeted exchange rate while the control process captures central bank intervention efforts. Jeanblanc-Picqué [JP] considers the problem of controlling in an impulsive way an exchange rate, modelled by geometric Brownian motion with drift, so that it is confined within a given interval $]A, B[$. Mundaca and Øksendal [MØ] and Cadenillas and Zapatero [CZ1, CZ2] carried out further research in this area incorporating an additional central bank intervention policy that takes the form of absolutely continuous control of the drift of the underlying exchange rate dynamics. In these references, the objective criteria considered are discounted expectations. With regard to the associated economics theory, the significance of the discounting factors appearing in these references is not fully justified.

Central banks typically attempt to control exchange rates by influencing the supply and demand of the foreign currency relative to the domestic currency through the purchase and sale of foreign government securities in the open market. A long-term average criterion may be more appropriate for assessing the efficacy of these central bank interventions.

Another potential application of the problem we study is associated with controlling inflation through monetary policy. Historically, the primary function of central banks in free market economies was to promote stability in the banking system, while political authorities were responsible for monetary policy. More recently, central banks have been given responsibility for controlling inflation through the exercise of monetary policy that is independent from other government policies and objectives. In this new regime, most central banks target explicit or implicit inflation levels and may face statutory penalties if inflation falls below or (more often) rises above certain thresholds. In this context, the state process represents the targeted inflation rate while the control process captures central bank intervention efforts. Chiarolla and Haussmann [CH] study a model for the control of an inflation rate by means of an intervention policy that results in a singular stochastic control problem, using discounted expectations as the objective criteria. With respect to the problem of controlling inflation, central banks typically attempt to control the inflation rate either by changing the domestic interest rate or by altering money supply through the issuance or withdrawal of domestic government securities. Again in this case, a long-term average criterion seems more appropriate than discounted expectations for assessing the efficacy of these central bank interventions.

A third potential application of the stochastic control problem that we study arises in the context of defined benefit pension fund management. In this context, the state process would represent the surplus or deficit of the fund, while the impulse control process would capture the contributions that the sponsoring organisation would be required to make in order to maintain the funding position of the plan within an appropriate range. In this case, discounted expectations may be more appropriate for assessing the sponsoring company's contribution policy from a cash flow perspective. However, a long-term average criterion would be more appropriate for assessing contribution policy from an accounting, regulatory, and solvency perspective as contributions are effectively subtracted from retained earnings and therefore affect directly the book value of equity, which only changes through new stock issuance or through earnings retention over time.

Controlling the price of energy commodities (oil, natural gas, etc.) through coordinated action on supply undertaken by members of the Organisation of Petroleum Exporting Countries (OPEC) presents another possible practical application of the optimal control problem that we study. In this context, the state process represents the price of the targeted commodity, while the control process reflects coordinated changes in the supply of the targeted commodity.

A potential shortcoming with respect to the practical application of the stochastic optimal control problem that we study is the time lag between control impulse and state process response observed in some practical applications. This is particularly true for inflation rate control applications, where the state process typically responds to impulse control efforts with a lag of nine to twelve months. However, this is less of a problem in exchange rate and commodity price control applications, where even the suspicion or announcement of intervention efforts may be sufficient to move the targeted exchange rate or commodity price in the desired direction. In any case, we would view the stochastic optimal control problem that we solve as a general approximation of these specific practical applications.

Jack and Zervos [JZ] carried out research that provided a first step in the direction of developing a new approach to the problem of controlling an exchange rate or an inflation rate. In their model, they assume general dynamics for the underlying state process and consider a long-term average criterion. Our study is a further step in the same direction and differs from Jack and Zervos [JZ] in that we assume an externally specified target range $]A, B[$ while Jack and Zervos [JZ] assumed that the target range was part of the optimisation problem.

