Irreversible capacity expansion with proportional and fixed costs *

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Abstract

We consider the problem of determining the optimal capacity expansion strategy that a firm operating within a random economic environment should adopt. We model market uncertainty by means of a geometric Brownian motion. The objective is to maximise a performance criterion that involves a general running payoff function and associates a fixed and a proportional cost with each capacity increase. The resulting optimisation problem takes the form of a two-dimensional impulse control problem that we explicitly solve.

1 Introduction

We consider the problem of determining in a dynamical way the optimal capacity level of a given investment project, the operation of which is associated with random cash flows. In particular, we consider an investment project that yields payoff at a rate that is dependent

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on its installed capacity level and on an underlying economic indicator such as the price of or the demand for the project's unique output commodity, which we model by a geometric Brownian motion. The project's capacity level can only be increased, and each capacity increase is associated with a fixed and a proportional cost. The objective is to determine the capacity expansion strategy that maximises the resulting expected, discounted payoff flow. The resulting optimisation problem takes the form of a two-dimensional impulse control problem that we solve explicitly.

The model studied by Davis, Dempster, Sethi and Vermes [DDSV87] (see also Davis [D93]) is among the first ones to address the issue of optimally determining the timing and the size of capacity increases that can be associated with the operation of a given investment project in the presence of random economic fluctuations. Since then, irreversible capacity expansion models have attracted significant interest in the literature. Important contributions include Kobila [K93], Øksendal [Ø00], Wang [W03], Chiarolla and Haussmann [CH05], Bank [B05]; see also the references therein. In these references, capacity increases are associated with proportional costs only, and result in optimisation problems of singular control type. At this point, we should observe that capacity expansion models in which the installed capacity level can be reduced as well as increased, namely, reversible capacity expansion models, have also attracted interest in the literature (see Abel and Eberly [AE96], Guo and Pham [GP05]).

The chapter is organised as follows. Section 2 is concerned with a rigorous formulation of the investment decision model that we study. In Section 3, we derive sufficient conditions, which conform with economic intuition, for the associated optimisation problem to possess a finite value function, and we establish a number of estimates that we use in our subsequent analysis. Finally, we solve the optimisation problem considered in Section 4.

2 Problem formulation

We fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by P-negligible sets, and carrying a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{A}$ the family of all càglàd, $(\mathcal{F}_t)$-adapted, increasing and piecewise constant processes Z such that $Z_0 = 0$.

We consider an investment project that produces a given commodity, and we assume that the project's capacity, namely its rate of output, can be increased at any given time and by any amount. We denote by Y_t the project's capacity at time t, and we model capacity increases by jumps of an impulse control process $Z \in \mathcal{A}$, i.e. every time Z_t jumps, there is a capacity increase and $\Delta Z_t = \Delta Y_t$ is the size of the jump at time t. The capacity process Y is therefore given by

$$Y_t = y + Z_t, \quad Y_0 = y \geq 0, \tag{1}$$

where $y \geq 0$ is the project's initial capacity. Every process $Z \in \mathcal{A}$ is characterised by the collection $\mathcal{Z} = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \Delta Z_{\tau_1}, \Delta Z_{\tau_2}, \ldots, \Delta Z_{\tau_n}, \ldots)$ where τ_n is the $(\mathcal{F}_t)$-stopping time at which the n-th jump of Z occurs, while ΔZ_{τ_n} is the associated jump size. If the project's management adopts the capacity expansion strategy modelled by Z, then the project's capacity is increased at the times τ_n, $n \geq 1$, by an amount $\Delta Y_t = \Delta Z_t > 0$.

We assume that all randomness associated with the project's operation can be captured by a state process X that satisfies the SDE

$$dX_t = bX_t\, dt + \sqrt{2}\sigma X_t\, dW_t, \quad X_0 = x > 0, \tag{2}$$

for some constants b and σ. In practice, X_t can be an economic indicator reflecting, e.g., the value of one unit of the output commodity or the output commodity's demand or both, at time t.

To simplify the notation, we define

$$\mathcal{S} = \left\{(x, y) \in \mathbb{R}^2 : \ x > 0, \ y \geq 0\right\},$$

so that $\mathcal{S}$ is the set of all possible initial conditions. With each decision policy Z we associate the performance criterion

$$J_{x,y}(Z) = \text{``}E\left[\int_0^\infty e^{-rt} h(X_t, Y_t)\, dt - \sum_{0 \leq t} e^{-rt}\left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}\right]\text{''}, \tag{3}$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given function and

$$r, K, c > 0 \tag{4}$$

are constants. Here, h models the running payoff resulting from the project's operation, and K, c provide a proportional and a fixed cost incurred each time that the project's capacity level is changed.

As it stands in (3), the performance index $J_{x,y}$ is not necessarily well-defined because the random variable inside the expectation may not be integrable or even well-defined. To address this issue, we define

$$U_T = \int_0^T e^{-rt} h(X_t, Y_t)\, dt - \sum_{0 \leq t \leq T} e^{-rt}\left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}, \quad \text{for } T \geq 0. \tag{5}$$

In the next section (see Lemma 4, in particular), we are going to impose assumptions on h such that U_T is well-defined, for all $T > 0$, and *either*

$$U_\infty = \lim_{T\to\infty} U_T \text{ exists in } \mathbb{R}, \; P\text{-a.s.}, \quad \text{and} \quad U_\infty \in L^1(\Omega, \mathcal{F}, P), \tag{6}$$

in which case, we naturally define

$$J_{x,y}(\xi^+, \xi^-) = E\left[U_\infty\right], \tag{7}$$

as in (3), *or* there exists an $(\mathcal{F}_t)$-adapted process V such that

$$U_T \le V_T, \text{ for all } T \ge 0, \quad \text{and} \quad \limsup_{T\to\infty} E\left[V_T\right] = -\infty, \tag{8}$$

in which case, we define

$$J_{x,y}(\xi^+, \xi^-) = -\infty. \tag{9}$$

The objective is to maximise this performance index over all admissible capacity expansion strategies $Z \in \mathcal{A}$. The value function of the resulting optimisation problem is defined by

$$v(x, y) = \sup_{Z\in\mathcal{A}} J_{x,y}(Z). \tag{10}$$

3 Assumptions and preliminary estimates

The purpose of this section is to establish conditions on the problem's data under which our control problem is well-posed and its value function is finite, and to prove certain estimates that will be used in our analysis. Before we address these issues, we first discuss an ODE that will play an instrumental role in the solution of the control problem considered.

Let $k :]0, \infty[\to \mathbb{R}$ be any measurable function such that

$$E\left[\int_0^\infty e^{-rt} \left|k(X_t)\right| dt\right] < \infty, \quad \text{for all } x > 0. \tag{11}$$

With reference to Proposition 4.1 of Knudsen, Meister and Zervos [KMZ98], the function $R(\cdot; k) :]0, \infty[\to \mathbb{R}$ given by

$$R(x; k) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1} k(s)\, ds + x^n \int_x^\infty s^{-n-1} k(s)\, ds\right] \tag{12}$$

is well-defined and

$$R(x;k) = E\left[\int_0^\infty e^{-rt} k(X_t)\, dt\right] \tag{13}$$

Moreover, every solution of the ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - ru(x) + k(x) = 0. \tag{14}$$

can be expressed by

$$u(x) = Ax^n + Bx^m + R(x;k), \tag{15}$$

for some $A, B \in \mathbb{R}$. Here, the constants $m < 0 < n$ are the solutions of the quadratic equation

$$\sigma^2\lambda^2 + (b - \sigma^2)\lambda - r = 0, \tag{16}$$

given by

$$m, n = \frac{-(b-\sigma^2) \pm \sqrt{(b-\sigma^2)^2 + 4\sigma^2 r}}{2\sigma^2}. \tag{17}$$

With regard to $R(\cdot;k)$,

$$\text{if } k \text{ is increasing, then } R(\cdot;k) \text{ is increasing,} \tag{18}$$

and

$$\inf_{x>0} k(x) \geq 0 \quad \Leftrightarrow \quad \inf_{x>0} R(x;k) \geq 0. \tag{19}$$

For future reference, note that, given $\lambda \in \mathbb{R}$,

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} X_t^\lambda\, dt\right] &= x^\lambda \int_0^\infty e^{\left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right]t} E\left[e^{-\sigma^2\lambda^2 t + \sqrt{2}\sigma\lambda W_t}\right] dt \\ &= \begin{cases} \infty, & \text{if } \lambda \leq m \text{ or } \lambda \geq n, \\ -x^\lambda / \left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right], & \text{if } \lambda \in]m, n[. \end{cases} \end{aligned} \tag{20}$$

We are going to need the following estimate that is related with the definitions above.

Lemma 1 *Given any $\lambda \in]0, n[$, there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that*

$$E\left[e^{-rt}\bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda e^{-\varepsilon_1 t} \quad \text{and} \quad E\left[\sup_{t\geq 0} e^{-rt}\bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda,$$

where $\bar{X}_t = \sup_{s\leq t} X_s$.

Proof. Since n is the positive solution of the quadratic equation (16), it follows that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$r - \varepsilon_1 > 0 \quad \text{and} \quad \sigma^2\lambda^2 + (b - \sigma^2)\lambda - (r - \varepsilon_1) = -\varepsilon_2.$$

Given such parameters, we define

$$\Psi = \sup_{t \geq 0} \left[-\frac{\sigma^2\lambda^2 + \varepsilon_2}{\sqrt{2}|\sigma|\lambda} t + W_t \right],$$

we calculate

$$\begin{aligned} e^{-rt}\bar{X}_t^\lambda &= x^\lambda e^{-\varepsilon_1 t} e^{-(r-\varepsilon_1)t} \sup_{s \leq t} \exp\big((r - \varepsilon_1)s - (\sigma^2\lambda^2 + \varepsilon_2)s + \sqrt{2}\sigma\lambda W_s\big) \\ &= x^\lambda e^{-\varepsilon_1 t} \sup_{s \leq t} \left[\exp\big(-(r - \varepsilon_1)(t - s)\big) \exp\left(-(\sigma^2\lambda^2 + \varepsilon_2)s + \sqrt{2}\sigma\lambda W_s\right) \right] \\ &\leq x^\lambda e^{-\varepsilon_1 t} e^{\sqrt{2}|\sigma|\lambda\Psi}, \end{aligned}$$

and we observe that

$$\sup_{t \geq 0} e^{-rt}\bar{X}_t^\lambda \leq x^\lambda e^{\sqrt{2}|\sigma|\lambda\Psi}.$$

Since Ψ is exponentially distributed with parameter $2\left(\sigma^2\lambda^2 + \varepsilon_2\right) / \left(\sqrt{2}|\sigma|\lambda\right)$ (see Karatzas and Shreve [KS88, Exercise 3.5.9], the two bounds follow by a simple integration. □

The following assumption ensures that the control problem formulated in Section 2 is well-posed and its value function is finite and identifies with an appropriate solution of the associated Hamilton-Jacobi-Bellman equation.

