Axiomatic Interest Rate theory

Lane P. Hughston

Professor of Financial Mathematics
Department of Mathematics, King's College London
The Strand, London WC2R 2LS, UK
(lane.hughston@kcl.ac.uk, www.mth.kcl.ac.uk)

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References on which this presentation is based:

L. P. Hughston and A. Rafailidis (2004) A Chaotic Approach to Interest Rate Modelling, to appear in *Finance and Stochastics*
(http://www.mth.kcl.ac.uk/research/finmath/Hughston-Rafailidis-15Feb04.pdf)

D. C. Brody and L. P. Hughston (2004) Chaos and Coherence: a New Framework for Interest Rate Modelling, Proc. Roy. Soc. Lond. A, Vol. 460, 85-110.
(http://theory.ic.ac.uk/~brody/DCB/dcb36.pdf)

L. P. Hughston (2003) The Past, Present, and Future of Term Structure Modelling, chapter 7 in Modern Risk Management: A History, introduced by Peter Field, Risk Publications.

Contents

1. General requirements for asset pricing models

The aim of this presentation is to lay out the foundations for a unified theory of interest rate dynamics and stochastic volatility.

For this we need a class of flexible pricing models that can be adapted to various different categories of financial instruments.

We do not wish to assume that the market is complete, or that positions in derivatives are necessarily hedgeable.

We want a framework that is equally applicable both to (a) derivatives pricing and hedging problems, and to (b) general risk management problems.

For this reason, we shall always stick with the "real" probability measure. Other measures can be introduced when necessary for solving particular problems.

2. The axiomatic approach

Given these criteria, for many purposes the most effective way forward is the use of the so-called "pricing kernel" method.

According to this method, the absence of arbitrage in financial markets is represented by the existence of a universal "pricing kernel" that establishes the intertemporal relations between asset prices.

We shall adopt an axiomatic approach.

The setup is as follows.

We model the financial markets with the specification of a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which we denote as Π.

We assume that Π is equipped with the standard augmented filtration $\Phi = (\mathcal{F}_t)$ generated by a system of one or more independent Wiener processes W_t^α $(\alpha = 1, \cdots, k)$ over an infinite time horizon.

We assume that the random trajectories followed by asset prices are continuous semimartingales on Π.

The absence of arbitrage in the resulting models will be characterised according to the following axiomatic scheme:

We assume the existence of a pricing kernel π_t, satisfying $\pi_t > 0$, such that:

(A1) There exists an absolutely continuous, strictly increasing "risk-free" asset B_t (the money-market account).

(A2) If S_t is the price-process of any asset, and D_t is the associated continuous dividend rate, then the process M_t defined by

$$M_t = \pi_t S_t + \int_0^t \pi_s D_s ds \tag{1}$$

is a martingale.

(A3) There exists an asset (a floating rate note) that offers a continuous dividend rate sufficient to ensure that the value of the asset remains constant.

(A4) A system of discount bonds P_{tT} exists for all $0 \leq t \leq T < \infty$ satisfying $\lim_{T\to\infty} P_{tT} = 0$.

It turns out that this simple system of axioms is sufficient to allow us to develop a very general framework for interest rate modelling.

At the same time, we gain some new insights into the nature of stochastic volatility.

3. Dynamical equations for risky assets

First, let's see how various familiar results concerning the dynamics of assets can be reconstructed from the axiomatic scheme.

Since B_t is strictly increasing and absolutely continuous, there exists an adapted short rate process $r_t > 0$ such that

$$B_t = B_0 \exp\left(\int_0^t r_s ds\right). \tag{2}$$

Because the money market account is non-dividend-paying, it follows from (A1) and (A2) that there exists a martingale ρ_t such that

$$\pi_t B_t = \rho_t. \tag{3}$$

Since ρ_t is positive, there exists an adapted vector-valued process λ_t such that

$$d\rho_t = -\rho_t \lambda_t dW_t. \tag{4}$$

Axiom (A2) implies, in the case of a risky asset that pays no dividend, that S_t can be written in the form

$$S_t = \frac{B_t M_t}{\rho_t} \tag{5}$$

where M_t is a martingale.

