

Statistical Physics Approach to Models of Risk

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with

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To access published results go to

<http://www.mth.kcl.ac.uk/~kuehn/riskmodeling.html>

Fundamental Problem of Risk Analysis

- Estimation of risk
 - **Market:** potential negative fluctuation of portfolio-value (stock-prices, exchange rates, interest rates, economic indices)
 - **Credit:** potential change of credit quality, including default (asset values of firms, ratings, stock-prices)
 - **Operational:** potential losses incurred by process failures (human errors, hardware/software-failures, lack of communication, fraud, external catastrophes)
 - **Liquidity:** potential losses incurred by rare fluctuations in cash-flows, requiring short term acquisition of funds to maintain liquidity

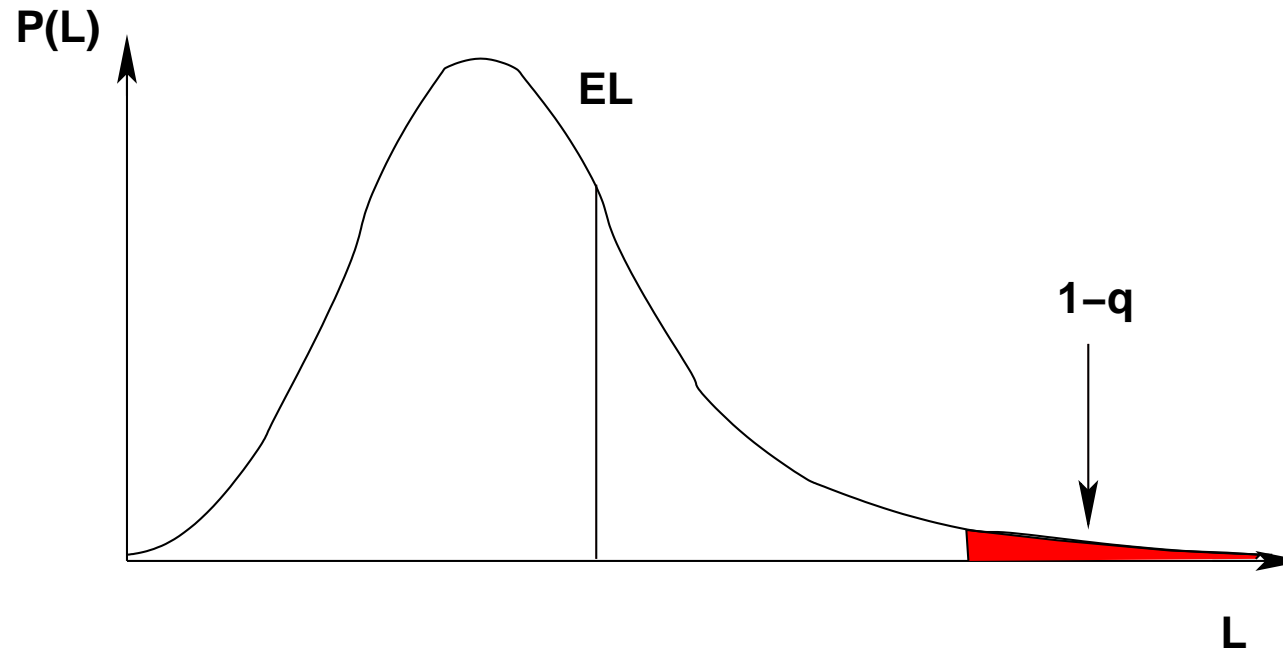
Task

- Estimate PDF of
 - Portfolio value at time T , $PV(T)$, given $PV(0)$
 - Market value of assets of obligors at time T , $A_i(T)$, given $A_i(0)$
 - Losses $L(T)$ due to process failures incurred during risk horizon T
 - Losses $L(T)$ incurred during risk horizon T by need to maintain liquidity in situations of stress

- Quantity of interest: **Value at Risk**

$$\text{VaR}_{q,T} = (Q_q[L(T)] - EL) e^{-rT} \quad \text{Prob}(L(T) \leq Q_q[L(T)]) = q$$

- Quantity of interest: Value at Risk



$$\text{VaR}_{q,T} = (Q_q[L(T)] - EL) e^{-rT} \quad \text{Prob}(L(T) \leq Q_q[L(T)]) = q$$

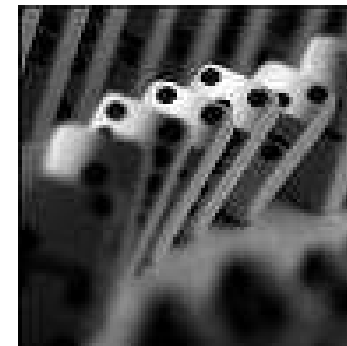
- Requires good knowledge of tails of loss distributions.