Assumption 1 The problem's data satisfy the following conditions:

(a) The function h is C^3, and

$$h_x(x, y) \geq 0, \text{ for all } y \geq 0. \tag{21}$$

If we define

$$H(x, y) = h_y(x, y), \quad (x, y) \in \mathcal{S}, \tag{22}$$

then, given any $y > 0$,

$$H_x(x, y) > 0, \text{ for all } x > 0, \quad \text{and} \quad \lim_{x \to \infty} H(x, y) = \infty, \tag{23}$$

and, given any $x > 0$,

$$H_y(x, y) < 0, \text{ for all } y > 0. \tag{24}$$

(b) The constants r, K, C are strictly positive (see (4)), and there exist constants

$$\alpha > 0, \;\; \beta \in]0, 1[, \;\; \vartheta_1 \in]0, n[, \;\; \vartheta_2 \in]0, K] \text{ and } C > 0,$$

where $n > 0$ is given by (17), such that

$$\frac{\alpha}{1-\beta} \in]0, n[\quad \Leftrightarrow \quad \frac{n\beta}{n-\alpha} \in]0, 1[, \tag{25}$$

$$-C(1+y) \leq h(x, y) \leq C(1 + x^{n-\vartheta_1}) + Cx^\alpha y^\beta + r(K - \vartheta_2)y, \quad \text{for all } (x, y) \in \mathcal{S}. \tag{26}$$

$$-C \leq H(x, y) \leq \beta C x^\alpha y^{-(1-\beta)} + r(K - \vartheta_2), \quad \text{for all } (x, y) \in \mathcal{S}. \tag{27}$$

(c) There exist constants $y_1 > 0$ and Λ such that

$$\Lambda > \frac{K}{\vartheta_2} \frac{n\beta}{n-\alpha} C, \tag{28}$$

where α, β, ϑ_2, C are as in (b) above, and

$$H(x, y) \geq \beta \Lambda x^\alpha y^{-(1-\beta)}, \quad \text{for all } x > 0 \text{ and } y \geq y_1. \tag{29}$$

(d) Given any $y > 0$,

$$\int_0^x s^{-m-1} \left| H_y(s, y) \right| ds + \int_x^\infty s^{-n-1} \left| H_y(s, y) \right| ds < \infty.$$

□

Some of the conditions appearing in this assumption have a natural economical interpretation. Indeed, we can think of $H(x, y)\Delta y$ as the *additional* running payoff that we are faced with if we increase the project's capacity level from y to $y + \Delta y$, for small Δy, and the underlying state process X assumes the value x. In view of this observation, (23) reflects the idea that, given y, a small amount of extra capacity should be associated with increasing values of additional running payoff as the value of x, which, e.g., models the price of or the demand for the project's output commodity, is increasing. Similarly, (24) reflects the fact that, for a given value x of the underlying state process, the extra running payoff resulting from a small amount of additional capacity is decreasing as the level of the already installed capacity y increases. Also, (21) admits a similar, but simpler to express, economical interpretation.

The rest of the assumptions are of a technical nature. However, some of them cannot be significantly relaxed without losing the well-posedness of our control problem (see Lemma 3 below). Also, it is worth noting that part (c) of the assumption is rather weak because it only involves the tail of the function H as y tends to ∞; indeed, y_1 can be chosen arbitrarily large.

Example 1 A choice for the running payoff function h that has been widely considered in the economics literature is the so-called Cobb-Douglas production function given by

$$h(x, y) = x^{\alpha} y^{\beta}, \quad \text{for some constants } \alpha > 0 \text{ and } \beta \in \,]0, 1[. \tag{30}$$

It is straightforward to verify that this choice for the running payoff function h satisfies all of our assumptions if and only if the parameters α and β appearing in (30) satisfy the inequality (25). To this end, it suffices to take $\vartheta_2 = K$, $C = 1$, and any $\Lambda \in \left]\frac{n\beta}{n-\alpha}, 1\right]$. □

It is a straightforward exercise to show that the bounds in (26)–(27) imply the following estimates.

Lemma 2 *With reference to the notation in (12), let $R^{[h]}, R^{[H]} : \mathcal{S} \to \mathbb{R}$ be the functions defined by $R^{[h]}(x, y) = R(x; h(\cdot, y))$, $R^{[H]}(x, y) = R(x; H(\cdot, y))$, respectively. The bounds provided by (26) and (27) in Assumption 1 imply that there exists a constant $C_1 > 0$ such that*

$$-C_1(1 + y) \le R^{[h]}(x, y) \le C_1 \left(1 + y + x^{n-\vartheta_1} + x^{\alpha} y^{\beta}\right), \quad \text{for all } (x, y) \in \mathcal{S},$$
$$-C_1 \le R^{[H]}(x, y) \le C_1 \left(1 + x^{\alpha} y^{-(1-\beta)}\right), \quad \text{for all } (x, y) \in \mathcal{S}.$$

A bound such as the one in (26)–(27) is essential for the value function to be finite. Indeed, we can prove the following result.

Lemma 3 *Consider the control problem formulated in Section 2 that arises if the running payoff function h is the Cobb-Douglas production function defined in Example 1, and suppose that $\frac{\alpha}{1-\beta} > n > \alpha$ and $r > b$. Then, under any well-posed definition of the performance index $J_{x,y}$ that is consistent with (3), $v(x, y) = \infty$, for every initial condition $(x, y) \in \mathcal{S}$.*

Proof. Define $\lambda = \frac{n-\alpha}{\beta} > 0$ and note that the assumption that $\frac{\alpha}{1-\beta} > n$ implies that $\lambda < n$. Consider the capacity expansion strategy defined by

$$\tilde{Z}_t = \sum_{j=1}^{\infty} 2^{\lambda j} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}}, \quad \text{for } t \geq 0,\ j = 1, 2, \ldots, \tag{31}$$

where $\bar{X}_t = \sup_{s \leq t} X_s$, and note that the associated capacity level process satisfies

$$\begin{aligned} \tilde{Y}_t^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}} &= \left[y + 2^{\lambda j}\right]^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}} \\ &\geq \left[y + (\bar{X}_t + 1)^{\lambda}\right]^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}} \\ &\geq X_t^{n-\alpha} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}}. \end{aligned}$$

With reference to (20), it follows that

$$E\left[\int_0^{\infty} e^{-rt} X_t^{\alpha} \tilde{Y}_t^{\beta}\, dt \right] \geq E\left[\int_0^{\infty} e^{-rt} X_t^{n}\, dt\right] = \infty. \tag{32}$$

To proceed further, we define the sequence of stopping times

$$\begin{aligned} \tau_j &= \inf\left\{t \geq 0 :\ X_t \geq 2^j\right\}, \\ &= \inf\left\{t \geq 0 :\ \frac{b-\sigma^2}{\sqrt{2}|\sigma|} + \frac{\sigma}{|\sigma|} W_t \geq \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right)\right\}, \quad \text{for } j = 1, 2, \ldots. \end{aligned}$$

Since the process $\left(\frac{\sigma}{|\sigma|} W_t,\ t \geq 0\right)$ is a standard Brownian motion, we can use the result of Exercise 3.5.10 in Karatzas and Shreve [KS88] and the definition of $n > 0$ given by (17) to calculate

$$\begin{aligned} E[e^{-r\tau_j}] &= \exp\left(\frac{b-\sigma^2}{\sqrt{2}|\sigma|} \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right) - \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right) \sqrt{\frac{(b-\sigma^2)^2}{2\sigma^2} + 2}\right) \\ &= \left(\frac{x}{2^j}\right)^n. \end{aligned} \tag{33}$$

In view of this calculation, we can see that

$$
\begin{aligned}
E\Bigg[\int_0^\infty & e^{-rt}(K\Delta\tilde{Z}_t + c)\mathbf{1}_{\{\Delta\tilde{Z}_t>0\}}\Bigg] \\
&= K\left(2^\lambda - 1\right) + c + E\left[\sum_{j=1}^\infty e^{-r\tau_j}\left[K\left(2^{\lambda(j+1)} - 2^{\lambda j}\right) + c\right]\right] \\
&\le K2^\lambda + c + \sum_{j=1}^\infty \left[K\left(2^\lambda - 1\right)2^{\lambda j} + c\right] E\left[e^{-r\tau_j}\right] \\
&= K2^\lambda + c + K\left(2^\lambda - 1\right)x^n \sum_{j=1}^\infty \left(\frac{1}{2^{n-\lambda}}\right)^j + cx^n \sum_{j=1}^\infty \left(\frac{1}{2^n}\right)^j \\
&< \infty,
\end{aligned}
\tag{34}
$$

the last equality following thanks to (33) and the inequality being true because $n - \lambda > 0$. However, combining this result with (32), we can see that

$$
E\left[\int_0^\infty e^{-rt}h(X_t, \tilde{Y}_t)\,dt - \sum_{0\le t} e^{-rt}\left(K\Delta\tilde{Z}_t + c\right)\mathbf{1}_{\{\Delta\tilde{Z}_t>0\}}\right]
$$

is well-defined and infinite, so, $J_{x,y}(\tilde{Z}) = \infty$, and the proof is complete. □

We can now prove that our assumptions are sufficient for the optimisation problem considered to be well-posed and for its value function to be finite.

Lemma 4 *Suppose that the running payoff function h satisfies (26) in Assumption 1 and that (4) is true. Given any initial condition $(x, y) \in \mathcal{S}$, (6)–(9) provide a well-posed definition of the performance criterion $J_{x,y}$, and the following statements hold true:*

(a) *Given any capacity expansion strategy $Z \in \mathcal{A}$, $J_{x,y}(Z) \in \mathbb{R}$ if and only if*

$$
E\left[\int_0^\infty e^{-rt}Y_t\,dt + \sum_{0\le t} e^{-rt}(K\Delta Z_t + c)\mathbf{1}_{\{\Delta Z_t>0\}}\right] < \infty. \tag{35}
$$

(b) *Condition (35) implies*

$$
\liminf_{T\to\infty} e^{-rT}E\left[Y_{T+}\right] = 0. \tag{36}
$$

(c) $v(x, y) \in \mathbb{R}$.

Proof. Fix any initial condition $(x, y) \in \mathcal{S}$ and any strategy $Z \in \mathcal{A}$. Since Z is an increasing càglàd process with $Z_0 = 0$, we use the integration by parts formula and (1) to calculate

$$\begin{aligned} -K \sum_{0 \le t \le T} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} &= -K \int_{[0,T]} e^{-rt} \, dZ_t \\ &= -K \left[e^{-rT} Z_{T+} + r \int_0^T e^{-rt} Z_t \, dt \right] \\ &= -rK \int_0^T e^{-rt} Y_t \, dt - K e^{-rT} Y_{T+} + Ky. \end{aligned} \tag{37}$$

This inequality and (26) in Assumption 1, imply that the random variables U_T defined by (5) satisfy

$$\begin{aligned} U_T &\le Ky + \int_0^T e^{-rt} [h(X_t, Y_t) - rKY_t] \, dt \\ &\le Ky + C \int_0^T \left(1 + X_t^{n - \vartheta_1}\right) dt - \hat{V}_T, \end{aligned} \tag{38}$$

where

$$\hat{V}_T = \int_0^T e^{-rt} \left[r\vartheta_2 Y_t - C X_t^\alpha Y_t^\beta \right] dt, \quad \text{for } T \ge 0.$$

With reference to (20),

$$\begin{aligned} I_1(x) &:= E \left[C \int_0^\infty e^{-rt} \left(1 + X_t^{n - \vartheta_1}\right) dt \right] \\ &= \frac{C}{r} - \frac{C x^{n - \vartheta_1}}{\sigma^2 (n - \vartheta_1)^2 + (b - \sigma^2)(n - \vartheta_1) - r} \in \,]0, \infty[. \end{aligned} \tag{39}$$

Now, suppose that $Z \in \mathcal{A}$ is associated with

$$E \left[\int_0^\infty e^{-rt} Y_t \, dt \right] = \infty. \tag{40}$$

With regard to (25) in Assumption 1 and (20), we observe that

$$I_2(x) := E \left[\int_0^\infty e^{-rt} X_t^{\alpha/(1-\beta)} \, dt \right] < \infty. \tag{41}$$

Therefore, given any constant $\mu > 0$,

$$E \left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{ Y_t < \mu X_t^{\alpha/(1-\beta)} \right\}} \, dt \right] \le \mu I_2(x) < \infty. \tag{42}$$

It follows that (40) is true if and only if

$$E\left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \geq \mu X_t^{\alpha/(1-\beta)}\right\}}\, dt\right] = \infty. \tag{43}$$

Now, fix any $\mu > 0$ such that $r\vartheta_2 - C\mu^{-(1-\beta)} > 0$, where the constants $\vartheta_2, C > 0$ and $\beta \in]0,1[$ are as in Assumption 1, and note that

$$\begin{aligned} E\left[\hat{V}_T\right] \geq & - C\mu^\beta E\left[\int_0^T e^{-rt} X_t^{\alpha/(1-\beta)} \mathbf{1}_{\left\{Y_t < \mu X_t^{\alpha/(1-\beta)}\right\}}\, dt\right] \\ & + \left(r\vartheta_2 - C\mu^{-(1-\beta)}\right) E\left[\int_0^T e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \geq \mu X_t^{\alpha/(1-\beta)}\right\}}\, dt\right]. \end{aligned}$$

In view of (42)–(43) and the monotone convergence theorem, the right hand side of this inequality converges to ∞, which implies that $\lim_{T\to\infty} E[\hat{V}_T] = \infty$. However, this conclusion, (38) and (39) imply that there exists a process V such that (8) is satisfied and, therefore, $J_{x,y}(Z) = -\infty$.