Then if we write $dM_t = \theta_t dW_t$ it is easy to verify that

$$dS_t = (r_t S_t + \lambda_t \psi_t)dt + \psi_t dW_t, \tag{6}$$

where ψ_t is defined by

$$\psi_t = \frac{B_t \theta_t}{\rho_t} + \lambda_t S_t. \tag{7}$$

Thus ψ_t is the "absolute" volatility of S_t.

If S_t is positive, then we can write $\psi_t = \sigma_t S_t$, and the dynamical equation satisfied by S_t can be written

$$\frac{dS_t}{S_t} = (r_t + \lambda_t \sigma_t)dt + \sigma_t dW_t. \tag{8}$$

It is thus evident that λ_t is the market risk premium, and we recognise (8) as the dynamics of a risky asset with limited liability in a market with no arbitrage.

4. Risky assets with dividends

In the case of a dividend paying asset these formulae need to be modified slightly, and as a consequence of (A2) we obtain

$$dS_t = (r_t S_t - D_t + \lambda_t \psi_t)dt + \psi_t dW_t. \tag{9}$$

If S_t is positive we can introduce a proportional dividend rate f_t by the relation $D_t = f_t S_t$, and we obtain

$$\frac{dS_t}{S_t} = (r_t - f_t + \lambda_t \sigma_t)dt + \sigma_t dW_t, \tag{10}$$

where σ_t is defined by $\psi_t = \sigma_t S_t$.

This shows how we obtain the dynamics of a dividend or interest paying asset with limited liability.

For example, if S_t is the price of a foreign currency, then f_t corresponds to the overnight rate for that currency.

5. Price processes for discount bonds

To proceed further we introduce a system of discount bonds. Let us write P_{tT} for the price of a T-maturity bond at time t. We require $P_{TT} = 1$ for all T.

Since discount bonds are non-dividend-paying, it follows from axiom (A2) that $\pi_t P_{tT}$ is a martingale. Therefore there exists a family of martingales M_{tT} such that

$$P_{tT} = \frac{B_t M_{tT}}{\rho_t}. \tag{11}$$

Since M_{tT} is a positive martingale for each T, there exists a process Ω_{tT} such that

$$\frac{dM_{tT}}{M_{tT}} = (\Omega_{tT} - \lambda_t)dW_t. \tag{12}$$

The dynamical equations of the discount bond system are then given by

$$\frac{dP_{tT}}{P_{tT}} = (r_t + \lambda_t \Omega_{tT})dt + \Omega_{tT} dW_t. \tag{13}$$

Thus we recognise Ω_{tT} as the discount bond volatility process.

6. The volatility structure approach

If we make use of the relation $P_{tt} = 1$, we can integrate the dynamical equations of the discount bond system to deduce that

$$P_{tT} = P_{0tT} \frac{\exp\left(\int_0^t \lambda_s \Omega_{sT} ds + \int_0^t \Omega_{sT} dW_s - \frac{1}{2}\int_0^t \Omega_{sT}^2 ds\right)}{\exp\left(\int_0^t \lambda_s \Omega_{st} ds + \int_0^t \Omega_{st} dW_s - \frac{1}{2}\int_0^t \Omega_{st}^2 ds\right)}. \tag{14}$$

The money market account is then given by

$$B_t = \frac{B_0}{P_{0t} \exp\left(\int_0^t \lambda_s \Omega_{st} ds + \int_0^t \Omega_{st} dW_s - \frac{1}{2}\int_0^t \Omega_{st}^2 ds\right)}. \tag{15}$$

Here $P_{0tT} = P_{0T}/P_{0t}$ denotes the t-forward price made at time 0 for a T-maturity bond.

