Von Neumann-Gale Dynamical Systems and their Applications in Finance

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Multivalued dynamical systems

Given:

Set A_t (*state space* at time t), $t = 0, 1, 2, \ldots$;

multivalued mapping

$$a \mapsto G_t(a), \quad a \in A_{t-1}, \quad G_t(a) \subseteq A_t$$

(*transition mapping*).

Paths, or *trajectories*: sequences $a_0, a_1, \ldots$ such that

$$a_t \in G_t(a_{t-1}).$$

Von Neumann-Gale dynamical systems

A_t are **cones** in linear spaces;

for each t, the graph of the transition mapping $G_t(\bullet)$,

$$Z_t = \{(a, b) \in A_{t-1} \times A_t : b \in G_t(a)\},$$

is a **cone**.

Equivalent description in terms of transition cones:

Given: *transition cones* Z_t; *paths* are sequences $a_0, a_1, \ldots$ such that

$$(a_{t-1}, a_t) \in Z_t.$$

Autonomous systems

A_t and $G_t(\bullet)$ (or Z_t) do not depend on t.

Example: **von Neumann (1937) model of economic growth**

$$A_t = R^n_+,$$

$$Z_t \subseteq R^n_+ \times R^n_+ \text{ polyhedral cones.}$$

States $a = (a^1, \ldots, a^n) \geq 0$ are *commodity vectors*. The process of economic growth: dynamics of a_t in time. Feasible growth paths $a_0, a_1, \ldots$:

$$(a_{t-1}, a_t) \in Z_t\,.$$

Elements $(a, b) \in Z_t$ are feasible *input-output pairs*, or *technological processes* (for the time period $t-1, t$). Z_t are termed *technology sets*.

The cone Z_t is *polyhedral*: there is a finite set of *basic* technological processes

$$(a^1_{t-1}, b^1_t), \ldots, (a^m_{t-1}, b^m_t)$$

and

$$(a, b) \in Z_t \quad \Leftrightarrow \quad (a, b) = \sum_{j=1}^{m} d^j (a^j_{t-1}, b^j_t),$$

where

$$d^1 \geq 0, \ldots, d^m \geq 0$$

are *intensities* of operating the technological processes

$$(a^1_{t-1}, b^1_t), \ldots, (a^m_{t-1}, b^m_t).$$

Gale (**1956**): general, not polyhedral, cones.

Stochastic von Neumann–Gale dynamical systems

Pioneering work of Dynkin, Radner and their research groups in the 1970s.

Probability space $(\Omega, \mathcal{F}, P)$;

filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_t \subseteq \ldots \subseteq \mathcal{F}$;

set-valued *transition mappings*

$$(\omega, a) \mapsto G_t(\omega, a) \subseteq R_+^n$$

assigning to each $\omega \in \Omega$ and $a \in R_+^n$ a set $G_t(\omega, a) \subseteq R_+^n$ such that

(i) for each ω, the graph

$$Z_t(\omega) := \{(a, b) : b \in G_t(\omega, a)\}$$

of the mapping $G_t(\omega, \bullet)$ is a closed convex cone (*transition cone*);

(ii) the set-valued mapping $Z_t(\omega)$ is $\mathcal{F}_t$-measurable.

Paths (trajectories) $y_0(\omega), y_1(\omega), \ldots$

$$y_t(\omega) \in G_t(\omega, y_{t-1}(\omega)) \text{ (a.s.)},$$

or, equivalently,

$$(y_{t-1}(\omega), y_t(\omega)) \in Z_t(\omega) \text{ (a.s.)}$$

and

$$y_t \text{ is } \mathcal{F}_t\text{-measurable.}$$

Applications of von Neumann-Gale systems in finance: a dynamic securities market model

n *assets*;

(contingent) *portfolio* of assets

$$y_t(\omega) = (y_t^1(\omega), \ldots, y_t^n(\omega)),\ \mathrm{F}_t\text{-measurable;}$$

trading strategy: a sequence of contingent portfolios

$$y_0, y_1, y_2, \ldots;$$

self-financing strategy:

$$(y_{t-1}(\omega), y_t(\omega)) \in Z_t(\omega) \text{ (a.s.).} \qquad (*)$$

Given: closed convex cone $Z_t(\omega)$ (*transition cone*) depending F_t-measurably on ω.