Assessment of PDFs (tails, VaR)

- adjustment of business model, (re)design of processes
 - charged fees, interest rates, rating of clients
 - activities on derivative markets (hedging)
 - insurance policies
- Proper risk control and management is
 - *demanded* by international banking supervision (BASEL)
 - *recognised* by rating agencies, analysts

Our Main Interest and Concern: Interactions

- Traditional approaches treat risk elements as **independent** or *at best statistically correlated*
- Misses **functional & dynamic nature** of relations: terminal–mainframe/input errors–results/manufacturer–supplier relations . . .
- Effect of interactions between risk elements
 - Possibility of avalanches of risk events (process failures, defaults)
 - Fat tails in loss distributions
 - Volatility clustering in markets (intermittency)



The Case of Operational Risks — Interacting Processes

- Conceptualise organisation as a network of processes
- Two state model: processes either up and running ($n_i = 0$) or down ($n_i = 1$)
- Reliability of processes and degree of functional interdependence heterogeneous across the set of processes (quenched disorder); connectivity functionally defined
 - ⇒ (lattice gas) model defined on random graph
- losses determined (randomly) each time a process goes down (annealed disorder)

Dynamics

- processes need support to keep running (energy, human resources, material, information, input from other processes, etc.)
- h_{it} total support received by process i at time t

$$h_{it} = \vartheta_i - \sum_j J_{ij} n_{jt} - \eta_{it}$$

with η_{it} random (e.g. Gaussian white noise).

- process i will fail, if the total support for it falls below a critical threshold

$$n_{it+\Delta t} = \Theta \left(\sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it} \right)$$

Meaning of Parameters

- Probability of failure in given 'situation'

$$\text{Prob}(n_{it+\Delta t} = 1 | \mathbf{n}(t)) = \Phi \left(\sum_j J_{ij} n_{jt} - \vartheta_i \right)$$

with

$$\Phi(x) = \frac{1}{2} [1 + \text{erf}(x/\sqrt{2})]$$

- **unconditional** and **conditional** probability of failure

$$p_i = \Phi(-\vartheta_i)$$

$$p_{i|j} = \Phi(J_{ij} - \vartheta_i)$$

$$\Rightarrow \vartheta_i = -\Phi^{-1}(p_i) , \quad J_{ij} = \Phi^{-1}(p_{i|j}) - \Phi^{-1}(p_i)$$

Key Features

- For sufficiently strong cooperative interactions, get first order phase transition: coexistence of 'functioning state and state of 'catastrophic breakdown (dominoes)
- Critical point at sufficiently high p_i .
- Of special relevance: resilience to (external) stress can be tested.

The Case of Credit Risks — Interacting Companies

- Risk arising from the possibility of obligors going bankrupt or from changes in 'credit quality' (\Rightarrow credit trading)
- Here only influence of defaults
- Two state model: company up and running ($n_i = 0$) or down ($n_i = 1$)
- Probabilities of default and mutual impacts of defaults **heterogeneous** across the set of companies (quenched disorder); connectivity **functionally** defined
 - \implies (lattice gas) model defined on random graph
- Losses determined (randomly: recovery process) when a company defaults (annealed disorder)

Dynamics

- Companies need “orders” (support, cash inflow) to maintain wealth and avoid default

- h_{it} total **wealth** of company i at time t ,

$$h_{it} = \vartheta_i - \sum_j J_{ij} n_{jt} - \eta_{it}$$

with η_{it} random (e.g. Gaussian white noise).

- company i defaults, if the total wealth falls below zero

$$n_{it+1} = n_{it} + (1 - n_{it}) \Theta \left(\sum_j J_{ij} n_{jt} - \vartheta_i + \eta_{it} \right)$$

- **No recovery** within ‘risk horizon’ T : $n_i = 1$ is absorbing state.
Time unit: 1 month; $T = 12 \Leftrightarrow 1$ year.