An interesting feature of these expressions is that the discount bond system and the money market account can be represented in terms of the risk premium λ_t and the volatility Ω_{tT}, together with the initial discount function P_{0t}, without reference to the short rate r_t.

We may thus regard λ_t and Ω_{tT} as being subject to an exogenous specification.

Historically this observation forms the basis of the approach to interest rate derivatives pricing widely used in practice according to which one models the volatility structure.

7. The bond pricing formula

Because the discount bonds pay no dividend, it follows from (A2) that the martingale relation

$$\pi_t P_{tT} = \mathbb{E}_t \left[\pi_u P_{uT} \right] \tag{16}$$

holds for all $0 \leq t \leq u \leq T < \infty$.

Here $\mathbb{E}_t[-]$ denotes conditional expectation with respect to $\mathcal{F}_t$.

In particular, setting $u = T$ we obtain the so-called bond pricing formula

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}_t \left[\pi_T \right]. \tag{17}$$

Thus if Π supports a discount bond system, then (A1) and (A2) imply that P_{tT} takes this form.

On the other hand, it follows from (A1) and (A2) that $\mathbb{E}[\pi_t] < \infty$ for all $t \in [0, \infty)$.

This, together with the fact that $\pi_t > 0$, ensures the existence of the conditional expectation $\mathbb{E}_t[\pi_T]$ for $0 \leq t \leq T < \infty$, and hence the existence of a discount bond system.

8. Relation to the HJM framework

We note that $\pi_t = \rho_t / B_t$.

Since B_t is $\mathcal{F}_t$-measurable and increasing, we deduce that $\mathbb{E}_t[\pi_T] < \pi_t$, and thus we see that π_t is a supermartingale.

It follows then from the bond pricing formula that $P_{tT} < 1$ for all $t < T$.

Next we observe that axiom (A3) implies that the following process is a martingale:

$$N_t = \pi_t + \int_0^t r_s \pi_s ds. \tag{18}$$

This follows from the fact that the value of a perpetual floating rate note is constant, and that the dividend is the short rate.

The resulting martingale relation allows us to deduce that

$$P_{tT} = 1 - \frac{1}{\pi_t} \mathbb{E}_t \left[\int_t^T r_s \pi_s ds \right]. \tag{19}$$

Equation (19) says that ownership of a T-maturity discount bond is equivalent to ownership of one unit of the floating rate note, but without the right to the dividend flow of the floating rate note from time t to time T.

It follows that for any two maturity dates T_1 and T_2 we have

$$P_{tT_1} - P_{tT_2} = \frac{1}{\pi_t}\mathbb{E}_t\left[\int_{T_1}^{T_2} r_s\pi_s ds\right]. \tag{20}$$

Therefore if $T_2 > T_1$, we infer that $P_{tT_2} < P_{tT_1}$, and hence the positivity of forward rates.

A variant of this relation follows if we use Fubini's theorem to obtain

$$P_{tT} = 1 - \frac{1}{\pi_t}\int_t^T E_t\left[r_s\pi_s\right] ds, \tag{21}$$

which shows that the discount bond system is differentiable in the maturity date.

Writing $f_{tT} = -\partial_T \ln P_{tT}$ for the instantaneous forward rate system we conclude that

$$f_{tT}P_{tT} = \frac{\mathbb{E}_t\left[r_T\pi_T\right]}{\pi_t}. \tag{22}$$

This relation shows that the instantaneous forward rate f_{tT} can be interpreted as the value, at t, future-valued to T, of the contingent claim that pays r_T at T on a unit principal.

The significance of the existence of the instantaneous forward rates is that the class of interest rate models under consideration here is equivalent to the family of all models of the HJM type defined over the given time horizon, in a general (incomplete) market setting, with positive interest.