Apart from its several potential applications, the problem that we solve is also interesting from the perspective of the general theory of stochastic optimal control because it provides a rare non-trivial example of a control problem that admits a solution of an explicit analytic nature without imposing any limiting assumptions. A similar problem involving singular instead of impulse control was solved by Karatzas [Ka] and was later generalised considerably by Weerasinghe [W]. In view of the relationship of singular control with impulse control, it is of interest to note that our assumptions are very similar to the ones imposed by Weerasinghe [W]. Other notable contributions to the theory of continuous time stochastic control with an ergodic criterion include Borkar [B], Borkar and Ghosh [BG], Kurtz and Stockbridge [KuS], Kushner [Ku] and others.

2 Problem formulation

The state process X of the stochastic system that we study is driven by a Brownian motion W and a controlled process Z that affects the system's dynamics in an impulsive way. The associated one-dimensional SDE that describes the dynamics of this system is

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x \in [A, B], \tag{1}$$

where $b, \sigma : [A, B] \to \mathbb{R}$ are given functions, W is a standard, one-dimensional Brownian motion and $-\infty < A < B < \infty$. The impulse control process Z is a finite variation, càglàd, piece-wise constant process. The time evolution of this process is determined by the system's controller, who is given the two bounds $A, B \in \mathbb{R}$ and has the task of applying a strictly negative impulse whenever the state process X assumes the value $x = B$ and a strictly positive impulse when the state process X assumes the value $x = A$ so as to keep X inside the band $[A, B]$ at all times. Such a process can also be described by the collection

$$\mathcal{Z} = \left(\tau_1, \tau_2, \ldots, \tau_n, \ldots; \Delta Z_{\tau_1}, \Delta Z_{\tau_2}, \ldots, \Delta Z_{\tau_n}, \ldots\right),$$

where τ_n is the random time at which the n-th jump of Z occurs and $\Delta Z_{\tau_n} := Z_{\tau_n^+} - Z_{\tau_n}$ is the size of the corresponding jump. In view of this characterisation, an admissable choice of Z should satisfy

$$X_{\tau_n} \mathbf{1}_{\{\Delta Z_{\tau_n} > 0\}} = A, \quad X_{\tau_n} \mathbf{1}_{\{\Delta Z_{\tau_n} < 0\}} = B \quad \text{and} \quad |\Delta Z_{\tau_n}| < B - A, \quad \text{for all } n \geq 1. \tag{2}$$

The controlled process Z affects the system's state process only by causing a jump of size $\Delta X_{\tau_n} = \Delta Z_{\tau_n}$ at each of the times τ_n. Indeed, the evolution of the state process between any two consecutive times at which Z has a discretionary jump is governed by the uncontrolled SDE

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0 = x \in \mathbb{R}. \tag{3}$$

We impose the following assumption that incorporates conditions required by our analysis of the control problem considered in this paper.

Assumption 1 The functions $b, \sigma : [A, B] \to \mathbb{R}$ are bounded and satisfy the following conditions:

$$\sigma^2(x) > 0, \text{ for all } x \in [A, B],$$

and

$$\int_A^B \sigma^{-2}(x)\, dx < \infty.$$

□

In the presence of this assumption both of conditions (ND)′ and (LI)′ in Section 5.5 of Karatzas and Shreve [KaS] are satisfied, and, therefore, (3) has a weak solution up to the exit time from $[A, B]$ that is unique in the sense of distribution law. Moreover, the scale function and the speed measure that characterise a one-dimensional diffusion such as the one in (3), given by

$$p_\gamma(\gamma) = 0, \quad p'_\gamma(x) = \exp\left(-2\int_\gamma^x \frac{b(s)}{\sigma^2(s)}\, ds\right), \quad \text{for } x \in [A, B], \tag{4}$$

and

$$m_\gamma(dx) = \frac{2}{p'_\gamma(x)\sigma^2(x)}\, dx, \tag{5}$$

respectively, for any given choice of $\gamma \in]A, B[$, are well-defined,

$$\sup_{x, \gamma \in [A,B]} p'_\gamma(x) < \infty \text{ and } m_\gamma([A, B]) < \infty. \tag{6}$$

We adopt a weak formulation of the control problem that we are going to study.