To proceed further, let us assume that

$$E\left[\int_0^\infty e^{-rt} Y_t\, dt\right] < \infty, \tag{44}$$

which is necessary for condition (35) to be satisfied. Since Y is a finite variation process, its sample paths can have at most countable discontinuities. Using Fubini's theorem, we can see that this observation and (44) imply

$$\int_0^\infty e^{-rt} E\left[Y_{t+}\right] dt = E\left[\int_0^\infty e^{-rt} Y_{t+}\, dt\right] = E\left[\int_0^\infty e^{-rt} Y_t\, dt\right] < \infty,$$

which proves that (35) implies (36).

Now, using Hölder's inequality, we calculate

$$E\left[\int_0^\infty e^{-rt} X_t^\alpha Y_t^\beta\, dt\right] \leq I_2^{1-\beta}(x) \left(E\left[\int_0^\infty e^{-rt} Y_t\, dt\right]\right)^\beta < \infty, \tag{45}$$

where $I_2(x)$ is given by (41). This inequality, (39), (44) and the bounds in (26) in Assumption 1 imply

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} \left|h(X_t, Y_t)\right| dt\right] &\leq E\left[\int_0^\infty e^{-rt} \left[C\left(1 + X_t^{n-\vartheta_1} + X_t^\alpha Y_t^\beta\right) + r(K^+ - \vartheta_2) Y_t\right] dt\right] \\ &< \infty, \end{aligned}$$

which combined with the dominated convergence theorem, implies that

$$\lim_{T\to\infty} E\left[\int_0^T e^{-rt} h(X_t, Y_t)\, dt\right] = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t)\, dt\right] \in \mathbb{R}. \tag{46}$$

This observation gives rise to two possibilities. The first one is associated with the inequality

$$E\left[\sum_{0\le t} e^{-rt} \left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}\right] < \infty.$$

In this case, $\lim_{T\to\infty} U_T$ exists, P-a.s., and belongs to $L^1(\Omega, \mathcal{F}, P)$, so $J_{x,y}(\xi^+, \xi^-)$ is finite and is given by (7). The second possibility is associated with

$$\lim_{T\to\infty} E\left[\sum_{0\le t\le T} e^{-rt} \left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t>0\}}\right] \equiv E\left[\sum_{0\le t} e^{-rt} \left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t>0\}}\right] = \infty,$$

which, combined with (46), implies that $\lim_{T\to\infty} E[U_T] = -\infty$, so (8) is satisfied for $V = U$ and $J_{x,y}(Z) = -\infty$.

The analysis above establishes the well-posedness of the definition of $J_{x,y}$ given by (6)–(9) as well as parts (a) and (b) of the lemma. To prove part (c) of the lemma, we first note that the results presented in (11)–(13) and the bounds in Lemma 2 imply

$$R^{[h]}(x, y) = E\left[\int_0^\infty e^{-rt} h(X_t, y)\, dt\right] \in \mathbb{R}.$$

However, this shows that our performance criterion is finite for the strategy that involves no capacity changes at any time, which proves that $v(x, y) > -\infty$. To show that $v(x, y) < \infty$, consider any capacity expansion strategy $Z \in \mathcal{A}$ such that $J_{x,y}(Z) > -\infty$. With reference to (44) and (45),

$$\begin{aligned} E&\left[\int_0^\infty e^{-rt} \left[r\vartheta_2 Y_t - C X_t^\alpha Y_t^\beta\right] dt\right] \\ &\ge r\vartheta_2 E\left[\int_0^\infty e^{-rt} Y_t\, dt\right] - C I_2^{1-\beta}(x) \left(E\left[\int_0^\infty e^{-rt} Y_t\, dt\right]\right)^\beta \\ &\ge -\frac{(1-\beta) r\vartheta_2}{\beta} \left(\frac{\beta C}{r\vartheta_2}\right)^{1/(1-\beta)} I_2(x), \quad \text{for all } T > 0, \end{aligned} \tag{47}$$

the second inequality following because, given any constants $\kappa, \lambda > 0$ and $\beta \in \,]0, 1[$,

$$\kappa Q - \lambda Q^\beta \ge -\frac{(1-\beta)\kappa}{\beta} \left(\frac{\beta\lambda}{\kappa}\right)^{1/(1-\beta)}, \quad \text{for all } Q \ge 0,$$

in particular, for $Q = E\left[\int_0^\infty e^{-rt} Y_t\, dt\right]$. However, (38), (39) and (47) imply

$$J_{x,y}(\xi^+, \xi^-) \leq I_1(x) + K^+ y + \frac{(1-\beta) r \vartheta_2}{\beta} \left(\frac{\beta C}{r \vartheta_2}\right)^{1/(1-\beta)} I_2(x),$$

which proves that $v(x, y) < \infty$ because the right hand side of this inequality is finite and independent of Z. □

4 The solution to the control problem

We now construct an explicit solution to the control problem formulated in Section 2 by constructing an appropriate solution to the associated Hamilton-Jacobi-Bellman (HJB) equation. To this end, we expect that the value function can be identified with a solution to the HJB equation

$$\max\bigl\{\sigma^2 x^2 w_{xx}(x, y) + bx w_x(x, y) - rw(x, y) + h(x, y), \\ - w(x, y) - c + \sup_{z>0} \left[w(x, y+z) - Kz\right]\bigr\} = 0, \quad (x, y) \in \mathcal{S}. \tag{48}$$

To get a qualitative feeling about the origins of this equation, observe that, at time 0, the project's management has two options. The first one is to wait for a short time Δt and then continue optimally. With respect to Bellman's principle of optimality, this option, which is not necessarily optimal, is associated with the inequality

$$v(x, y) \geq E\left[\int_0^{\Delta t} e^{-rt} h(X_t, y)\, dt + e^{-r\Delta t} v(X_{\Delta t}, y)\right].$$

Applying Itô's formula to the second term in the expectation, and dividing by Δt before letting $\Delta t \downarrow 0$, we obtain

$$\sigma^2 x^2 v_{xx}(x, y) + bx v_x(x, y) - rv(x, y) + h(x, y) \leq 0. \tag{49}$$

The second option is to increase capacity by $\Delta Z_0 = z > 0$, and then continue optimally. Since such a capacity increase is not necessarily optimal, this action is associated with the inequality

$$v(x, y) \geq v(x, y+z) - Kz - c,$$

which implies

$$\sup_{z>0} \left[v(x, y+z) - Kz\right] - v(x, y) - c \leq 0, \tag{50}$$

because $z > 0$ was arbitrary. Since these two are the only options available, we expect that, given any initial condition $(x, y) \in \mathcal{S}$, one of them should be optimal, so that one of the inequalities (49)–(50) should hold with equality. However, this observation and (49)–(50) suggest that the value function v should identify with a solution w to the HJB equation (48). Now, it turns out that the value function v is not C^2, so we need to consider the following definition.

Definition 1 A function $w : \mathcal{S} \to \mathbb{R}$ is a *classical* solution of the HJB equation (48) if w is $C^{2,1}$,

$$\sigma^2 x^2 w_{xx}(x, y) + bxw_x(x, y) - rw(x, y) + h(x, y) \leq 0, \quad \text{Lebesgue-a.e., for all } y > 0,$$
$$\sup_{z>0} [v(x, y + z) - Kz] - v(x, y) - c \leq 0, \quad \text{for all } (x, y) \in \mathcal{S},$$

and there exists a set $\mathcal{I} \subset \mathcal{S}$ such that

$$\sigma^2 x^2 w_{xx}(x, y) + bxw_x(x, y) - rw(x, y) + h(x, y) = 0,$$

is satisfied in the interior of $\mathcal{I}^{\mathrm{c}}$, Lebesgue-a.e., for all $y > 0$, and

$$\sup_{z>0} [v(x, y + z) - Kz] - v(x, y) - c = 0, \quad \text{for all } (x, y) \in \mathcal{I}.$$

□

To proceed further, we conjecture that the optimal strategy is characterised by a point $x^{\mathrm{o}} > 0$ and two strictly increasing functions $G_0, G_1 : [x^{\mathrm{o}}, \infty[\to \mathbb{R}_+$, such that $G_0(x) < G_1(x)$, for all $x \geq x^{\mathrm{o}}$, and $G_0(x^{\mathrm{o}}) = 0$. The function G_0 separates the state space $\mathcal{S}$ into two regions, the wait region $\mathcal{W}$ and the investment region $\mathcal{I}$, while the function G_1 provides the capacity level that should be reached whenever it is optimal to increase the project's capacity (see Figure 1).

With regard to the heuristic arguments considered above, we therefore look for a solution to the HJB equation (48) that satisfies

$$\sigma^2 x^2 w_{xx}(x, y) + bxw_x(x, y) - rw(x, y) + h(x, y) = 0, \quad \text{for } (x, y) \in \mathcal{W}, \tag{51}$$

and

$$w(x, y) = w(x, G_1(x)) - K [G_1(x) - y] - c, \quad \text{for } (x, y) \in \mathcal{I} \tag{52}$$

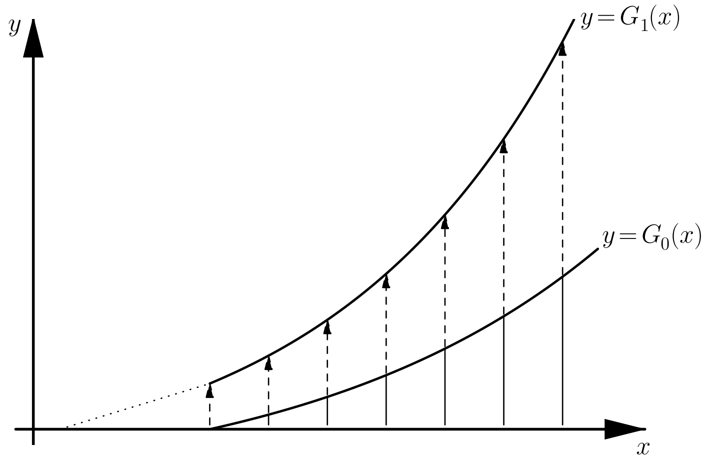

Figure 1: Illustration of a typical optimal capacity expansion strategy.

With regard to the discussion regarding the solvability of (14) in Section 3, every solution to equation (51) that remains bounded as $x \downarrow 0$ is given by

$$w(x, y) = A(y)x^n + R^{[h]}(x, y), \tag{53}$$

for some function A. Here,

$$R^{[h]}(x, y) = \frac{1}{\sigma^2(n-m)} \left[x^m \int_0^x s^{-m-1} h(s, y)\, ds + x^n \int_x^\infty s^{-n-1} h(s, y)\, ds \right]. \tag{54}$$

and the constants $m < 0 < n$ are given by (17). [1]

To determine $A(y)$, $G_0(x)$ and $G_1(x)$, we postulate that $w(x, \cdot)$ is C^1 at the free boundary point $G_0(x)$, which yields

$$\lim_{u \downarrow G_0(x)} w_y(x, u) \equiv A'(G_0(x))x^n + R_y(x, G_0(x)) = K \equiv \lim_{u \uparrow G_0(x)} w_y(x, u). \tag{55}$$

Also, in view of the inequality

$$w(x, G_0(x) + z) - w(x, G_0(x)) - Kz - c \le 0, \quad \text{for all } z > 0. \tag{56}$$

and the conjecture

$$w(x, G_1(x)) - w(x, G_0(x)) - K\left[G_1(x) - G_0(x)\right] = c, \tag{57}$$

we can see that the function

$$z \mapsto w(x, G_0(x) + z) - w(x, G_0(x)) - Kz - c$$

has a local maximum at $z^* = G_1(x) - G_0(x)$, which is associated with the equation

$$w_y(x, G_1(x)) \equiv A'(G_1(x))x^n + R_y^{[h]}(x, G_1(x)) = K. \tag{58}$$

Now, given any $y \geq 0$, (55) is equivalent to

$$A'(y)[G_0^{-1}(y)]^n + R_y^{[h]}(G_0^{-1}(y), y) = K, \tag{59}$$

while (58) is equivalent to

$$A'(y)[G_1^{-1}(y)]^n + R_y^{[h]}(G_1^{-1}(y), y) = K. \tag{60}$$

Eliminating $A'(y)$ from these two equations, and using the equality $\sigma^2 mn = -r$ as well as the definition of $R^{[h]}$, which implies that $R_y^{[h]} = R^{[H]}$, we can see that the points $G_0^{-1}(y)$ and $G_1^{-1}(y)$ should satisfy

$$F(G_1^{-1}(y), y, G_0^{-1}(y)) = 0, \tag{61}$$

where

$$F(x, y, z) = z^{-n} R^{[H(\cdot)-rK]}(z, y) - x^{-n} R^{[H(\cdot)-rK]}(x, y), \tag{62}$$

and

$$R^{[H(\cdot)-rK]}(x, y) = \frac{1}{\sigma^2(n-m)} \biggl[x^m \int_0^x s^{-m-1}[H(s, y) - rK]\, ds + x^n \int_x^\infty s^{-n-1}[H(s, y) - rK]\, ds \biggr]. \tag{63}$$