9. Asymptotic conditions on the discount bond system

Now we bring axiom (A4) into play and assume that

$$\lim_{T\to\infty} P_{tT} = 0. \tag{23}$$

It follows immediately that

$$\pi_t = \lim_{T\to\infty} \mathbb{E}_t\left[\int_t^T r_s\pi_s ds\right]. \tag{24}$$

Since the process N_t is a positive martingale, the martingale convergence theorem implies the existence of a random variable $A_\infty = \int_0^\infty r_s \pi_s ds$ to which the process

$$A_t = \int_0^t r_s \pi_s ds \tag{25}$$

converges, almost surely, in the limit $t \to \infty$, with the property that

$$E\left[\int_0^\infty r_s \pi_s ds\right] < \infty. \tag{26}$$

It then follows by the conditional form of the monotone convergence theorem that

$$\lim_{T\to\infty} E_t \left[\int_t^T r_s \pi_s ds\right] = E_t \left[\lim_{T\to\infty} \int_t^T r_s \pi_s ds\right] \tag{27}$$

and hence that

$$\pi_t = E_t \left[\int_t^\infty r_s \pi_s ds\right]. \tag{28}$$

This very useful relation says that a floating rate note paying the rate r_t on a unit principal in perpetuity has the value unity.

An alternative expression for π_t that follows immediately from (28) is given by

$$\pi_t = \mathbb{E}_t \left[A_\infty\right] - A_t. \tag{29}$$

We see therefore that our axioms imply that the pricing kernel is necessarily of this form.

Conversely, if π_t is of this form, then one can show that axioms (A1)–(A4) are satisfied.

In particular, we can show that the money market account, the floating rate note, and the discount bond system all exist, and satisfy the axioms.

10. Conditional variance representation for π_t

Now let η_t be a vector process satisfying

$$\eta_t^2 = r_t \pi_t. \tag{30}$$

Then we can define a random variable X_∞ by the formula

$$X_\infty = \int_0^\infty \eta_s dW_s. \tag{31}$$

The existence of X_∞ is guaranteed by virtue of axiom (A3) which implies

$$\mathbb{E}\left[\int_0^\infty r_s \pi_s ds\right] < \infty. \tag{32}$$

It follows immediately then by virtue of the Ito isometry that

$$\begin{aligned} \pi_t &= \mathbb{E}_t\left[\int_t^\infty \eta_s^2 ds\right] \\ &= \mathbb{E}_t\left[\left(\int_t^\infty \eta_s dW_s\right)^2\right] \\ &= \mathbb{E}_t\left[\left(\int_0^\infty \eta_s dW_s - \int_0^t \eta_s dW_s\right)^2\right]. \end{aligned} \tag{33}$$

However, because

$$\mathbb{E}_t\left[X_\infty\right] = \int_0^t \eta_s dW_s, \tag{34}$$

we deduce that

$$\pi_t = \mathbb{E}_t\left[\left(X_\infty - \mathbb{E}_t\left[X_\infty\right]\right)^2\right], \tag{35}$$

which we recognise as the conditional variance of X_∞ with respect to the σ-algebra $\mathcal{F}_t$.

11. Calibration of interest rate models by Wiener chaos coefficients

It follows from the foregoing analysis that we have a map from the set of all arbitrage-free positive interest rate models to the space of square-integrable random variables on the Wiener space Π.

This space has a very rich natural structure that can be exploited in the analysis of the associated interest rate systems.

The key point is that we can represent X_∞, and therefore characterise the corresponding interest rate system, by use of a *Wiener chaos expansion.*

In particular, X_∞ can be expanded in a unique way in a series of the form

$$X_\infty = \int_0^\infty \phi_s dW_s + \int_0^\infty \int_0^s \phi_{ss_1} dW_{s_1} dW_s + \cdots, \tag{36}$$

where the integrands here are deterministic tensor-valued functions.

The different interest rate models thus arising from the specification of X_∞ are nested in a natural way.

We shall call an interest rate model that only contains terms up to order n in the expansion of X_∞ an $n^{\rm th}$-order chaos model.