Key Features

- Interpretation of parameters ϑ_i and J_{ij} in terms of unconditional and conditional default probabilities
- For sufficiently strong interactions — in particular cooperative interactions — get possibility of collective acceleration of default rates in times of economic stress
- Typical behaviour less sensitive to interactions than rare events
⇒ fat tails in loss distribution; important for risk analysis where rare event asymptotics is relevant.
- Due to initial conditions $n_{i0} \equiv 0$ and absorbing state:
⇒ no equilibrium dynamics

Analysis for a Stochastic Setting

- Interactions on a **random graph**

$$J_{ij} = c_{ij} \left(\frac{J_0}{c} + \frac{J}{\sqrt{c}} x_{ij} \right)$$

with

$$P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N} \right) \delta_{c_{ij},0} ,$$

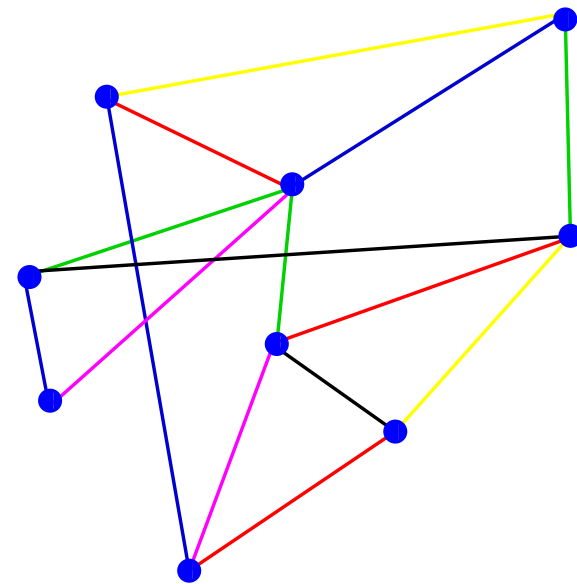
symmetric ($c_{ij} = c_{ji}$), and

$$\overline{x_{ij}} = 0 , \quad \overline{x_{ij}^2} = 1 , \quad \overline{x_{ij}x_{ji}} = \alpha .$$

Study the limit

$$c \gg 1 , \quad N \gg 1 .$$

Processes/organizations: small J !



Dynamical Mean-field Analysis — Generating Functions

- Generating function for correlation functions

$$Z[\psi] = \langle e^{-i \sum_{it} \psi_{it} n_{it}} \rangle = \sum_{\mathbf{n}_0, \dots, \mathbf{n}_T} P[\mathbf{n}_0, \dots, \mathbf{n}_T] e^{-i \sum_{i,t} \psi_{it} n_{it}}$$

- Correlation functions for typical disorder (\Leftrightarrow averaged correlation functions)

$$\overline{\langle n_{it} \rangle} = i \frac{\partial \overline{Z[\psi]}}{\partial \psi_{it}} \Big|_{\psi \equiv 0}, \quad \overline{\langle n_{is} n_{jt} \rangle} = i^2 \frac{\partial^2 \overline{Z[\psi]}}{\partial \psi_{is} \partial \psi_{jt}} \Big|_{\psi \equiv 0}$$

- Credit risk: decompose stochastic force

$$\eta_{it} = \sqrt{\rho} \eta_0 + \sqrt{1 - \rho} \xi_{it}$$

Global component η_0 slowly varying (e.g. fixed over risk horizon of one year); OR: $\rho = 0$

Generating Functions — Solution

- System represented by ensemble of effective single site processes parameterised by ϑ

– OR ($\rho = 0$)

$$n_{t+1} = \Theta\left(J_0 m_t + \alpha J^2 \sum_{s < t} G_{ts} n_s - \vartheta + \phi_t\right)$$

– CR ($\rho \neq 0$, $n_{it} = 1$ absorbing)

$$n_{t+1} = n_t + (1 - n_t) \Theta\left(J_0 m_t + \alpha J^2 \sum_{s < t} G_{ts} n_s + \sqrt{\rho} \eta_0 - \vartheta + \phi_t\right)$$

- Single-site processes exhibit

coloured noise $\{\phi_t\}$, and memory G_{ts}

Generating Functions — Self-Consistency

- Self-consistency equations

$$\langle \phi_s \phi_t \rangle = (1 - \rho) \delta_{st} + J^2 q_{st}$$

$$m_t = \langle \langle n_t \rangle \rangle_{\vartheta}$$

$$q_{st} = \langle \langle n_s n_t \rangle \rangle_{\vartheta}$$

$$G_{ts} = \left\langle \frac{d \langle n_t \rangle}{d h_s} \right\rangle_{\vartheta}$$