To proceed further, let us assume that G_0 and G_1 are C^1. In this case, we can differentiate (57) with respect to x, and use (55) and (58) to obtain

$$w_x(x, G_1(x)) = w_x(x, G_0(x)), \tag{64}$$

which in view of (53), implies

$$[A(G_1(x)) - A(G_0(x))]\, x^n = -\frac{x}{n} \left[R_x^{[h]}(x, G_1(x)) - R_x^{[h]}(x, G_0(x)) \right]. \tag{65}$$

However, combining this with (57) and (53), we can see that $G_0(x)$ and $G_1(x)$ should satisfy

$$-\frac{x}{n} \left[R_x^{[h]}(x, G_1(x)) - R_x^{[h]}(x, G_0(x)) \right] + \left[R^{[h]}(x, G_1(x)) - R^{[h]}(x, G_0(x)) \right] - K[G_1(x) - G_0(x)] - c = 0, \tag{66}$$

which, in view of the definition (54) of R and the equality $\sigma^2 mn = -r$, is equivalent to

$$\Phi(x, G_0(x), G_1(x)) = 0, \tag{67}$$

where

$$\Phi(x, y, p) = \int_y^p x^m \int_0^x s^{-m-1} \left[H(s,u) - rK\right] ds\, du + \frac{rc}{m}. \tag{68}$$

To summarise the heuristic discussion above, suppose that we can find a point $x^\circ > 0$ and two strictly increasing functions $G_0, G_1 : [x^\circ, \infty[\to \mathbb{R}_+$ satisfying (61) and (67). Since F and Φ are C^1 in each of their arguments, both of G_0 and G_1 are C^1. With regard to (59) or (60), if we choose

$$\begin{aligned} A(y) = \frac{1}{\sigma^2(n-m)} \Bigg[& \int_y^\infty \left(G_1^{[-1]}(u)\right)^{m-n} \int_0^{G_1^{[-1]}(u)} s^{-m-1}[H(s,u) - rK]\, ds\, du \\ & + \int_y^\infty \int_{G_1^{[-1]}(u)}^\infty s^{-n-1}[H(s,u) - rK]\, ds\, du \Bigg], \end{aligned} \tag{69}$$

then, assuming the integrals are well-defined and finite, (55) and (58) are true, which, along with the C^1 continuity of G_0 and G_1, imply that (65) is also satisfied. Moreover, (67) implies that (66) is true, which, combined with (65), implies that (57) is satisfied as well. In light of these observations, we can see that constructing a solution w to the HJB equation (48) amounts to finding functions G_0 and G_1 satisfying (61) and (67).

The next result is concerned with this construction and the associated solution to the HJB equation (48).

Lemma 5 *Suppose that Assumption 1 holds. The system of equations (61) and (67) define a point $x^\circ > 0$ and two C^1, increasing functions $G_0, G_1 : [x^\circ, \infty[\to \mathbb{R}_+$ such that $G_0(x) < G_1(x)$, for all $x \geq x^\circ$, $G_0(x^\circ) = 0$, and*

$$G_1(x) \leq C_2 x^{\frac{\alpha}{1-\beta}}, \quad \textit{for all } x \geq 0, \tag{70}$$

for some constant $C_2 > 0$. The function w defined by

$$w(x,y) = \begin{cases} A(y)x^n + R(x,y), & \textit{for } (x,y) \textit{ such that } y > G_0(x), \\ w(x, G_1(x)) - K\left[G_1(x) - y\right] - c, & \textit{for } (x,y) \textit{ such that } 0 \leq y \leq G_0(x), \end{cases} \tag{71}$$

where A is given by (69), is C^1, and, given any $y \geq 0$, $w(\cdot, y)$ is C^2 outside the graph of G_0. Also, w is a classical solution to the HJB equation (48), in the sense of Definition 1, and there exist constants $C_3 > 0$ and $\varepsilon_3 \in]0, n[$ such that

$$-C_3(1 + y + x^{\frac{\alpha}{1-\beta}}) \leq w(x, y) \leq C_3 \left(1 + y + [G_1^{[-1]}(y)]^{n-\varepsilon_3} + [G_1^{[-1]}(y)]^{\alpha} y^{\beta} + x^{n-\varepsilon_3} + x^{\frac{\alpha}{1-\beta}}\right), \tag{72}$$

for all $(x, y) \in \mathcal{S}$.

We can now prove the main result of the paper.

Theorem 6 *Consider the capacity control problem formulated in Section 2, and suppose that Assumption 1 holds. The value function v identifies with the solution to the HJB equation (48) constructed in Lemma 5. Apart from an initial jump of size $(G_1(x) - y)^+$ at time 0, the optimal capacity level process Y° has jumps of size provided by the function $G_1 - G_0$ that occur at the $(\mathcal{F}_t)$-stopping times when the process (X, Y°) hits the graph of G_0, and is constructed rigorously in the proof below.*

Proof. Fix any initial condition (x, y) and any admissible strategy $Z \in \mathcal{A}$ such that $J_{x,y}(Z) > -\infty$. Since Y is piecewise constant and $w(\cdot, y)$ is C^2 outside the graph of G_0, for all $y \geq 0$, we can use Itô's formula and the fact that X has continuous sample paths to obtain

$$\begin{aligned} e^{-rT} w(X_T, Y_{T+}) = w(x, y) &+ \int_0^T e^{-rt} \left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t)\right] dt \\ &+ M_T + \sum_{0 \leq t \leq T} e^{-rt} \left[w(X_t, Y_{t+}) - w(X_t, Y_t)\right], \end{aligned}$$

where

$$M_T = \sqrt{2}\sigma \int_0^T e^{-rt} X_t w_x(X_t, Y_t)\, dW_t, \quad T \geq 0. \tag{73}$$

Recalling the definition of U in (5), this implies

$$\begin{aligned} &U_T + e^{-rT} w(X_T, Y_{T+}) \\ &= w(x, y) + \int_0^T e^{-rt} \left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t) + h(X_t, Y_t)\right] dt \\ &\quad + M_T + \sum_{0 \leq t \leq T} e^{-rt} \left[w(X_t, Y_t + \Delta Z_t) - w(X_t, Y_t) - K\Delta Z_t - c\right] \mathbf{1}_{\{\Delta Z_t > 0\}}. \end{aligned} \tag{74}$$

Since w satisfies the HJB equation (48), it follows that

$$U_T + e^{-rT} w(X_T, Y_{T+}) \leq w(x, y) + M_T. \tag{75}$$

Now, in view of (37) and the assumption $K > 0$,

$$-e^{-rT} Y_{T+} \geq - \sum_{0 \leq t \leq T} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} - y,$$

which, combined with the lower bound in (72), yields

$$e^{-rT} w(X_T, Y_{T+}) \geq -C_3 \left(e^{-rT} + \sum_{0 \leq t \leq T} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} + e^{-rT} X_T^{\frac{\alpha}{1-\beta}} \right). \tag{76}$$

Furthermore, using (26) in Assumption 1, we obtain

$$\int_0^T e^{-rt} h(X_t, Y_t)\, dt \geq -C \int_0^T e^{-rt} Y_t\, dt - \frac{C}{r} \left(1 - e^{-rT}\right). \tag{77}$$

Combining (76) and (77) with (75), we can see that

$$\inf_{T \geq 0} M_T \geq -C_4 \left(1 + \int_0^\infty e^{-rt} Y_t\, dt + \sum_{[0,\infty[} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} + \sup_{T \geq 0} e^{-rT} \bar{X}_T^{\frac{\alpha}{1-\beta}} \right), \tag{78}$$

where $C_4 > 0$ is a constant and $\bar{X}_t = \sup_{s \leq t} X_s$. Recalling the assumption that $\frac{\alpha}{1-\beta} \in]0, n[$, we can see that the second bound in Lemma 1 and (35) in Lemma 4 imply that the random variable on the right hand side of this inequality has finite expectation. It follows that the stochastic integral M defined by (73) is a supermartingale, and therefore, $E[M_T] \leq 0$, for all $T > 0$. Taking expectations in (75), we therefore obtain

$$E[U_T] \leq w(x, y) + \liminf_{T \to \infty} e^{-rT} E[-w(X_T, Y_{T+})]. \tag{79}$$

Furthermore, since

$$U_T \geq -C_4 \left(1 + \int_0^\infty e^{-rt} Y_t\, dt + \sum_{[0,\infty[} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} \right),$$

and the random variable on the right hand side of this inequality has finite expectation, Fatou's lemma implies

$$J_{x,y}(\xi^+, \xi^-) \leq \liminf_{T \to \infty} E[U_T], \tag{80}$$

while (72) implies

$$\begin{aligned}\liminf_{T\to\infty} e^{-rT} E\left[-w(X_T, Y_{T+})\right] \leq & \lim_{T\to\infty} e^{-rT} C_3 + C_3 \liminf_{T\to\infty} e^{-rT} E\left[Y_{T+}\right] \\ & + C_3 \lim_{T\to\infty} e^{-rT} E\left[\bar{X}_T^{\alpha/(1-\beta)}\right] \\ = & \; 0,\end{aligned} \tag{81}$$

the equality being true thanks to the first bound in Lemma 1 and (35). However, (79)–(81) imply that $J_{x,y}(Z) \leq w(x,y)$, which proves that $v(x,y) \leq w(x,y)$.

Now, let us set

$$\tau_0 = 0 \quad \text{and} \quad Z_t^{(0)} = \left[G_1(x) - y\right] \mathbf{1}_{\{y < G_0(x)\}}, \tag{82}$$

and define iteratively the $(\mathcal{F}_t)$-stopping times τ_n and the processes $Z^{(n)}$ by

$$\begin{aligned}\tau_{k+1} &= \inf\left\{t \geq \tau_k : \; X_t \geq G_0^{[-1]}\left(y + Z_t^{(n)}\right)\right\}, \quad \text{for } k = 0, 1, \ldots, \\ Z_t^{(k+1)} &= Z_t^{(k)} + \left[G_1(X_{\tau_{k+1}}) - G_0(X_{\tau_{k+1}})\right] \mathbf{1}_{\{t > \tau_{k+1}\}}, \quad \text{for } k = 0, 1, \ldots.\end{aligned} \tag{83}$$

Observing that $\lim_{k\to\infty} \tau_k = \infty$, P-a.s., and that $Z_t^{(k)} = Z_t^{(k+1)}$, for all $t \in [0, \tau_{k+1}]$, and $k \geq 0$, we define the capacity expansion process Z^{o} by $Z_t^{\text{o}} = Z_t^{(k)}$ for $t < \tau_k$, and we note that the associated capacity process Y^{o} satisfies

$$Y^{\text{o}} \leq y \mathbf{1}_{\left\{\bar{X}_t \leq G_0^{[-1]}(y)\right\}} + G_1(\bar{X}_t) \mathbf{1}_{\left\{\bar{X}_t > G_0^{[-1]}(y)\right\}}. \tag{84}$$

This inequality and the upper estimate of w in (72) in Lemma 5 imply

$$e^{-rT} w(X_T, Y_T^{\text{o}}) \leq C_{53} \, e^{-rT} \left(1 + \bar{X}_T^{\frac{\alpha}{1-\beta}} + \bar{X}_T^{n-\varepsilon}\right). \tag{85}$$