The $n^{\rm th}$-order chaos models are contained as a subset of the $m^{\rm th}$-order chaos models, for all $n < m$.

The inputs of such models are the deterministic functions ϕ_s, ϕ_{ss_1}, $\phi_{ss_1s_2}$, and so on, which we call the chaos coefficients.

It follows that interest rate models can be classified according to their chaos structure.

12. Foreign exchange systems

The modelling framework described here generalises in a natural way to the situation where we also include a foreign exchange system, with a family of discount bonds associated with each currency.

In particular, a chaos representation exists for the dynamics of the entirety of such a system of interest rates and foreign exchange.

The resulting chaos coefficients parametrise both the interest rate models as well as the volatility structures of the exchange rate system.

Let us write S_t^{ij} for the price of one unit of currency i in units of currency j, where $i, j = 0, 1, \cdots, N$.

We may for convenience think of the label $i = 0$ as referring to the currency with respect to which the axioms (A1)–(A4) are formulated.

For each currency we assume that there exists a strictly increasing money-market asset B_t^i, with a corresponding strictly positive short rate process r_t^i such that

$$B_t^i = B_0^i \exp\left(\int_0^t r_s^i ds\right). \tag{37}$$

We also require the existence of a floating rate note in each currency.

That is to say, for each value of i we assume the existence of an asset of constant value in units of currency i, paying a dividend at the rate r^i_t.

If we write S^{i0}_t for the value of one unit of currency i in units of the base currency, we see that the product $S^{i0}_t B^i_t$ represents the base-currency price of a non-dividend-paying asset.

Therefore, by Axiom 2 we deduce for each value of i that $\pi_t S^{i0}_t B^i_t$ is a martingale, from which it follows that $\pi_t S^{i0}_t$ is a supermartingale.

Thus if we define $\pi^i_t = \pi_t S^{i0}_t$ to be the pricing kernel associated with currency i, then from the cyclic relation $S^{ij}_t S^{j0}_t = S^{i0}_t$ we deduce that

$$S^{ij}_t = \frac{\pi^i_t}{\pi^j_t}. \tag{38}$$

This gives us a general expression for the exchange-rate process as a ratio of pricing kernels.

As a consequence we find that the dynamics of S^{ij}_t are given by

$$\frac{dS^{ij}_t}{S^{ij}_t} = \left[r^j_t - r^i_t + \lambda^j_t \left(\lambda^j_t - \lambda^i_t\right)\right] dt + \left(\lambda^j_t - \lambda^i_t\right) dW_t. \tag{39}$$

Here λ^i_t denotes the market price of risk process associated with assets that are denominated in currency i.

Thus we see that the axiomatic scheme, when extended to include a money market account and a floating rate note for each foreign currency, is sufficient to generate the general foreign exchange dynamics between the currencies.

Consider now the discount bond system for foreign currency number i, and denote by P^i_{tT} the value at time t of a bond that pays one unit of currency i at time T.

In this case $S^{i0}_t P^i_{tT}$ is the price in base currency of an asset that pays no dividend, and therefore $\pi_t S^{i0}_t P^i_{tT}$ is a martingale by Axiom 2.

It follows then by use of the martingale relation that each foreign bond system admits a representation of the form

$$P^i_{tT} = \frac{\mathbb{E}_t\left[\pi^i_T\right]}{\pi^i_t}. \tag{40}$$

To proceed further we make the assumption that $\lim_{T\to\infty} P^i_{tT} = 0$ for all i.

Then a conditional variance representation exists for the pricing kernel associated with each currency.

In other words, there exists a system of random variables X^i_∞ $(i = 0, 1, \cdots, N)$, belonging to $L^2(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$\pi^i_t = \mathbb{E}_t\left[\left(X^i_\infty - \mathbb{E}_t\left[X^i_\infty\right]\right)^2\right], . \tag{41}$$

Each of these random variables admits a chaos representation.