Formula $(*)$ means that the portfolio y_{t-1} can be transformed to y_t under the self-financing constraint (with *transaction costs*).

The cones $Z_t(\omega)$ define a stochastic von Neumann-Gale dynamical system. Self-financing trading strategies: paths in this system.

To simplify presentation, in this talk we will assume that $y_t \geq 0$, i.e. $Z_t(\omega) \in R_+^n \times R_+^n$ (no short selling). This assumption is standard in capital growth theory – the main application of von Neumann-Gale systems which we consider in this talk.

Example of the transition cone $Z_t(\omega)$ in a financial market model

Let $S_t^1(\omega), \ldots, S_t^n(\omega)$ be asset prices.

The cone $Z_t(\omega)$ consists of (a, b) for which

$$\sum_{i=1}^{n} (1 + \lambda_{t,i}^{+})(b^i - \frac{S_t^i}{S_{t-1}^i} a^i)_+ \leq \sum_{i=1}^{n} (1 - \lambda_{t,i}^{-})(\frac{S_t^i}{S_{t-1}^i} a^i - b^i)_+,$$

where $a_+ := \max\{a, 0\}$.

Transaction cost rates for buying and selling: $\lambda_{t,i}^{+}(\omega) \geq 0$ and $1 > \lambda_{t,i}^{-}(\omega) \geq 0$.

Portfolio positions are measured in terms of asset values.

Asset purchases are made only at the expense of sales (with transaction costs).

This is a standard framework for the modelling of proportional transaction costs.

Rapid paths

The primary focus of the von Neumann-Gale theory is on paths growing faster, in a sense, than others. A central definition is as follows.

Definition. A path $x_0, x_1, \ldots$ is called *rapid* if there is a sequence of random vectors $p_0, p_1, \ldots$ ($p_t \geq 0$, $p_t \in \mathrm{F}_t$) such that

$$p_t x_t = 1 \text{ (a.s.)} \tag{1}$$

and for any

$$(x(\omega), y(\omega)) \in Z_t(\omega) \text{ (a.s.)},\ x \in \mathrm{F}_{t-1},\ y \in \mathrm{F}_t,$$

we have

$$E(p_t y | \mathrm{F}_{t-1}) \leq p_{t-1} x. \tag{2}$$

By virtue of (2), for any path $\{y_0, y_1, \ldots\}$, the sequence $p_0 y_0, p_1 y_1, p_2 y_2, \ldots$ is a *supermartingale*.

Terminology: **why "rapid"**? Suppose (1) holds. Then the supermartingale property (2) holds if and only if the path $x_0, x_1, \ldots$ **maximizes the expected growth rate**

$$E \frac{p_t x_t}{p_{t-1} x_{t-1}}$$

or, equivalently, **the expected logarithm of the growth rate**

$$E \ln \frac{p_t x_t}{p_{t-1} x_{t-1}}$$

over each time period $(t-1, t]$. Also: **asymptotic optimality** of rapid paths (will be discussed later).

Dual paths and rapid paths

Sequences $p_0, p_1, \ldots$ ($p_t \geq 0$, F_t-measurable) satisfying the supermartingale property

$$E(p_t y | \mathrm{F}_{t-1}) \leq p_{t-1} x. \qquad (2)$$

are called *dual paths*.

If the normalization property

$$p_t x_t = 1 \text{ (a.s.)} \qquad (1)$$

holds, (p_t) is said to *support* (x_t).

Thus, **rapid paths are those which are supported by dual paths**.

Consistent price systems

In applications, dual paths (p_t) are often interpreted as *price systems.*

In financial models, dual paths are analogues of *consistent price systems*.

If portfolio positions are measured in terms of asset values, they represent *consistent discount factors* (different for different assets).

Rapid paths are analogues of *numeraire portfolios* (Long 1990) or *benchmark strategies* (Platen, Heath and others, 2006).