Definition 1 Given an initial condition $x \in [A, B]$, an admissible impulse control of the stochastic system under consideration is any seven-tuple $\mathbb{C}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, P, W, Z, X)$ such that

$(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space satisfying the usual conditions,
W is a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion,
Z is a finite variation, piece-wise constant, càglàd, $(\mathcal{F}_t)$-adapted process,
X is a càglàd, $(\mathcal{F}_t)$-adapted process, and
(1) and (2) hold true.

We define $\mathcal{C}_x$ to be the family of all such admissible controls $\mathbb{C}_x$. □

With each control $\mathbb{C}_x \in \mathcal{C}_x$, we associate the long-term average *pathwise* performance criterion defined by

$$J^{\mathrm{P}}(\mathbb{C}_x) = \limsup_{T\to\infty} \frac{1}{T} \Bigg[\int_0^T h(X_t)\, dt + \sum_{t\in[0,T]} \left(K^+ \Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t > 0\}} + \sum_{t\in[0,T]} \left(-K^- \Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t < 0\}} \Bigg], \tag{7}$$

as well as the long term average *expected* criterion

$$J^{\mathrm{E}}(\mathbb{C}_x) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_x \Bigg[\int_0^T h(X_t)\, dt + \sum_{t\in[0,T]} \left(K^+ \Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t > 0\}} + \sum_{t\in[0,T]} \left(-K^- \Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t < 0\}} \Bigg], \tag{8}$$

Here, $h :]A, B[\to [0, \infty[$ is a given function that models the running cost resulting from the system's operation and $K^+, c^+, K^-, c^- > 0$, are given constants penalising the use of control effort. With regard to the control's contribution to the performance index, we observe that, each time the controller deploys an impulse to cause a repositioning of the state process in the positive (resp., negative) direction, a fixed cost c^+ (resp., c^-) and a proportional cost equal to K^+ (resp., K^-) multiplied by the size of the jump are incurred.

The objective of the optimisation problem is to minimise the performance criteria defined by (7) and (8) over all controls $\mathbb{C}_x \in \mathcal{C}_x$. For the control problem to be well posed, we impose the following additional assumption

Assumption 2 The function $h :]A, B[\to [0, \infty[$ is Lebesgue-measurable and satisfies the integrability condition

$$\int_A^B h(s)\, m_\gamma(ds) < \infty.$$

Also, $K^+, c^+, K^-, c^- > 0$.

It is worth noting that our assumptions on h are quite general. Indeed, we impose no monotonicity or convexity conditions on h, and we allow for the possibility that $\lim_{x\downarrow A} h(x) = \infty$ and / or $\lim_{x\uparrow B} h(x) = \infty$.

3 The solution of the control problem

In light of standard stochastic control theory, we will solve the control problem formulated in the previous section by finding a sufficiently, for an application of Itô's formula, smooth function w and a constant λ satisfying the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda = 0, \quad x \in]A, B[. \tag{9}$$

subject to the boundary conditions

$$\begin{aligned} w(A) &= \inf_{z\in [0,B-A[} \left[w(A+z) + K^+ z\right] + c^+, \quad (10)\\ w(B) &= \inf_{z\in [0,B-A[} \left[w(B-z) + K^- z\right] + c^-. \quad (11)\end{aligned}$$

If such a pair (w, λ) exists, then, subject to suitable technical conditions, we expect the following statements to be true. Given any initial condition $x \in [A, B]$,

$$\lambda = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{P}}(\mathbb{C}_x) = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{E}}(\mathbb{C}_x),$$

which reflects the fact that the optimal value of the performance criterion is independent of the system's initial condition. Equation (9) is associated with the requirement that the controller should take no action as long as the system's state process assumes values in the interval $]A, B[$. Also, the boundary condition (10) (resp., (11)) arises from the requirement that the controller should exercise immediate impulsive action so as to appropriately reposition the state process inside the interval $]A, B[$ whenever this hits the boundary point A (resp., B).