Thus, once the random variables X^i_∞ have been specified for $i = 0, 1, \cdots, N$ then the entire system of interest rates and foreign exchange is completely determined.

13. Stochastic volatility models for general assets

The line of reasoning presented in the previous section can be extended to characterise the price of any asset in terms of another, providing that these prices are always positive and that we interpret the associated short-rate systems as continuous dividend streams.

As a consequence we infer that the "generic" model for an asset price is a process of the form

$$S_t = \frac{\text{Var}_t\left[Y_\infty\right]}{\text{Var}_t\left[X_\infty\right]}. \tag{42}$$

In other words, the price S_t is given by a ratio of conditional variances, where X_∞ and Y_∞ are elements of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

It follows that the volatility of S_t is naturally 'parameterised' by the chaos coefficients of the random variables X_∞ and Y_∞.

We are thus led in this way to a natural family of nested stochastic volatility models for the dynamics of general assets.

One of the appealing features of such a scheme is the way in which it opens up the possibility of a consistent, unified approach to stochastic volatility modelling and interest rate modelling.

It remains to be seen whether empirical studies will support the conclusion that the Wiener chaos coefficients do indeed represent the appropriate functional degrees of freedom necessary for conditioning such models on the available market data.

14. Geometric Brownian motion and beyond

In conclusion, we shall generate the well-known Black-Scholes geometric Brownian motion process, which can be seen to arise in a natural way in the present context.

This is of course not a new asset price model as such, but it makes the point that familiar material can be recovered in the present framework, if so desired.

Indeed, by seeing an established idea in a new context, we actually learn something new about it as well.

For the Black-Scholes economy we have a constant interest rate r and a constant market price of risk λ.

The dynamical equation for the pricing kernel reads

$$d\pi_t = -r\pi_t dt - \lambda\pi_t dW_t. \tag{43}$$

The solution to this equation is

$$\pi_t = \pi_0 \exp\left(-rt - \lambda W_t - \tfrac{1}{2}\lambda^2 t\right). \tag{44}$$

The associated random variable X_∞ is given by

$$\begin{aligned} X_\infty &= \int_0^\infty \sqrt{r_t\pi_t}\, dW_t \\ &= \sqrt{r\pi_0}\int_0^\infty \exp\left(-\tfrac{1}{2}rt - \tfrac{1}{2}\lambda W_t - \tfrac{1}{4}\lambda^2 t\right) dW_t \end{aligned} \tag{45}$$

It is an exercise to check that

$$\pi_t = \mathbb{E}_t\left[X_\infty^2\right] - \left(\mathbb{E}_t\left[X_\infty\right]\right)^2. \tag{46}$$

Now we introduce a second pricing kernel ψ_t based on the same Brownian motion, satisfying

$$d\psi_t = -f\psi_t dt - \gamma\psi_t dW_t. \tag{47}$$

This process can be interpreted as the pricing kernel appropriate for use when the risky asset S_t is taken to be numeraire.

Then f is the dividend rate or foreign interest rate associated with S_t, and γ is the associated market price of risk for assets priced in units of S_t.

For ψ_t we obtain

$$\psi_t = \psi_0 \exp\left(-ft - \gamma W_t - \tfrac{1}{2}\gamma^2 t\right). \tag{48}$$

For the associated random variable Y_∞ we have

$$Y_\infty = \sqrt{f\psi_0}\int_0^\infty \exp\left(-\tfrac{1}{2}ft - \tfrac{1}{2}\gamma W_t - \tfrac{1}{4}\gamma^2 t\right) dW_t. \tag{49}$$