- Interpretation of variables

$$m_t = \frac{1}{N} \sum_i \overline{\langle n_{it} \rangle}$$

$$q_{st} = \frac{1}{N} \sum_i \overline{\langle n_{is} n_{it} \rangle} \quad , \quad G_{ts} = \frac{1}{N} \sum_i \frac{d}{d h_s} \overline{\langle n_{it} \rangle}$$

Generating Functions — Fully Asymmetric Network ($\alpha = 0$)

- At $\alpha = 0$, **no memory effects**, and m_t sufficient to describe the dynamics

– OR ($\rho = 0$)

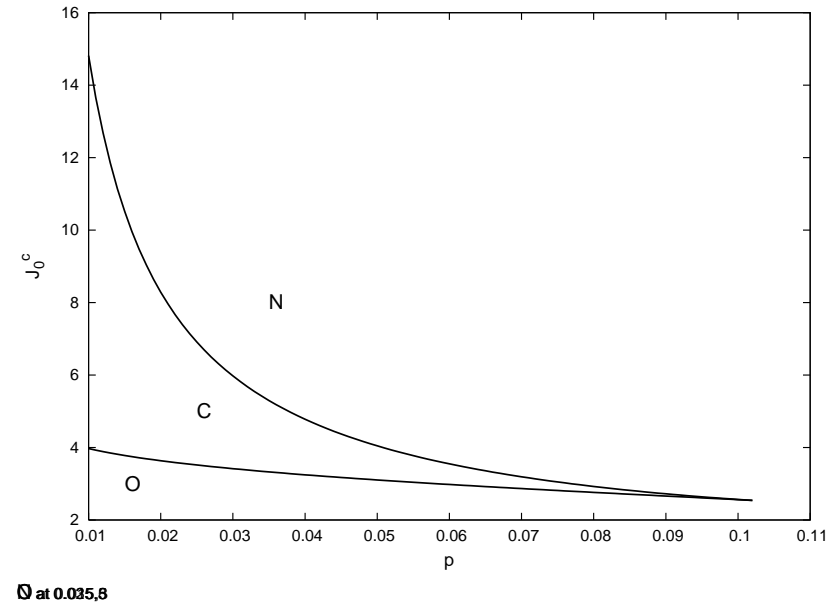
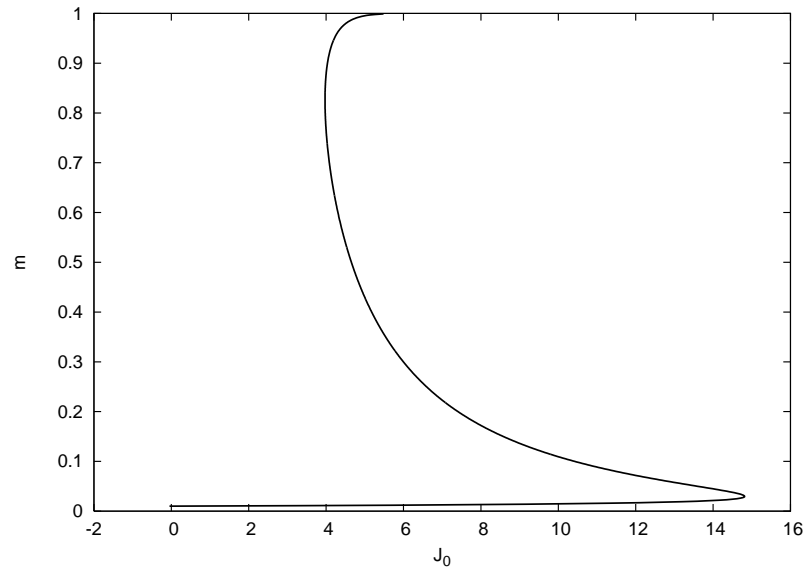
$$m_{t+1} = \left\langle \phi \left(\frac{J_0 m_t - \vartheta}{\sqrt{1 + J^2 m_t}} \right) \right\rangle_{\vartheta}$$

– CR ($\rho \neq 0$, $n_{it} = 1$ absorbing)

$$\langle n_{t+1} \rangle = \langle n_t \rangle + (1 - \langle n_t \rangle) \Phi \left(\frac{J_0 m_t + \sqrt{\rho} \eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right)$$

$$m_{t+1} = m_t + \left\langle (1 - \langle n_t \rangle) \Phi \left(\frac{J_0 m_t + \sqrt{\rho} \eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right) \right\rangle_{\vartheta}$$