Now, we conjecture that the optimal strategy is characterised by two unknown points α, β, such that $A < \alpha < \beta < B$, and takes a form that can be described as follows. If the state process X assumes the value $x = B$, then control should be exercised to "push" it instantaneously to the level β. Similarly, whenever the state process X assumes the value $x = A$, control action should be used to reposition it at α. As long as the state process is inside the interval $]A, B[$, the controller should take no action. Therefore, we look for a solution (w, λ) to (7)–(11) such that

$$w(B) = w(\beta) + K^-(B - \beta) + c^-, \tag{12}$$

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda = 0, \quad \text{for } x \in]A, B[, \tag{13}$$

$$w(A) = w(\alpha) + K^+(\alpha - A) + c^+. \tag{14}$$

Assuming that this strategy is indeed optimal, we need an appropriate system of equations to determine the free boundary points α, β and the constant λ. To this end, we first observe that the boundary condition (11) implies that the function w should satisfy

$$w(x) + K^{-}(B - x) + c^{-} - w(B) \geq 0, \qquad \text{for all } x \leq B.$$

Combining this inequality with (12), we can conclude that the function $x \mapsto w(x) - K^{-}x$ has local minimum at β, which implies

$$w'(\beta) = K^{-} \tag{15}$$

A similar argument involving (14) and the boundary condition (10) implies

$$w'(\alpha) = -K^{+} \tag{16}$$

In addition to these conditions, we can see that (12) and (14) imply that the free boundary points β and α should satisfy

$$\int_{\beta}^{B} w'(s)\, ds = K^{-}(B - \beta) + c^{-},$$

$$\int_{A}^{\alpha} w'(s)\, ds = -K^{+}(\alpha - A) - c^{+},$$

respectively. Now, equations (15) and (16), the fact that K^{+}, K^{-} are strictly positive and the continuity of w', which is required by (13) imply that there exists a point $\gamma \in \,]\alpha, \beta[$ such that $w'(\gamma) = 0$. It is straightforward to show that any solution of the ODE (9) that satisfies this boundary condition is given by

$$w'(x) = p'_{\gamma}(x) \int_{\gamma}^{x} [\lambda - h(s)]\, m_{\gamma}(ds), \quad x \in [A, B], \tag{17}$$

where p_{γ} and m_{γ} are the scale function and the speed measure of the uncontrolled diffusion (3), defined by (4) and (5), respectively.

Summarising the analysis so far, in order to determine the four unknown parameters $\alpha < \gamma < \beta$ and λ, we have to solve the following system of four non-linear algebraic equations:

$$g(\beta, \lambda, \gamma) = K^{-}, \tag{18}$$

$$g(\alpha, \lambda, \gamma) = -K^{+}, \tag{19}$$

$$\int_{\beta}^{B} g(s, \lambda, \gamma)\, ds = K^{-}\,(B - \beta) + c^{-}, \tag{20}$$

$$\int_{A}^{\alpha} g(s, \lambda, \gamma)\, ds = -K^{+}\,(\alpha - A) - c^{+}, \tag{21}$$

where g is defined by

$$g(x,\lambda,\gamma) = p'_\gamma(x) \int_\gamma^x [\lambda - h(s)]\, m_\gamma(ds), \quad \text{for } x,\gamma \in [A,B] \text{ and } \lambda \in \mathbb{R}. \tag{22}$$

The following result is concerned with showing that the system of equations (18)–(21) has a solution of the required form.

Lemma 1 *Suppose that Assumptions 1 and 2 hold. The system of equations (18)–(21), where g is defined by (22), has a solution $(\alpha,\gamma,\beta,\lambda)$ such that $A < \alpha < \gamma < \beta < B$. Moreover, if w is a function defined by (17), then w is twice differentiable in the classical sense, w' is bounded in $[A,B]$ and the pair (w,λ) satisfies (9)–(11).*

Proof. We start by observing that

$$g_x(x,\lambda,\gamma) = -\frac{2}{\sigma^2(x)}\big[h(x) + b(x)g(x,\lambda,\gamma) - \lambda\big], \quad \text{for all } x,\gamma \in]A,B[\text{ and } \lambda \in \mathbb{R}. \tag{23}$$