Also, this inequality and the upper bound on h in Assumption 1.(26), imply

$$\begin{aligned}\int_0^T e^{-rt} h(X_t, Y_t^{\text{o}}) \, dt &\leq C_{54} \left(1 + \int_0^\infty e^{-rt} \left[X_t^{n-\vartheta_1} + X_t^{\alpha} (Y_t^{\text{o}})^{\beta} + Y_t^{\text{o}}\right] dt\right) \\ &\leq C_{55} \left(1 + \int_0^\infty e^{-rt} \left[\bar{X}_t^{n-\vartheta_1} + \bar{X}_t^{\alpha/(1-\beta)}\right] dt\right).\end{aligned} \tag{86}$$

However, these inequalities and the estimates in Lemma 1 imply

$$E\left[\sup_{T>0} \left(\int_0^T e^{-rt} h(X_t, Y_t^{\text{o}}) \, dt + e^{-rT} w(X_T, Y_T^{\text{o}})\right)\right] < \infty. \tag{87}$$

Now, with regard to the construction of Y°, we can see that (74) implies

$$\int_0^T e^{-rt} h(X_t, Y_t^{\circ})\, dt - \sum_{0\le t\le T} e^{-rt}(K\Delta Z_t + c)\mathbf{1}_{\{\Delta Z_t>0\}} \\ + e^{-rT} w(X_T, Y_{T+}^{\circ}) = w(x,y) + M_T^{\circ}, \tag{88}$$

where M° is defined as in (73) with $Y_t = Y_t^{\circ}$. This identity and (87) imply that $E\left[\sup_{T>0} M_T^{\circ}\right] < \infty$, so the stochastic integral M is a submartingale. In view of this observation, we can take expectations in (88) and pass to the limit to obtain

$$\begin{aligned} J_{x,y}(Z^{\circ}) &\ge w(x,y) + \liminf_{T\to\infty} e^{-rT} E[-w(X_T, Y_T^{\circ})] \\ &\ge w(x,y) + C_{56} \liminf_{T\to\infty} e^{-rT} E\left[1 + \bar{X}_t^{n-\vartheta_1} + \bar{X}_t^{\alpha/(1-\beta)}\right] \\ &= w(x,y). \end{aligned} \tag{89}$$

Here, the second inequality follows from the upper bound of w in (72), (84) and (70). However, combining with the inequality $v(x,y) \le w(x,y)$ that we proved above, we deduce that $v(x,y) = w(x,y)$ and that Z° is optimal, and the proof is complete. □

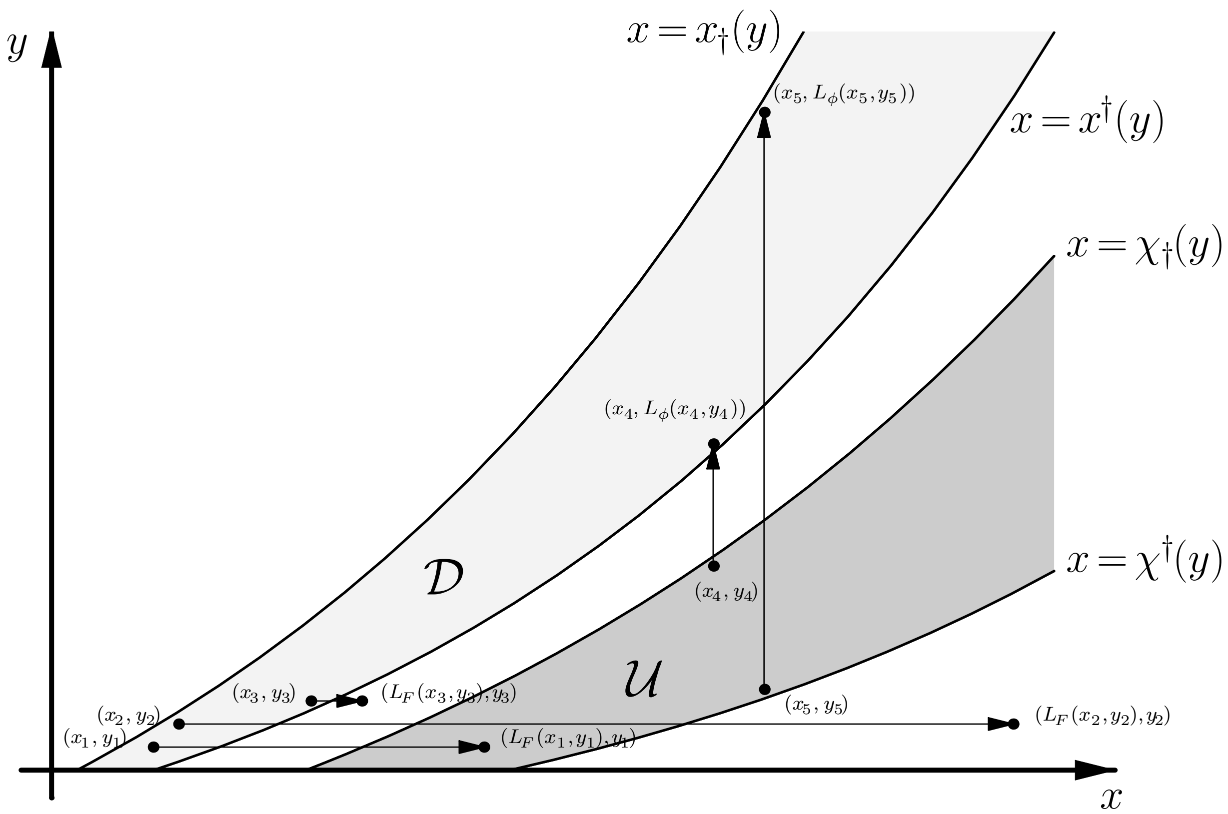

Figure 2: Illustration of the functions L_F and L_Φ constructed in Lemmas 7 and 8.

5 Appendix: Proof of Lemma 5

To establish Lemma 5, we first need to prove a number of preliminary results. The next one is concerned with a study of the function F defined by (61).

Lemma 7 *The equations*

$$q_\dagger(x,y) := -x^{m-n}\int_0^x s^{-m-1}[H(s,y)-rK]\,ds - \int_x^\infty s^{-n-1}[H(s,y)-rK]\,ds = 0, \tag{90}$$

$$q^\dagger(x,y) := \int_0^x s^{-m-1}[H(s,y)-rK]\,ds = 0, \tag{91}$$

define uniquely two strictly increasing, C^1 functions $x_\dagger, x^\dagger : [0,\infty[\to [0,\infty[$, respectively, such that

$$\left[\frac{\sigma^2\vartheta_2}{\beta C}(\alpha-m)(n-\alpha)\right]^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}} \le x_\dagger(y) < x^\dagger(y), \quad \textit{for all } y \ge 0. \tag{92}$$

Also, if we define

$$\mathcal{D} = \left\{(x,y) \in \mathcal{S} : \; x \in [x_\dagger(y), x^\dagger(y)]\right\}, \tag{93}$$

then the following statements are true:

(*a*) *For* $(x,y) \in \mathcal{S}\backslash\mathcal{D}$, *the equation* $F(x,y,z) = 0$ *has no solution* $z > x$.

(*b*) *There exists a unique mapping* $L_F : \mathcal{D} \to [0,\infty[$ *such that*

$$x < L_F(x,y) \quad \text{and} \quad F(x,y,L_F(x,y)) = 0. \tag{94}$$

Moreover,

$$\frac{\partial}{\partial x} L_F(x,y) < 0 \quad \text{and} \quad \frac{\partial}{\partial y} L_F(x,y) > 0, \quad \text{for all } (x,y) \in \operatorname{int}\mathcal{D}, \tag{95}$$

$$\lim_{x \downarrow x_\dagger(y)} L_F(x,y) = \infty \quad \text{and} \quad \lim_{x \uparrow x^\dagger(y)} L_F(x,y) = x^\dagger(y). \tag{96}$$

Proof. Consider (91), fix any $y > 0$, and observe that the upper bound in (27) in Assumption 1 implies that

$$\inf_{x>0} H(x,y) - rK = -r\vartheta_2 < 0. \tag{97}$$

Combining this inequality with (23) in Assumption 1, we can see that there exists a unique point $x_* = x_*(y) > 0$ such that

$$\frac{\partial}{\partial x} q^\dagger(x,y) = x^{-m-1}\left[H(x,y) - rK\right] \begin{cases} < 0, & \text{for all } x \in]0, x_*[, \\ > 0, & \text{for all } x > x_*. \end{cases} \tag{98}$$

In view of this calculation, we combine the fact that $q^\dagger(\cdot,y)$ is strictly decreasing in $]0,x_*[$ and strictly increasing in $]x_*,\infty[$, with $q^\dagger(0,y) = 0$, to see that $q^\dagger(x,y) < 0$, for all $x \le x_*$, in particular, $q^\dagger(x_*,y) < 0$. Therefore, if $q^\dagger(x,y) = 0$ has a solution $x > 0$, then this must satisfy $x > x_*$. Also, given that it exists, this solution is unique because $q^\dagger(\cdot,y)$ is strictly increasing in $]x_*,\infty[$. To prove that the required solution indeed exists, it suffices to show that $\lim_{x\to\infty} q^\dagger(x,y) = \infty$. The assumption that $\lim_{x\to\infty} H(x,y) = \infty$ implies that, given any constant $M > 0$, there exists $\beta > x_*$ such that $H(x,y) - rK \ge M$, for all $x \ge \beta$. However, given any such choice of these constants, we calculate

$$\begin{aligned} \lim_{x\to\infty} q^\dagger(x,y) &= \lim_{x\to\infty}\left[q^\dagger(\beta,y) + \int_\beta^x s^{-m-1}\left[H(s,y) - rK\right] ds\right] \\ &\ge \lim_{x\to\infty}\left[q^\dagger(\beta,y) + \frac{M}{m}\beta^{-m} - \frac{M}{m}x^{-m}\right] \\ &= \infty. \end{aligned}$$

It follows that equation (91) defines uniquely a continuous function $x^\dagger :]0,\infty[\, \to]0,\infty[$ such that

$$\text{given any } y > 0, \quad q^\dagger(x,y) \begin{cases} < 0, & \text{for } x < x^\dagger(y), \\ = 0, & \text{for } x = x^\dagger(y), \\ > 0, & \text{for } x > x^\dagger(y). \end{cases} \tag{99}$$

Moreover, the arguments above imply that

$$H(x^\dagger(y), y) - rK > 0, \quad \text{for all } y > 0. \tag{100}$$

To see that $x^\dagger$ is C^1 and strictly increasing, we differentiate $q^\dagger(x^\dagger(y), y) = 0$ with respect to y to obtain

$$\begin{aligned} \frac{\partial}{\partial y} x^\dagger(y) &= -\frac{1}{(x^\dagger(y))^{-m-1}[H(x^\dagger(y), y) - rK]} \int_0^{x^\dagger(y)} s^{-m-1} H_y(s,y)\, ds \\ &> 0, \end{aligned} \tag{101}$$

the inequality following from (100) and (24) in Assumption 1. Since $x^\dagger :]0,\infty[\to]0,\infty[$ is increasing, we can extend its domain by defining $x^\dagger(0) = \lim_{y\downarrow 0} x^\dagger(y)$.