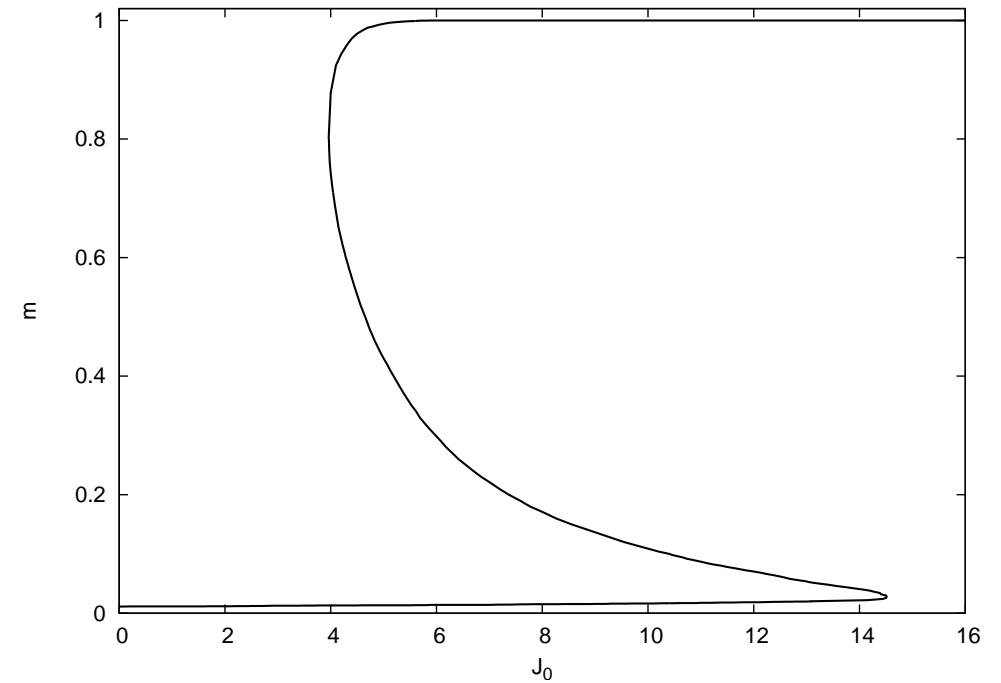
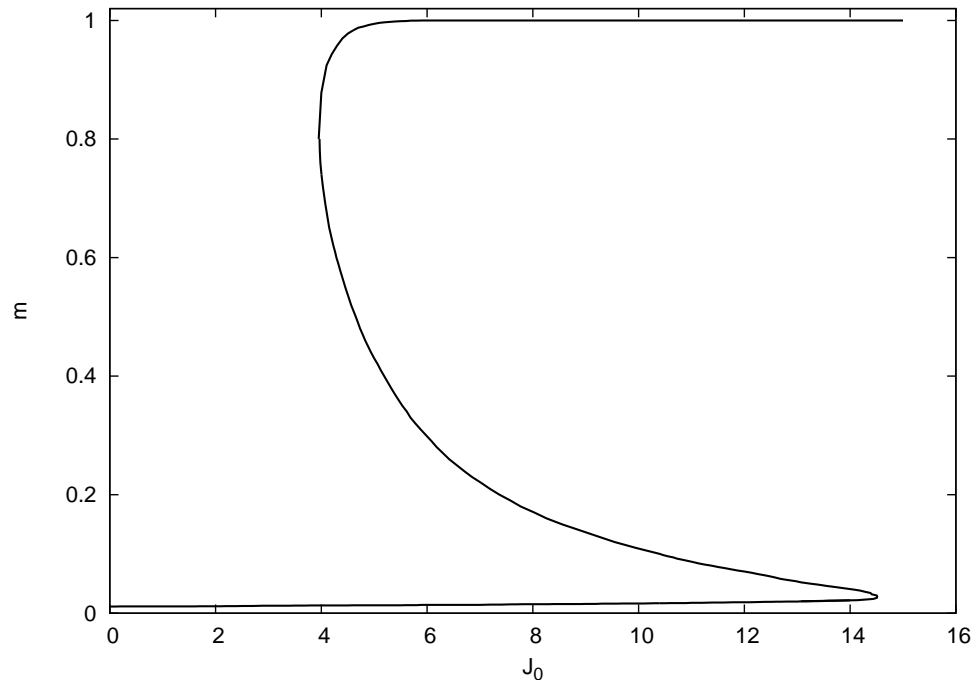
Results OR — Stationary Solution at $\alpha = 0$