Also, given any $\gamma \in]A,B[$ and $x \in [A,B]$, we calculate

$$g_\lambda(x,\lambda,\gamma) = p'_\gamma(x) \int_\gamma^x m_\gamma(ds) \begin{cases} < 0, & \text{if } x < \gamma, \\ > 0, & \text{if } x > \gamma, \end{cases} \tag{24}$$

which implies

$$\lim_{\lambda\to\infty} g(x,\lambda,\gamma) = \begin{cases} -\infty, & \text{if } x < \gamma, \\ \infty, & \text{if } x > \gamma. \end{cases}$$

Combining these calculations with the observation that

$$g(x,0,\gamma) = -p'_\gamma(x) \int_\gamma^x h(s)\, m_\gamma(ds) \begin{cases} > 0, & \text{if } x < \gamma, \\ < 0, & \text{if } x > \gamma. \end{cases}$$

and the positivity of the function h and the constants K^+, K^-, we can see that

$$^*\lambda(\gamma) := \inf\left\{\lambda \in \mathbb{R} \mid \inf_{x\le\gamma} g(x,\lambda,\gamma) \le -K^+\right\} \in]0,\infty[, \text{ for all } \gamma \in]A,B[\tag{25}$$

$$\lambda^*(\gamma) := \inf\left\{\lambda \in \mathbb{R} \mid \sup_{x\ge\gamma} g(x,\lambda,\gamma) \ge K^-\right\} \in]0,\infty[, \text{ for all } \gamma \in]A,B[\tag{26}$$

where we adopt the usual convention that $\inf\emptyset = \infty$. It follows that the sets

$$\mathcal{S}_l = \{(\lambda,\gamma) \in]\alpha,\beta[\times \mathbb{R} \mid \ \lambda \ge \lambda^*(\gamma)\},$$
$$\mathcal{S}_r = \{(\lambda,\gamma) \in]\alpha,\beta[\times \mathbb{R} \mid \ \lambda \ge^* \lambda(\gamma)\}$$

are non-empty and that the functions $\mathcal{A} : \mathcal{S}_l \mapsto]A, B[$ and $\mathcal{B} : \mathcal{S}_r \mapsto]A, B[$ given by

$$\mathcal{A}(\lambda, \gamma) = \max \left\{ x \leq \gamma \mid \ g(x, \lambda, \gamma) = -K^+ \right\}, \tag{27}$$

$$\mathcal{B}(\lambda, \gamma) = \min \left\{ x \geq \gamma \mid \ g(x, \lambda, \gamma) = K^- \right\} \tag{28}$$

are well-defined and real valued. Now, differentiating $g(\beta(\lambda, \gamma), \lambda, \gamma) - K^- = 0$ with respect to λ, we can see that

$$\mathcal{B}_\lambda(\lambda, \gamma) = - \frac{g_\lambda(\mathcal{B}(\lambda, \gamma), \lambda, \gamma)}{g_x(\mathcal{B}(\lambda, \gamma), \lambda, \gamma)} > 0, \quad \text{for all } \gamma \in]A, B[\text{ and } \lambda \geq \ \lambda^*(\gamma).$$

the inequality following thanks to (24) and the inequality $g_x(\beta(\lambda, \gamma), \lambda, \gamma) \geq 0$, which follows from the definition of $\mathcal{B}$ in (28), the definition of $\lambda^*(\gamma)$ in (26) and the fact that $\lambda \mapsto g(x, \lambda, \gamma)$ is continuous and strictly increasing for all $x > \gamma$ fixed. Making a similar observation relative to $\mathcal{A}$, we reach the conclusion that, given any $\gamma \in]A, B[$,

$$\mathcal{A}(\cdot, \gamma) \quad \text{is strictly increasing in }]\lambda^*(\gamma), \infty[\tag{29}$$

and

$$\mathcal{B}(\cdot, \gamma) \quad \text{is strictly decreasing in }]\lambda^*(\gamma), \infty[\tag{30}$$