Finally, for S_t we deduce that

$$\begin{aligned} S_t &= \frac{\mathrm{Var}_t\left[Y_\infty\right]}{\mathrm{Var}_t\left[X_\infty\right]} \\ &= \frac{\psi_t}{\pi_t} \\ &= \frac{\psi_0 \exp\left(-ft - \gamma W_t - \frac{1}{2}\gamma^2 t\right)}{\pi_0 \exp\left(-rt - \lambda W_t - \frac{1}{2}\lambda^2 t\right)} \\ &= S_0 \exp\left((r-f)t + \lambda\sigma t + \sigma W_t - \tfrac{1}{2}\sigma^2 t\right), \end{aligned} \tag{50}$$

where

$$S_0 = \frac{\psi_0}{\pi_0}, \quad \text{and} \quad \sigma = \lambda - \gamma. \tag{51}$$

Thus we recover the standard geometric Brownian motion model for an asset paying dividends at the rate f, where σ is the volatility.

The slightly surprising feature that arises here even in the case of the elementary geometric Brownian motion model is the fact that the asset price process can be expressed as the quotient of the pricing kernels for two different numeraire systems.

In particular, we note that the volatility of the asset price arises as the difference of the two risk premiums.

The example above, based on a single Brownian factor, with constant parameters, can be generalised to encompass significantly wider classes of models.

References

1 K.I. Amin and R. Jarrow (1992) Pricing options on risky assets in a stochastic interest rate economy, *Math. Finance* **2**, 217.

2 M.W. Baxter (1992) General interest-rate models and the universality of HJM. In: Dempster, M.A.H., Pliska, S.R. (eds): *Mathematics of Derivative Securities*. Cambridge University Press.

3 D.C. Brody and L.P. Hughston (2001) Interest rates and information geometry, *Proc. Roy. Soc. London* **457**, 1343.

4 D.C. Brody and L.P. Hughston (2002) Entropy and information in the interest rate term structure, *Quant. Finance* **2**, 70.

5 D.C. Brody and L.P. Hughston (2004) Chaos and Coherence: a New Framework for Interest Rate Modelling, *Proc. Roy. Soc. London* **460**, 85.

6 B. Flesaker and L.P. Hughston (1996) Positive interest *Risk* **9**, 46 (Reprinted in: Hughston, L. P. (ed.): Vasicek and beyond: approaches to building and applying interest rate models. London: Risk Publications 1996).

7 B. Flesaker and L.P. Hughston (1997) International models for interest rates and foreign exchange, *Net exposure* **1**, 55 (Reprinted in: Hughston, L. P. (ed.): The new interest rate models. Risk Publications 2000).

8 B. Flesaker and L.P. Hughston (1998) Positive interest: an afterword. In: Broadie, M., Glasserman, P. (eds): Hedging with trees: advances in pricing and risk managing derivatives. Risk Publications (p. 120).

9 M.R. Grasselli and T.R. Hurd (2004) Wiener chaos and the Cox-Ingersoll-Ross model, to appear in *Proc. Roy. Soc. Lond.*

10 D. Heath, R. Jarrow, and A. Morton (1992) Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation, *Econometrica* **60**, 77.

11 Y. Jin and P. Glasserman (2001) Equilibrium positive interest rates: a unified view. *Rev. Fin. Studies* **14**, 187.

12 L.P. Hughston (2003) The Past, Present, and Future of Term Structure Modelling, chapter 7 in Modern Risk Management: A History, introduced by Peter Field, Risk Publications.

13 L.P. Hughston and A. Rafailidis (2004) A Chaotic Approach to Interest Rate Modelling, to appear in *Finance and Stochastics*.

14 P.J. Hunt and J.E. Kennedy (2000) *Financial derivatives in theory and practice*, Chichester: Wiley.

15 M. Musiela and M. Rutkowski (1997) *Martingale methods in financial modelling*, Berlin: Springer.

16 L.C.G. Rogers (1997) The potential approach to the term structure of interest rates and foreign exchange rates, *Math. Finance* **7**, 157.

17 M. Rutkowski (1997) A note on the Flesaker-Hughston model on the term structure of interest rates, *Applied Math. Finance* **4**, 151.