(K Anand and RK, Phys Rev E75 (2007))

Stationary fraction of down-processes (left) and phase diagram (right) for $p_i = 0.001$, $\alpha = 0$, $J = 0.2$

Results OR — Stationary Solution at $\alpha \neq 0$



(K Anand and RK, Phys Rev E75 (2007))

Stationary fraction of down-processes for $\alpha = 0.5$ (left) $\alpha = 1.0$ (right) at $J = 0.2$

Results CR — Defaulted Fraction and Loss Distribution

- Recall effective single node process:

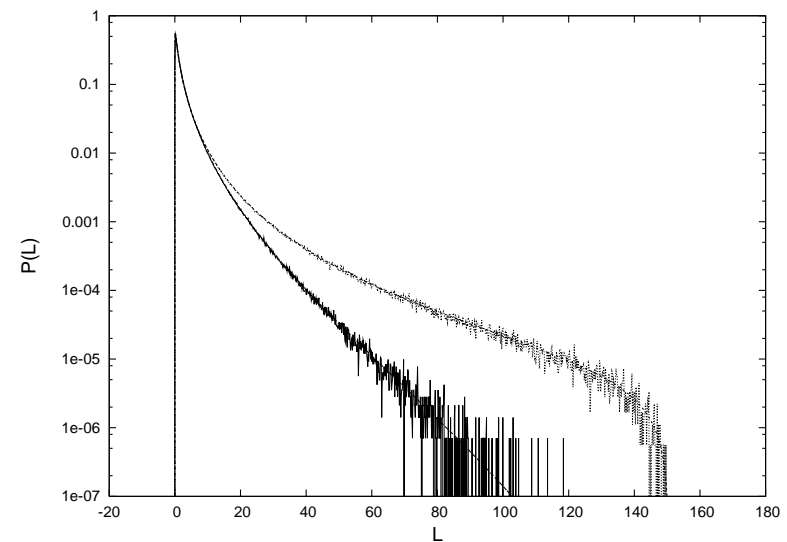
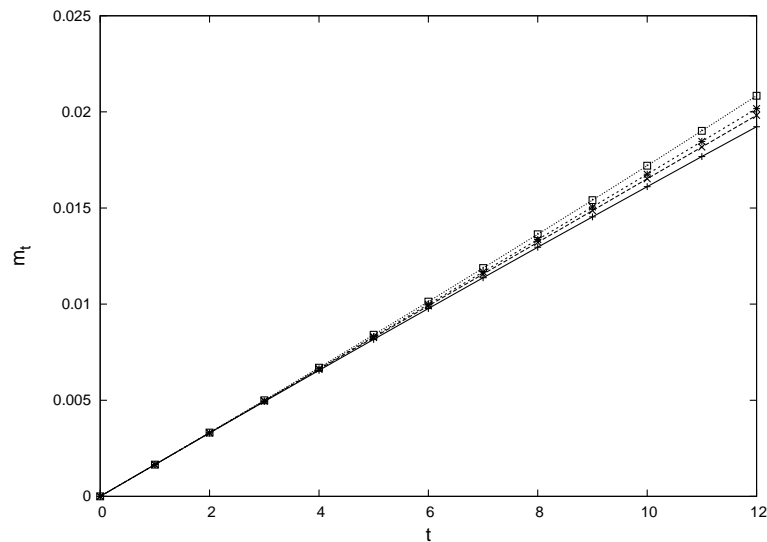
$$n_{t+1} = n_t + (1 - n_t)\Theta\left(J_0 m_t + \alpha J^2 \sum_{s < t} G_{ts} n_s + \sqrt{\rho} \eta_0 - \vartheta + \phi_t\right)$$

- Memory term vanishes as long as $n_t = 0$, and becomes **irrelevant**, once $n_t = 1$.
- Can use $\alpha = 0$ theory:

$$m_{t+1} = m_t + \left\langle (1 - \langle n_t \rangle) \Phi \left(\frac{J_0 m_t + \sqrt{\rho} \eta_0 - \vartheta}{\sqrt{1 - \rho + J^2 m_t}} \right) \right\rangle_{\vartheta}$$