To proceed further, we define

$$^*q(\lambda, \gamma) = \int_A^{\mathcal{A}(\lambda, \gamma)} \left[g(s, \lambda, \gamma) + K^+ \right] ds + c^+, \quad \text{for } (\lambda, \gamma) \in \ \mathcal{S}_l \tag{31}$$

$$q^*(\lambda, \gamma) = \int_{\mathcal{B}(\lambda, \gamma)}^B \left[g(s, \lambda, \gamma) - K^- \right] ds - c^-, \quad \text{for } (\lambda, \gamma) \in \ \mathcal{S}_r \tag{32}$$

With regard to (31) and the definition (28) of $\mathcal{B}$, we calculate

$$q^*_\lambda(\lambda, \gamma) = \int_{\mathcal{B}(\lambda, \gamma)}^B g_\lambda(s, \lambda, \gamma) ds > 0, \quad \text{for all } (\lambda, \gamma) \in \ \mathcal{S}_r \tag{33}$$

the inequality following thanks to (24) and the fact that $\mathcal{B}(\lambda, \gamma) > \gamma$. Combining this with (30), we can see that given $\lambda_1 > \lambda^*(\gamma)$,

$$q^*_\lambda(\lambda, \gamma) > q^*_\lambda(\lambda_1, \gamma), \quad \text{for all } \lambda > \lambda_1 \text{ and } \gamma \in]A, B[.$$

Since the right hand side in this inequality is a strictly positive constant, independent of λ, it follows that $\lim_{\lambda \to \infty} q^*(\lambda, \gamma) = \infty$, for all $\gamma \in]A, B[$. However, this limit, (33), and the observation that

$$q^*(\lambda^*(\gamma), \gamma) \leq -c^- < 0,$$

which follows from the definition of λ^* in (26), imply that there exists a unique function $\Lambda^* > \lambda^*$ such that

$$q^*(\Lambda^*(\gamma), \gamma) = 0, \quad \text{for all } \gamma \in]A, B[.$$

Using similar arguments, we can also show that there exists a unique function ${}^*\Lambda >^* \lambda$ such that

$${}^*q({}^*\Lambda(\gamma), \gamma) = 0, \quad \text{for all } \gamma \in]A, B[.$$

To complete the proof, we must show that there exists a unique $\gamma \in]A, B[$ such that $\Lambda^*(\gamma) =^* \Lambda(\gamma)$, which amounts to showing that the equation $Q(\gamma) = 0$, where $Q(\gamma) = \Lambda^*(\gamma) -^* \Lambda(\gamma)$, has a unique solution $\gamma \in]A, B[$. To this end, we first observe that the definition of g in (22), (6), and the definition (25) of ${}^*\lambda$ imply that $\lim^*_{\gamma\downarrow A} \lambda(\gamma) = \infty$, which combined with the inequality ${}^*\Lambda >^* \lambda$, yields $\lim^*_{\gamma\downarrow A} \Lambda(\gamma) = \infty$.

$$\lim_{\gamma\to B} \Lambda^*(\gamma) = \infty \text{ and } \lim_{\gamma\to B} {}^*\Lambda(\gamma) < \infty \text{ therefore } \lim_{\gamma\to B} Q(\gamma) = \infty$$

$$\lim_{\gamma\to A} \Lambda^*(\gamma) < \infty \text{ and } \lim_{\gamma\to A} {}^*\Lambda(\gamma) = \infty \text{ therefore } \lim_{\gamma\to A} Q(\gamma) = -\infty$$

This proves that $Q(\gamma) = 0$ has solutions. Now, if we prove that $Q_\gamma(\gamma) > 0$ then $Q(\gamma) = 0$ has unique solution.

$$Q_\gamma(\gamma) = \Lambda^*_\gamma(\gamma) -^* \Lambda_\gamma(\gamma) = -\frac{q^*_\gamma(\Lambda^*(\gamma), \gamma)}{q^*_\lambda(\Lambda^*(\gamma), \gamma)} + \frac{{}^*q_\gamma({}^*\Lambda(\gamma), \gamma)}{{}^*q_\lambda({}^*\Lambda(\gamma), \gamma)}$$