Now, fix any $y > 0$, consider (90) and observe that (91) and (99) imply

$$\begin{aligned} q_\dagger(x^\dagger(y), y) &= -(x^\dagger(y))^{m-n} q^\dagger(x^\dagger(y), y) - \int_{x^\dagger(y)}^\infty s^{-n-1}[H(s,y) - rK]\, ds \\ &= -\int_{x^\dagger(y)}^\infty s^{-n-1}[H(s,y) - rK]\, ds \\ &< 0, \end{aligned} \tag{102}$$

the inequality following thanks to (100) and the assumption that $H(\cdot, y)$ is strictly increasing. Also, note that

$$\begin{aligned} \frac{\partial}{\partial x} q_\dagger(x,y) &= (n-m) x^{m-n-1}\, q^\dagger(x,y) \\ &< 0, \quad \text{for all } x \in \,]0, x^\dagger(y)[. \end{aligned} \tag{103}$$

To proceed further, we fix any ϵ, $x_\epsilon > 0$ such that $H(s,y) - rK < -\epsilon$, for all $s \le x_\epsilon$. For such a choice of parameters, we obtain

$$\begin{aligned} \lim_{x\downarrow 0} q_\dagger(x,y) &\ge \lim_{x\downarrow 0} \epsilon \left[x^{m-n} \int_0^x s^{-m-1}\, ds + \int_x^{x_\epsilon} s^{-n-1}\, ds \right] - \int_{x_\epsilon}^\infty s^{-n-1}[H(s,y) - rK]\, ds \\ &= \infty. \end{aligned} \tag{104}$$

However, combining this calculation with (102) and (103), we can see that, there exists a unique $x_\dagger(y) \in]0, x^\dagger(y)[$ such that $q_\dagger(x_\dagger(y), y) = 0$, for all $y > 0$. Moreover,

$$q_\dagger(x,y) \begin{cases} > 0, & \text{for } x \in]0, x_\dagger(y)[, \\ < 0, & \text{for } x \in]x_\dagger(y), x^\dagger(y)[. \end{cases} \tag{105}$$

To see that $x_\dagger$ is C^1 and strictly increasing, we differentiate $q_\dagger(x_\dagger(y), y) = 0$ with respect to y to obtain

$$\begin{aligned} \frac{\partial}{\partial y} x_\dagger(y) &= (n-m)^{-1}\, x_\dagger(y)^{n-m+1}\, \frac{1}{q^\dagger(x_\dagger(y), y)} \\ &\quad \times \left[x_\dagger(y)^{m-n} \int_0^{x_\dagger(y)} s^{-m-1} H_y(s,y)\, ds + \int_{x_\dagger(y)}^\infty s^{-n-1} H_y(s,y)\, ds \right] \\ &> 0, \end{aligned} \tag{106}$$

the inequality following from (99) and (24) in Assumption 1. Furthermore, the conclusion that $x_\dagger :]0, \infty[\to]0, x^\dagger(y)[$ is increasing implies that the definition $x_\dagger(0) = \lim_{y \downarrow 0} x_\dagger(y)$ is well-posed.

To see (92), we use the upper bound in (27) in Assumption 1 to obtain

$$q_\dagger(x,y) \geq x^{-n} \left[\frac{-(n-m)}{(\alpha - m)(n-\alpha)} \beta C y^{-(1-\beta)} x^\alpha - r\vartheta_2 \frac{n-m}{nm} \right]. \tag{107}$$

This inequality implies that $x_\dagger(y)$ is greater than the unique point at which the right hand side vanishes, namely,

$$x_\dagger(y) \geq \left[\frac{\sigma^2 \vartheta_2}{\beta C} (\alpha - m)(n-\alpha) \right]^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}}, \tag{108}$$

which establishes (92).

Now, fix any $(x,y) \in \mathcal{S}$ with $y > 0$, and consider the equation $F(x,y,z) = 0$ for $z > x$. Plainly,

$$F(x,y,x) = 0, \tag{109}$$

while a straightforward calculation involving the definition (62) of F and the definition of $R^{[H(\cdot) - rK]}(x,y)$ as in (63) yields

$$\frac{\partial}{\partial z} F(x,y,z) = -\frac{1}{\sigma^2} z^{m-n-1} q^\dagger(z,y). \tag{110}$$

However, these observations and (99) show that the equation $F(x, y, z) = 0$ has no solution $z > x$ if $x > x^\dagger(y)$. In view of this observation, (109), (110) and (99), we will prove part (a) of the lemma and the existence of a unique mapping $L_F : \mathcal{D} \to [0, \infty[$ such that $x < L_F(x, y)$ and $F(x, y, L_F(x, y)) = 0$ if we show that

$$\lim_{z\to\infty} F(x,y,z) \begin{cases} > 0, & \text{for } x \in \,]0, x_\dagger(y)[, \\ < 0, & \text{for } x \in \,]x_\dagger(y), x^\dagger(y)[. \end{cases} \tag{111}$$

To this end, we use the upper bound in (27) in Assumption 1 and the fact that $q^\dagger(x^\dagger(y), y) = 0$ to calculate

$$\begin{aligned} &\lim_{z\to\infty} z^{m-n} \int_0^z s^{-m-1}[H(s,y) - rK]\, ds \\ &\quad = \lim_{z\to\infty} z^{m-n} \int_{x^\dagger(y)}^z s^{-m-1}[H(s,y) - rK]\, ds \\ &\quad \le \lim_{z\to\infty} z^{m-n} \int_{x^\dagger(y)}^z s^{-m-1}[\beta C s^\alpha y^{-(1-\beta)} - r\vartheta_2]\, ds \\ &\quad = \lim_{z\to\infty} \left[\frac{\beta C y^{-(1-\beta)}}{\alpha - m} \left(z^{\alpha-n} - (x^\dagger(y))^{\alpha-m} z^{m-n}\right) + \frac{r\vartheta_2}{m}\left(z^{-n} - (x^\dagger(y))^{-m} z^{m-n}\right) \right] \\ &\quad = 0, \end{aligned} \tag{112}$$

the last equality following because $m < 0 < \alpha < n$. Similarly, we use the lower bound in (27) in Assumption 1 to obtain

$$\begin{aligned} &\lim_{z\to\infty} z^{m-n} \int_0^z s^{-m-1}[H(s,y) - rK]\, ds \\ &\quad \ge \lim_{z\to\infty} -[C + rK] z^{m-n} \int_0^z s^{-m-1}\, ds \\ &\quad = \lim_{z\to\infty} \frac{C + rK}{m} z^{-n} \\ &\quad = 0. \end{aligned} \tag{113}$$

These calculations and the definition of $R^{[H(\cdot)-rK]}$ in (63) imply that $\lim_{z\to\infty} z^{-n} R^{[H(\cdot)-rK]}(z, y) = 0$, which, combined with the definition of F and $q_\dagger$, implies

$$\lim_{z\to\infty} F(x,y,z) = \frac{1}{\sigma^2(n-m)} q_\dagger(x,y). \tag{114}$$

However, this limit and (105) imply (111). Furthermore, a careful inspection of these arguments reveals that (96) is true.

Finally, to show (95), we first differentiate $F(x, y, L_F(x, y)) = 0$ with respect to x to obtain

$$\frac{\partial}{\partial x} L_F(x, y) = \frac{x^{m-n-1} q^\dagger(x, y)}{L_F^{m-n-1}(x, y) q^\dagger(L_F(x, y), y)} < 0, \quad \text{for all } (x, y) \in \operatorname{int} \mathcal{D} \tag{115}$$

the inequality following because $x < x^\dagger(y) < L_F(x, y)$ and $q^\dagger(x^\dagger(y), y) = 0$. Next, we observe that the definition of F in (62) implies

$$F_y(x, y, z) = z^{-n} R^{[H_y]}(z, y) - x^{-n} R^{[H_y]}(x, y), \tag{116}$$

while the definition of $R^{[H_y]}$, which can be easily deduced from (63), implies

$$\begin{aligned} \frac{d}{dx} x^{-n} R^{[H_y]}(x, y) &= -\frac{1}{\sigma^2} x^{m-n-1} \int_0^x s^{-m-1} H_y(s, y)\, ds \\ &> 0, \end{aligned} \tag{117}$$

the inequality following thanks to (24) in Assumption 1. However, these calculations and the fact that $x < x^\dagger(y) < L_F(x, y)$ show that

$$F_y(x, y, L_F(x, y)) > 0, \quad \text{for all } (x, y) \in \operatorname{int} \mathcal{D}, \tag{118}$$

while (110) and (99) imply

$$\begin{aligned} F_z(x, y, L_F(x, y)) &= -\frac{1}{\sigma^2} L_F^{m-n-1}(x, y) q^\dagger(L_F(x, y), y) \\ &< 0, \quad \text{for all } (x, y) \in \operatorname{int} \mathcal{D}. \end{aligned} \tag{119}$$

In view of these inequalities, we can differentiate $F(x, y, L_F(x, y)) = 0$ with respect to y to derive the second inequality in (95), and the proof is complete. □

The next result is concerned with a similar study of the function Φ defined by (68).

Lemma 8 *There exists a strictly increasing, C^1 function $\chi_\dagger : [0, \infty[\to \mathbb{R}_+$ such that given any $(x, y) \in \mathcal{S}$, the equation $\Phi(x, y, p) = 0$ has a solution $p > y$ if and only if $x > \chi_\dagger(y)$. In particular, there exists a unique mapping*

$$L_\Phi : \{(x, y) \in \mathcal{S} : \ x > \chi_\dagger(y)\} \to \mathcal{S} \tag{120}$$

such that

$$(x^\dagger)^{[-1]}(x) \le L_\Phi(x, y) \quad \text{and} \quad \Phi(x, y, L_\Phi(x, y)) = 0, \tag{121}$$

where the function $x^\dagger$ is as in Lemma (7). This mapping satisfies

$$\frac{\partial}{\partial x} L_\Phi(x,y) > 0 \quad \textit{and} \quad \frac{\partial}{\partial y} L_\Phi(x,y) < 0, \quad \textit{for all } y \geq 0 \textit{ and } x > \chi_\dagger(y), \tag{122}$$

and there exists a function $\chi^\dagger : [0,\infty[\to \mathbb{R}_+$ such that $\chi_\dagger(y) < \chi^\dagger(y)$, for all $y \geq 0$, and, if we define

$$\mathcal{U} = \left\{ (x,y) \in \mathcal{S} : \; x \in [\chi_\dagger(y), \chi^\dagger(y)] \right\},$$

then $L_\Phi(x,y) \in \mathcal{D}$, where $\mathcal{D}$ is defined by (93), if and only if $(x,y) \in \mathcal{U}$.

Proof. Fix any $(x,y) \in \mathcal{S}$, and define

$$\mathcal{D}_l = \{(x,y) \in \mathcal{S} : \; x \leq x^\dagger(y)\}, \tag{123}$$

where $x^\dagger$ is as in Lemma 7. With regard to this definition, the fact that $m < 0$, (91) and (99), we calculate

$$\Phi(x,y,y) = \frac{rc}{m} < 0, \tag{124}$$

$$\frac{\partial}{\partial p} \Phi(x,y,p) = x^m q^\dagger(x,p) \begin{cases} < 0, & \text{for } x < x^\dagger(p), \\ = 0, & \text{for } x = x^\dagger(p), \\ > 0, & \text{for } x > x^\dagger(p), \end{cases} \tag{125}$$

and

$$\begin{aligned} \frac{\partial}{\partial x} \Phi(x,y,p) &= \int_y^p \left\{ m x^{m-1} \int_0^x s^{-m-1} [H(s,u) - rK] \, ds + x^{-1} [H(x,u) - rK] \right\} du \\ &= \int_y^p x^{m-1} \int_0^x s^{-m} H_x(s,u) \, ds \, du \\ &> 0, \end{aligned} \tag{126}$$

the inequality following thanks to (23) in Assumption 1. Now, (125) implies that the function $\Phi(x,y,\cdot)$ has a global maximum at y if $(x,y) \in \mathcal{D}_l$ and at $x^\dagger(p)$ if $(x,y) \in \mathcal{S} \backslash \mathcal{D}_l$. Combining this observation with (124), we can see that the equation $\Phi(x,y,p) = 0$ has a solution $p > y$ if and only if $(x,y) \in \mathcal{S} \backslash \mathcal{D}_l$ and $\Phi(x,y,(x^\dagger)^{[-1]}(x)) > 0$. However, this conclusion and (125) imply that there exists a strictly increasing function $\chi_\dagger : [0,\infty[\to \mathbb{R}$ and a mapping L_Φ satisfying (120)–(121) if and only if

$$\frac{\partial}{\partial x}\Phi(x, y, (x^\dagger)^{[-1]}(x)) > 0, \tag{127}$$

and

$$\lim_{x\to\infty} \Phi(x, y, (x^\dagger)^{[-1]}(x)) > 0. \tag{128}$$

Inequality (127) follows immediately once we observe that (125) and (126) imply

$$\frac{\partial}{\partial x}\Phi(x, y, (x^\dagger)^{[-1]}(x)) = \int_y^{(x^\dagger)^{[-1]}(x)} x^{m-1} \int_0^x s^{-m} H_x(s, u)\, ds\, du > 0. \tag{129}$$

To see (128), we note that (23) in Assumption 1 implies that, given any constant $N > 0$, there exists $x_1 > 0$ such that $H(x, y) - rK \geq N$, for all $x \geq x_1$. Given any such choice of constants,

$$\begin{aligned}
&\lim_{x\to\infty} x^m \int_0^x s^{-m-1}[H(s, y) - rK]\, ds \\
&\quad \geq \lim_{x\to\infty} x^m \left[\int_0^{x_1} s^{-m-1}[H(s, y) - rK]\, ds - \frac{N}{m}x^{-m} + \frac{N}{m}x_1^{-m} \right] \\
&\quad = -\frac{N}{m},
\end{aligned} \tag{130}$$

the last equality following because $m < 0$. Since $N > 0$ is arbitrary, this calculation implies

$$\lim_{x\to\infty} x^m \int_0^x s^{-m-1}[H(s, y) - rK]\, ds = \infty. \tag{131}$$

However, in view of this limit and the fact that $\lim_{x\to\infty}(x^\dagger)^{[-1]}(x) = \infty$ (recall that the domain of $x^\dagger$ is the whole of $\mathbb{R}_+$), we can deduce that

$$\begin{aligned}
\lim_{x\to\infty} \Phi(x, y, (x^\dagger)^{[-1]}(x)) &= \lim_{x\to\infty} \left[\int_y^{(x^\dagger)^{[-1]}(x)} x^m \int_0^x s^{-m-1}[H(s, u) - rK]\, ds\, du + \frac{rc}{m} \right] \\
&= \infty,
\end{aligned} \tag{132}$$

which establishes (128).