Results CR — Defaulted Fraction and Loss Distribution

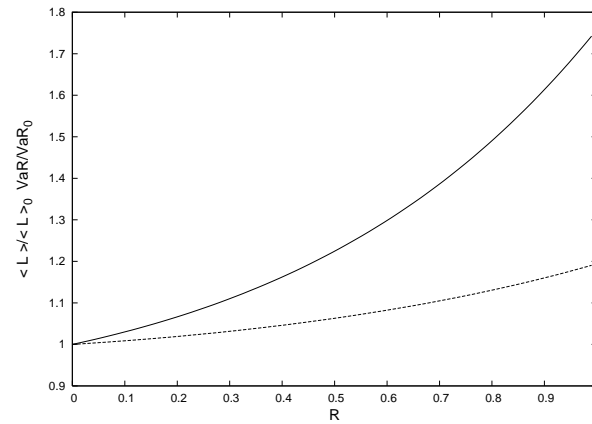
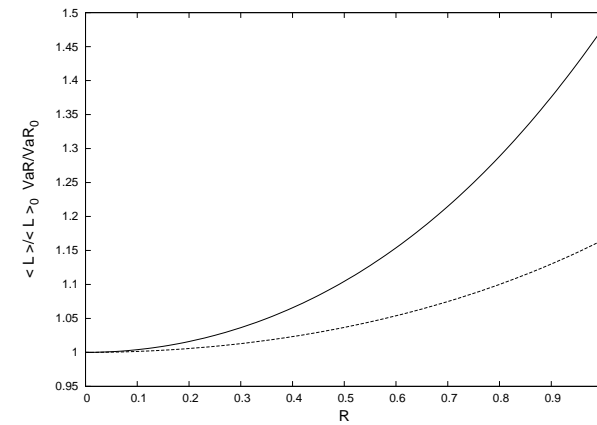
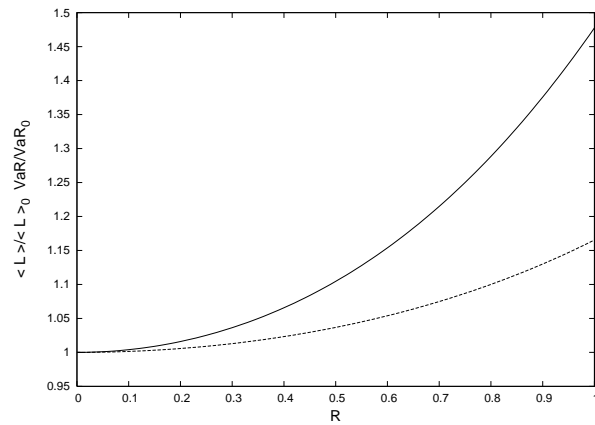
- Loss distribution driven by fluctuating macro-economic condition η_0



(JPL Hatchett and RK, J Phys A**39** (2006))

Fraction of defaulted firms for neutral macro-economic conditions $\eta_0 = 0$ at $(J_0, J) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$ (left, bottom to top; Loss distribution for a system with $(J_0, J) = (0, 0)$ and $(1, 1)$ and $\bar{\ell}(\vartheta) = 1/(\varepsilon + p_d(\vartheta))$ (right)

Results CR — Value at Risk



(JPL Hatchett and RK, J Phys A39 (2006))

Ratios of value at risk (upper curves) and average losses (lower curves) for systems with and without functional interaction evaluated along straight lines in the J_0 - J plane, with $R = \sqrt{J_0^2 + J^2}$. Left: $J_0 = 0$; right: $J = 0$; lower: $J_0/J = 1$.

Functional Dependencies in Market Risk

- Standard approach: Geometric Brownian motion (GBM) for Log-returns of risk elements i (stocks, bonds ...)

$$\frac{1}{S_i(t)} \frac{dS_i(t)}{dt} = \mu_i + \sigma_i \eta_i(t)$$

- Statistical dependencies via correlated Gaussian noises

$$\langle \eta_i(t) \eta_j(t') \rangle = \rho_{ij} \delta(t - t')$$

- \Rightarrow no functional dependencies, no collective behaviour, no market bubbles, crashes

Functional Dependencies in Market Risk — iGBM

- **Functional** dependencies between risk elements e.g. via recommendations of analysts, economic dependencies.