Equation (23) implies that $g_x(\gamma, \Lambda^*(\gamma), \gamma) = -h(\gamma) + \Lambda^*(\gamma) > 0$, therefore

$$h(\gamma) < \Lambda^*(\gamma)$$

Given this inequality and the calculation

$$q^*_\gamma(\Lambda^*, \gamma) = 2\frac{h(a) - \Lambda^*}{\sigma^2(\gamma)} \int_\beta^B p'_\gamma(s)\, ds.$$

it follows that $\Lambda^*_\gamma(\gamma) > 0$, for all $\gamma \in]A, B[$. Using similar reasoning, we can show that ${}^*\Lambda_\gamma(\gamma) < 0$, for all $\gamma \in]A, B[$. This demonstrates that $Q_\gamma(\gamma) > 0$ and therefore $Q(\gamma) = 0$ has unique solution. □

Theorem 2 *Suppose that Assumptions 1 and 2 hold, and let $(\alpha, \gamma, \beta, \lambda)$ be the solution of (18)–(21), where g is defined by (22), constructed in Lemma 1. Given any initial condition $x \in [A, B]$,*

$$\lambda = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{P}}(\mathbb{C}_x) = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{E}}(\mathbb{C}_x),$$

and the points α, β determine the optimal strategy that has the qualitative nature discussed above.

Proof. Throughout this proof, we fix a solution (w, λ) of (9)–(11) that is constructed as in Lemma 1. We also fix any initial condition $x \in [A, B]$. Given any admissible control $\mathbb{C}_x \in \mathcal{C}_x$, we can use Itô's formula for general semimartingales to obtain

$$w(X_T) = w(x) + \int_0^T \left[\frac{1}{2}\sigma^2(X_t)w''(X_t) + b(X_t)w'(X_t)\right] dt + M_T + \sum_{t \le T} \Delta w(X_t)$$

where M is the stochastic integral defined by

$$M_t = \int_0^t \sigma(X_s)w'(X_s)\, dW_s.$$

With regard to (2) and the fact that w satisfies (9), this implies

$$\begin{aligned}
&\frac{1}{T}\left[\int_0^T h(X_t)\, dt + \sum_{t \in [0,T]} \left(K^+\Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t > 0\}} + \sum_{t \in [0,T]} \left(-K^-\Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t < 0\}}\right] \\
&\qquad = \frac{w(x)}{T} + \lambda + \frac{M_T}{T} \\
&\qquad + \left[\sum_{t \in [0,T]} \left[w(A + \Delta Z_t) - w(A) + K^+\Delta Z_t + c^+\right] \mathbf{1}_{\{\Delta Z_t > 0\}}\right] \\
&\qquad + \left[\sum_{t \in [0,T]} \left[w(B + \Delta Z_t) - w(B) - K^-\Delta Z_t + c^-\right] \mathbf{1}_{\{\Delta Z_t < 0\}}\right] \\
&\qquad \ge \frac{w(x)}{T} + \lambda + \frac{M_T}{T}
\end{aligned} \tag{34}$$

the inequality following because w satisfies (10) and (11). Since ???

$$\lim_{T \to \infty} \frac{M_T}{T} = \infty$$

In view of this observation, we can take limits in (34) to obtain

$$J(\mathbb{C}_x) \ge \lambda.$$

To prove the reverse inequality, let $\mathbb{C}_x^\circ$ be a weak solution to (1), in the sense of Definition 1, that satisfies

$$\Delta Z^\circ_{\tau_n} \mathbf{1}_{\{\Delta Z^\circ_{\tau_n} > 0\}} = \alpha - A \quad \text{and} \quad \Delta Z^\circ_{\tau_n} \mathbf{1}_{\{\Delta Z^\circ_{\tau_n} < 0\}} = B - \beta$$

Such a weak solution can be constructed as in the proof of Theorem 2 in Jack and Zervos [JZ]. For this choice of a control strategy, (12) and (14) imply that (34) holds with equality, which, by passing to the limit as $T \to \infty$, implies $J(\mathbb{C}_x^\circ) = \lambda$, and the proof is complete. □

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