Now, to establish (122), we first differentiate $\Phi(x, y, L_\Phi(x, y)) = 0$ with respect to x to obtain, for all $y \geq 0$ and $x > \chi_\dagger(y)$,

$$\frac{\partial}{\partial x}L_\Phi(x, y) = -\frac{\Phi_x(x, y, L_\Phi(x, y))}{\Phi_p(x, y, L_\Phi(x, y))} > 0, \tag{133}$$

the inequality following thanks to (125) and (126) and the fact that $L_\Phi(x,y) \in \operatorname{int} \mathcal{D}_l$, which implies that $x < x^\dagger(L_\Phi(x,y))$. Also, differentiating $\Phi(x, y, L_\Phi(x,y)) = 0$ with respect to y yields

$$\frac{\partial}{\partial y} L_\Phi(x,y) = \frac{q^\dagger(x,y)}{q^\dagger(x, L_\Phi(x,y))} < 0, \tag{134}$$

the inequality following thanks to (99) and the inequalities $y < x_\dagger^{[-1]}(x) < L_\Phi(x,y)$.

With regard to the structure of the function Φ that we have studied above, the existence of the function $\chi^\dagger$ will follow if we prove that, given any $y \geq 0$,

$$\lim_{x \to \infty} \Phi(x, y, x_\dagger^{[-1]}(x)) > 0. \tag{135}$$

To this end, we consider any $x > 0$ and $p > y > y_1$, where y_1 is as in (29) in Assumption 1, and we calculate

$$\begin{aligned} \Phi(x,y,p) &\geq \int_y^p x^m \int_0^x s^{-m-1} [\beta \Lambda s^\alpha u^{-(1-\beta)} - rK] ds\, du + \frac{rc}{m} \\ &= \left[\frac{\Lambda\zeta}{\alpha - m} x^\alpha p^{-(1-\beta)} + \frac{rK}{m} \right] p + \left[\frac{\Lambda(1-\zeta)}{\alpha - m} p^\beta - \frac{\Lambda}{\alpha - m} y^\beta \right] x^\alpha - \frac{rK}{m} y + \frac{rc}{m}, \end{aligned} \tag{136}$$

where ζ is a constant. Now, (92) and the identity $\sigma^2 mn = -r$ imply

$$\frac{\Lambda\zeta}{\alpha - m} x^\alpha [x_\dagger^{[-1]}(x)]^{-(1-\beta)} + \frac{rK}{m} \geq \left[\Lambda\zeta - \frac{K}{\vartheta_2} \frac{n\beta}{n - \alpha} C \right] \frac{\sigma^2 \vartheta_2 (n - \alpha)}{\beta C}, \tag{137}$$

while (28) in Assumption 1 implies that there exists $\zeta \in]0,1[$ such that the right hand side of this inequality is strictly positive. However, for such a choice of ζ, (136) and the fact that $\lim_{x\to\infty} x_\dagger^{[-1]}(x) = \infty$ imply (135), and the proof is complete. □

Proof of Lemma 5. With regard to the definitions and the properties of the sets $\mathcal{D}$, $\mathcal{U}$ and the mappings L_F, L_Φ in Lemmas 7 and 8, we define

$$\begin{gathered} \mathcal{L}_F A = \{(L_F(x,y), y) : \ (x,y) \in A\}, \quad \text{for } A \subseteq \mathcal{D}, \\ \mathcal{L}_\Phi B = \{(L_\Phi(x,y), y) : \ (x,y) \in B\}, \quad \text{for } B \subseteq \mathcal{U}, \\ x_1^{(0)} = x_\dagger, \quad x_2^{(0)} = x^\dagger, \quad x_3^{(0)} = \chi_\dagger, \quad x_4^{(0)} = \chi^\dagger, \\ \mathcal{D}^{(0)} = \mathcal{D} \equiv \left\{ (x,y) \in \mathcal{S} : \ x \in [x_1^{(0)}(y), x_2^{(0)}(y)] \right\} \end{gathered}$$

and

$$\mathcal{U}^{(0)} = \mathcal{U} \equiv \left\{ (x,y) \in \mathcal{S} : \; x \in [x_3^{(0)}(y), x_4^{(0)}(y)] \right\},$$

and we observe that

$$x_1^{(0)}(y) < x_2^{(0)}(y) < x_3^{(0)}(y) < x_4^{(0)}(y), \quad \text{for all } y \geq 0, \tag{138}$$

$$\lim_{y\to\infty} x_i^{(0)}(y) = \infty, \quad \text{for } i = 1,2,3,4, \tag{139}$$

$$\mathcal{L}_F \mathcal{D}^{(0)} \supset \mathcal{U}^{(0)} \text{ and } \mathcal{L}_\Phi \mathcal{U}^{(0)} = \left\{ (x,y) \in \mathcal{D}^{(0)} : \; x \geq x_3^{(0)}(0) \right\}. \tag{140}$$

To proceed further, we appeal to an inductive argument, and we assume that we have found strictly increasing functions $x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$x_1^{(k)}(y) < x_2^{(k)}(y) < x_3^{(k)}(y) < x_4^{(k)}(y), \quad \text{for all } y \geq 0, \tag{141}$$

$$\lim_{y\to\infty} x_i^{(k)}(y) = \infty, \quad \text{for } i = 1,2,3,4, \tag{142}$$

and, if $\mathcal{D}^{(k)}$, $\mathcal{U}^{(k)}$ are the sets defined by

$$\mathcal{D}^{(k)} = \left\{ (x,y) \in \mathcal{S} : \; x \in [x_1^{(k)}(y), x_2^{(k)}(y)] \right\},$$

$$\mathcal{U}^{(k)} = \left\{ (x,y) \in \mathcal{S} : \; x \in [x_3^{(k)}(y), x_4^{(k)}(y)] \right\},$$

then

$$\mathcal{L}_F \mathcal{D}^{(k)} \supset \mathcal{U}^{(k)} \quad \text{and} \quad \mathcal{L}_\Phi \mathcal{U}^{(k)} = \left\{ (x,y) \in \mathcal{D}^{(k)} : \; x \geq x_3^{(k)}(0) \right\}. \tag{143}$$

With regard to the properties of the function $L_F(\cdot, y)$ established in Lemma 7, there exist functions $x_1^{(k+1)}, x_2^{(k+1)} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$x_1^{(k)}(y) < x_1^{(k+1)}(y) < x_2^{(k+1)}(y) < x_2^{(k)}(y), \quad \text{for all } y \geq 0, \tag{144}$$

$$\lim_{y\to\infty} x_1^{(k+1)}(y) = \lim_{y\to\infty} x_2^{(k+1)}(y) = \infty, \tag{145}$$

and, if we define

$$\mathcal{D}^{(k+1)} = \left\{ (x,y) \in \mathcal{S} : \; x \in [x_1^{(k+1)}(y), x_2^{(k+1)}(y)] \right\}, \tag{146}$$

then

$$\mathcal{L}_F \mathcal{D}^{(k+1)} = \mathcal{U}^{(k)}. \tag{147}$$

Similarly, the properties of the function L_Φ in Lemma 8 imply that there exist functions $x_2^{(k+1)}, x_3^{(k+1)} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$x_3^{(k)}(y) < x_3^{(k+1)}(y) < x_4^{(k+1)}(y) < x_4^{(k)}(y), \quad \text{for all } y \geq 0, \tag{148}$$

$$\lim_{y\to\infty} x_3^{(k+1)}(y) = \lim_{y\to\infty} x_4^{(k+1)}(y) = \infty, \tag{149}$$

$$\mathcal{L}_\Phi \mathcal{U}^{(k+1)} = \left\{(x,y) \in \mathcal{D}^{(k+1)} :\ x \geq x_3^{(k+1)}(0)\right\}, \tag{150}$$

where $\mathcal{U}^{(k+1)}$ is given by

$$\mathcal{U}^{(k+1)} = \left\{(x,y) \in \mathcal{S} :\ x \in [x_3^{(k+1)}(y), x_4^{(k+1)}(y)]\right\}. \tag{151}$$

By construction, the functions $x_i^{(k+1)}$, $i = 1,2,3,4$, and the sets $\mathcal{D}^{(k+1)}$, $\mathcal{U}^{(k+1)}$ have all of the properties assumed for the functions $x_i^{(k)}$, $i = 1,2,3,4$, and the domains $\mathcal{D}^{(k)}$, $\mathcal{U}^{(k)}$ (see (141)–(143)), and which are shared by the corresponding entities when $k = 0$ (see (138–(140)). By induction, it follows that there exist sequences of functions $(x_1^{(k)})$, $(x_2^{(k)})$, $(x_3^{(k)})$, $(x_4^{(k)})$ and subsets $(\mathcal{D}^{(k)})$, $(\mathcal{U}^{(k)})$ of $\mathcal{S}$ satisfying (141)–(143). Since $(x_1^{(k)})$, $(x_3^{(k)})$ (resp., $(x_2^{(k)})$, $(x_4^{(k)})$) are strictly increasing (resp., decreasing) sequences of functions,

$$\hat{x}_i(y) = \lim_{k\to\infty} x_i^{(k)}(y), \quad \text{for } y \geq 0 \text{ and } i = 1,2,3,4, \tag{152}$$

define increasing functions $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\hat{x}_1$, $\hat{x}_3$ are lower semi-continuous, $\hat{x}_2$, $\hat{x}_4$ are upper semi-continuous,

$$\begin{gathered} \hat{x}_1(y) \leq \hat{x}_2(y) < \hat{x}_3(y) \leq \hat{x}_4(y), \quad \text{for all } y \geq 0, \\ \lim_{y\to\infty} \hat{x}_i(y) = \infty, \quad \text{for } i = 1,2,3,4, \end{gathered} \tag{153}$$

and the sets

$$\begin{aligned} \mathbb{G}_1 &= \bigcap_{k=0}^{\infty} \mathcal{D}^{(k)} \equiv \{(x,y) \in \mathcal{S} :\ \hat{x}_1(y) \leq x \leq \hat{x}_2(y)\}, \\ \mathbb{G}_0 &= \bigcap_{k=0}^{\infty} \mathcal{U}^{(k)} \equiv \{(x,y) \in \mathcal{S} :\ \hat{x}_3(y) \leq x \leq \hat{x}_4(y)\} \end{aligned} \tag{154}$$

are non-empty and closed. Moreover, (147) and (150) imply

$$\mathcal{L}_F \mathbb{G}_1 = \mathbb{G}_0 \quad \text{and} \quad \mathcal{L}_\Phi \mathbb{G}_0 = \left\{(x,y) \in \mathcal{S} :\ x \geq \lim_{k\to\infty} x_3^{(k)}(0)\right\},$$

respectively, while the fact that $L_F(\cdot, y)$ and $L_\Phi(x, \cdot)$ are both decreasing implies

$$\mathcal{L}_F \operatorname{graph} \hat{x}_1 = \operatorname{graph} \hat{x}_4, \qquad \mathcal{L}_F \operatorname{graph} \hat{x}_2 = \operatorname{graph} \hat{x}_3,$$
$$\mathcal{L}_\Phi \operatorname{graph} \hat{x}_3 \subset \operatorname{graph} \hat{x}_2 \quad \text{and} \quad \mathcal{L}_\Phi \operatorname{graph} \hat{x}_4 \subset \operatorname{graph} \hat{x}_1.$$

These inclusions imply that, if we define

$$x^{\mathrm{o}} = \hat{x}_3(0), \quad G_1(x) = \hat{x}_2^{[-1]}(x) \quad \text{and} \quad G_0(x) = \hat{x}_3^{[-1]}(x), \quad \text{for } x \geq x^{\mathrm{o}},$$

then G_1 and G_0 satisfy (61), for all $y \geq 0$, and (67), for all $x \geq x^{\mathrm{o}}$. Moreover, since the functions F and Φ are C^1, the functions G_1, G_0 are C^1. At this point, we should note that the choice of G_1, G_0 made above plainly appears to be non-unique, which is due to the fact that we have not managed to prove that the sets $\mathbb{G}_0$ and $\mathbb{G}_1$ have empty interior.