- GBM in terms of $h_i(t) = \log(S_i(t)/S_{i0})$

$$\frac{dh_i(t)}{dt} = \mu_i - \frac{1}{2}\sigma_i^2 + \sigma_i\eta_i(t)$$

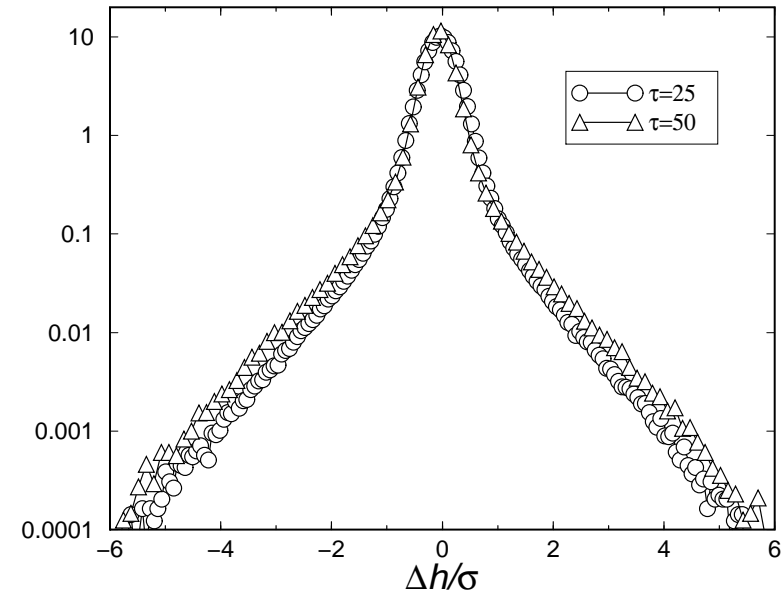
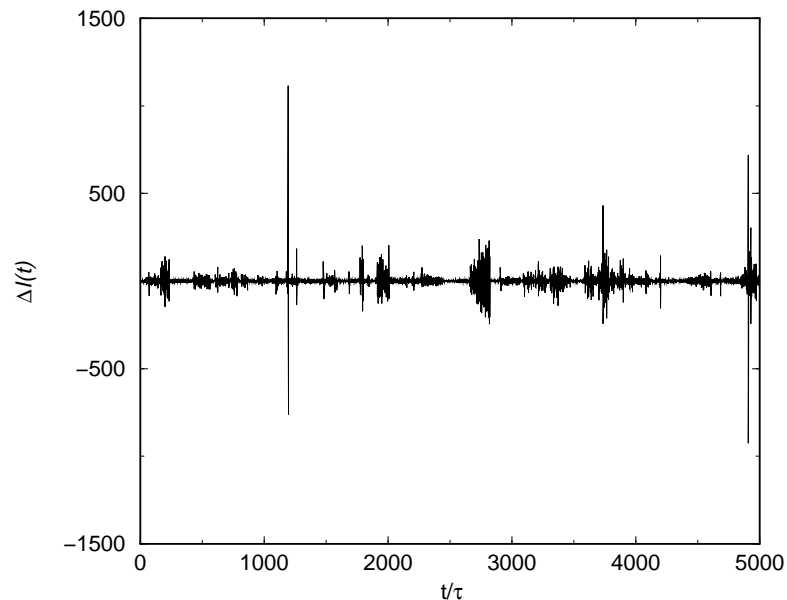
- Minimal interacting generalisation (iGBM): **stabilisation and non-linear feedback** (e.g. $g(x) = \tanh(x)$)

$$\frac{dh_i(t)}{dt} = -\kappa h_i(t) + \mu_i - \frac{1}{2}\sigma_i^2 + \sum_j J_{ij}g(h_j(t)) + \sigma_i\eta_i(t)$$

- \Leftrightarrow **dynamics of of graded-response neurons**
many meta-stable states; transitions between them \Rightarrow intermittent dynamics.

Functional Dependencies in Market Risk – iGBM

- Solve in stochastic setting using GFA as for OR/CR
- Here MC study; trigger transitions by ‘unexpected news’



(RK and P Neu, J Phys A41 (2008))

Change of index $I(t) = N^{-1} \sum_i S_i(t)$ over time increment $\tau = 25$ (left) and normalised distribution of log-returns for $\tau = 25$, and $\tau = 50$ (right).

Summary

- Standard risk models based on statistically correlated risk elements miss dynamically generated **functional** correlations
- Interacting processes capture functional dependencies in OR.
- Similarly: economic interactions \Rightarrow credit contagion in CR
- Physics analogy: lattice gas model on (random) graph.
- Describes bursts and avalanches of risk events
- Important: coexistence with phases of catastrophic breakdown — **no noticeable precursors !**
- Analogue Neuron model for MR — intermittent dynamics, fat tails.