Now, consider (69), and note that the upper bound in (27) in Assumption 1 and the inequalities $\alpha < \frac{\alpha}{1-\beta} < n$ imply

$$0 < A(y) \leq \frac{\beta C}{\sigma^2(\alpha - m)(n - \alpha)} \int_y^\infty u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du \tag{155}$$

Now, fix any $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < n - \frac{\alpha}{1-\beta} < n - \alpha.$$

Using the fact that $G_1^{[-1]}$ is increasing and the estimate provided by (70), we calculate

$$\begin{aligned}\int_y^\infty u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du &\leq \left(G_1^{[-1]}(y)\right)^{-\varepsilon_0} \int_y^\infty u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha-\varepsilon_0)} du \\ &\leq \frac{\alpha C_{71}}{(1-\beta)(n-\varepsilon_0) - \alpha} \left(G_1^{[-1]}(y)\right)^{-\varepsilon_0} y^{-[(1-\beta)(n-\varepsilon_0)-\alpha]/\alpha},\end{aligned}$$

where $C_{71} > 0$ is a constant, which implies

$$\int_y^\infty u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du \leq \frac{\alpha C_{71}}{(1-\beta)(n-\varepsilon_0) - \alpha} \left(G_1^{[-1]}(y)\right)^{-\varepsilon_0}, \quad \text{for all } y \geq 1. \tag{156}$$

Also, the fact that $G_1^{[-1]}$ is increasing implies that, given any $y < 1$,

$$\begin{aligned}\left(G_1^{[-1]}(y)\right)^n \int_y^1 u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du &\leq \left(G_1^{[-1]}(y)\right)^\alpha \int_y^1 u^{-(1-\beta)} du \\ &\leq \frac{1}{\beta} \left(G_1^{[-1]}(1)\right)^\alpha.\end{aligned} \tag{157}$$

However, (155)–(157) imply

$$\begin{aligned}A(y)\left(G_1^{[-1]}(y)\right)^n \leq & \frac{\beta C}{\sigma^2(n-m)(n-\alpha)}\left[\frac{\alpha C_{71}}{(1-\beta)(n-\varepsilon_0)-\alpha}\left(\left(G_1^{[-1]}(y)\right)^{n-\varepsilon_0}\mathbf{1}_{\{y\geq 1\}}\right)\right.\\ & \left.+\left(G_1^{[-1]}(1)\right)^{n-\varepsilon_0}\mathbf{1}_{\{y<1\}}+\frac{1}{\beta}\left(G_1^{[-1]}(1)\right)^{\alpha}\mathbf{1}_{\{y<1\}}\right]\\ = & C_{72}\left(1+\left(G_1^{[-1]}(y)\right)^{n-\varepsilon_0}\right), \quad \text{for all } y\geq 0,\end{aligned} \tag{158}$$

where $C_{72} > 0$ is a constant.

To proceed further, fix any $(x, y) \in \mathcal{W}$ such that $x \leq G_1^{[-1]}(0)$ or $x > G_1^{[-1]}(0)$ and $y > G_1(x)$. For such a point,

$$\begin{aligned}w(x,y) &\leq w(G_1^{[-1]}(y),y)\\ &= A(y)\left[G_1^{[-1]}(y)\right]^n + R^{[h]}(G_1^{[-1]}(y),y)\\ &\leq C_{73}\left(1+y+\left[G_1^{[-1]}(y)\right]^{n-\varepsilon_0\wedge\vartheta_1}+\left[G_1^{[-1]}(y)\right]^{\alpha}y^{\beta}\right),\end{aligned} \tag{159}$$

the first inequality following because $w(\cdot, y)$ is increasing, and the second one following thanks to (158) and the upper bound in Lemma 2.

For $(x, y) \in \mathcal{W}$ such that $y < G_1(x)$, the fact that w satisfies the HJB equation (48) implies

$$w(x, G_0(x)) \geq w(x,y) - K[y - G_0(x)] - c, \tag{160}$$

which, combined with the identity

$$w(x, G_0(x)) = w(x, G_1(x)) - K[G_1(x) - G_0(x)] - c \tag{161}$$

that is true by construction, implies

$$\begin{aligned}w(x,y) &\leq w(x,G_1(x)) + Ky\\ &= A(G_1(x))x^n + R(x,G_1(x)) + Ky\\ &\leq C_{74}\left(1+x^{n-\varepsilon_0\wedge\vartheta_1}+G_1(x)+x^{\alpha}[G_1(x)]^{\beta}\right)\\ &\leq C_{74}\left(1+x^{n-\varepsilon_0\wedge\vartheta_1}+x^{\frac{\alpha}{1-\beta}}\right),\end{aligned} \tag{162}$$

where $C_{74} > 0$ is a constant. The second inequality here follows thanks to (158) and Lemma 2, while the third one is true because of the estimate for G_1 provided by (70).

Also, if $(x, y) \in \mathcal{I}$, then the expression for w given by (71) implies that $w(x, y)$ satisfies (162) as well. However, (159) and (162) establish the upper estimate in (72).

To show that w satisfies the lower bound in (72), we first observe that the positivity of A and the lower bound in Lemma 2 imply that

$$w(x, y) \geq -C_1(1+y), \quad \text{for all } (x, y) \in \overline{\mathcal{W}}. \tag{163}$$

This estimate and the definition of w in $\mathcal{I}$, provided by (71), imply

$$\begin{aligned} w(x, y) &\geq -(C_1 + K)\, G_1(x) - C_1 \\ &\geq -C_{75}\,(1 + x^{\alpha/(1-\beta)}), \quad \text{for all } (x, y) \in \mathcal{I}, \end{aligned} \tag{164}$$

the second inequality following thanks to (70). However, (163)–(164) establish the lower bound in (72).

To see that w is $C^{1,1}$ along the boundary G_1, we use the second identity in (71) along with (58) to calculate

$$\begin{aligned} w_x(x, y) &= w_x(x, G_1(x)) + [w_y(x, G_1(x)) - K] \frac{dG_1(x)}{dx} \\ &= w_x(x, G_1(x)). \end{aligned} \tag{165}$$

By construction, we will prove that w satisfies the HJB equation (48) if we show that

$$\sigma^2 x^2 w_{xx}(x, y) + bxw_x(x, y) - rw(x, y) + h(x, y) \leq 0, \quad \text{for all } (x, y) \in \mathcal{I}, \tag{166}$$

$$-w(x, y) - c + \sup_{z>0}[w(x, y+z) - Kz] \leq 0, \quad \text{for all } (x, y) \in \mathcal{W} \tag{167}$$

To this end, we fix any $(x, y) \in \mathcal{W}$ and we observe that (59) and the definition of F in (62) imply

$$\begin{aligned} w_y(x, y) - K &= A'(y)x^n + R^{[H(\cdot)-rK]}(x, y) \\ &= \left[-(G_0^{[-1]}(y))^{-n} R^{[H(\cdot)-rK]}(G_0^{[-1]}(y), y) + x^{-n} R^{[H(\cdot)-rK]}(x, y)\right] x^n \\ &= -F(x, y, G_0^{[-1]}(y))x^n \\ &\begin{cases} < 0, & \text{for } x < G_1^{[-1]}(y), \\ > 0, & \text{for } x \in]G_1^{[-1]}(y), G_0^{[-1]}(y)[. \end{cases} \end{aligned} \tag{168}$$

To see how the inequalities follow, we note that

$$\frac{\partial F(x, y, z)}{\partial x} = \frac{1}{\sigma^2} x^{m-n-1} q^{\dagger}(x, y), \tag{169}$$

and that $q(x^\dagger(y), y) = 0$. Then, since $G_1^{[-1]}(y) < x^\dagger(y) < G_0^{[-1]}(y)$, we must have

$$\frac{\partial}{\partial x} F(x, y, G_0^{[-1]}(y)) \begin{cases} < 0, & \text{for } x < G_1^{[-1]}(y) \text{ and } x \in \left]G_1^{[-1]}(y), x^\dagger(y)\right[, \\ > 0, & \text{for } x \in \left]x^\dagger(y), G_0^{[-1]}(y)\right[. \end{cases} \tag{170}$$

Combining this with $F(G_0^{[-1]}(y), y, G_0^{[-1]}(y)) = F(G_1^{[-1]}(y), y, G_0^{[-1]}(y)) = 0$, we deduce that

$$F(x, y, G_0^{[-1]}(y)) \begin{cases} > 0, & \text{for } x < G_1^{[-1]}(y), \\ < 0, & \text{for } x \in \left]G_1^{[-1]}(y), G_0^{[-1]}(y)\right[. \end{cases} \tag{171}$$

Using (168), it is a tedious but straightforward exercise to show that (167) is satisfied.

To establish (166), and in view of the $C^{2,2}$ continuity of w in the interior of $\mathcal{W}$, we can differentiate $w_y(x, G_1(x)) = K$ with respect to x to obtain

$$w_{xy}(x, G_1(x)) = -w_{yy}(x, G_1(x))\, G_1'(x) \leq 0, \tag{172}$$

the inequality following because $w_{yy}(x, G_1(x)) \geq 0$, which is true thanks to (168), and the fact that G_1 is strictly increasing. Now, with regard to (165), we can see that

$$\begin{aligned} w_{xx}(x, y) &= w_{xx}(x, G_1(x)) + w_{xy}(x, G_1(x))\, G_1'(x) \\ &\leq w_{xx}(x, G_1(x)), \quad \text{for all } (x, y) \in \mathcal{I}. \end{aligned} \tag{173}$$

combining this inequality with (165) and (71), we can see that (166) is implied by

$$\begin{aligned} \sigma^2 x^2 w_{xx}(x, G_1(x)) + bxw_x(x, G_1(x)) - rw(x, G_1(x)) \\ + rK[G_1(x) - y] + h(x, y) + rc \leq 0, \quad \text{for } (x, y) \in \mathcal{I}, \end{aligned} \tag{174}$$

which is equivalent to

$$-\int_y^{G_1(x)} [H(x, u) - rK]\, du + rc \leq 0. \tag{175}$$

However, this is true because, by (67), we have

$$\begin{aligned} rc &= -m \int_{G_0(x)}^{G_1(x)} x^m \int_0^x s^{-m-1} [H(s, u) - rK]\, ds\, du \\ &\leq -m \int_{G_0(x)}^{G_1(x)} x^m \int_0^x s^{-m-1} [H(x, u) - rK]\, ds\, du \\ &= \int_{G_0(x)}^{G_1(x)} [H(x, u) - rK]\, du, \end{aligned} \tag{176}$$

the inequality here following from (23). Substituting this into (175) yields

$$-\int_y^{G_0(x)} [H(x,u) - rK]\, du \le -\int_y^{G_0(x)} [H(x, G_0(x)) - rK]\, du$$
$$\le 0, \tag{177}$$

the first inequality due to (24) and the second to the fact that $H(x, G_0(x)) - rK \geq H(x, (x^\dagger)^{[-1]}) - rK = 0$, and the proof is complete. □

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