Existence of rapid paths

Let the transition cones $Z_t(\omega)$ satisfy:

(**A1**) $Z_t(\omega)$ contains with every (a,b) all (a',b') such that $a' \geq a$ and $0 \leq b' \leq b$.

(**A2**) There exists a constant M such that $|b| \leq M|a|$ for all $(a,b) \in Z_t(\omega)$.

(**A3**) There exists a constant $\gamma > 0$ such that $(e, \gamma e) \in Z_t(\omega)$, where $e = (1,1,\ldots,1)$.

(**A4**) There exists an integer $l \geq 1$ such that for every $t \geq 0$ and $i = 1,\ldots,n$ there is a path $y_{t,i},\ldots,y_{t+l,i}$ satisfying

$$y_{t,i} = e_i, \ldots, y_{t+l,i} \geq \gamma e,$$

where $e_i = (0,0,\ldots,1,\ldots,0)$ (ith coordinate is 1).

Theorem 1. *For each $x_0(\omega) \geq \kappa e$ ($0 < \kappa = const$), there exists an infinite rapid path with initial state x_0.*

Under the above general assumptions, the result has been obtained quite recently:

W. Bahsoun, I. Evstigneev and M. Taksar, *Stochastics*, 2008, v. 80, 129-142.

Earlier results (under stronger assumptions):

I. Evstigneev and M. Taksar, *Stochastics and Dynamics*, 2001, v. 1, 493-509.

These stronger assumptions guarantee the *uniqueness* of an infinite rapid path with the given initial state.

Autonomous systems

Autonomous random dynamical systems (e.g. monograph by Arnold 1998) serve as a framework for *stationary models* in economics and finance. In an autonomous von Neumann-Gale system, we are given additionally an automorphism of the probability space (Ω, F, P) (*time shift*) — a measure preserving one-to-one transformation $T : \Omega \to \Omega$ such that

(a) the filtration $\ldots \subseteq \mathrm{F}_{-1} \subseteq \mathrm{F}_0 \subseteq \mathrm{F}_1 \subseteq \ldots \subseteq \mathrm{F}_t \subseteq \ldots$ (defined here for all $t = 0, \pm 1, \ldots$) is invariant

$$T^{-1}\mathrm{F}_t = \mathrm{F}_{t+1},$$

(b) for each t, we have $Z_t(T\omega) = Z_{t+1}(\omega)$.

Condition (b) means that the cone-valued process Z_t is *stationary*, stationarity being understood in terms of *ergodic theory* of dynamical systems.

Example: **stationary securities market model**.

The von Neumann-Gale system on which the standard model with proportional transaction costs is based is autonomous if the *asset returns*

$$R_t(\omega) = \left(\frac{S_t^1(\omega)}{S_{t-1}^1(\omega)}, \ldots, \frac{S_t^n(\omega)}{S_{t-1}^n(\omega)}\right),$$

and *transaction cost rates*

$$\lambda_t^{\pm}(\omega) = (\lambda_{t,1}^{\pm}(\omega), \ldots, \lambda_{t,n}^{\pm}(\omega))$$

form (jointly) stationary processes; Ω is the space of trajectories of these processes; T is the forward shift of trajectories; and F_t is generated by $R_k, \lambda_k^{\pm}$, $k \le t$.

Balanced paths

In autonomous systems, a central role is played by *balanced paths*, i.e. paths of the form

$$y_t := \lambda(T\omega)\lambda(T^2\omega)\ldots\lambda(T^t\omega)y(T^t\omega),$$

where the scalar function

$$\lambda(\omega) > 0$$

is F_0-measurable and the vector function

$$y(\omega) \geq 0$$

is F_0-measurable and satisfies

$$|y(\omega)| = 1.$$

A balanced path *grows with stationary proportions* $y(T^t\omega)$ and *at a stationary rate* $\lambda(T^t\omega)$.

In the deterministic case:

$$y_t = \lambda^t y.$$

Von Neumann path

A balanced path maximizing

$$E\log\lambda(\omega)$$

among all balanced paths is called a *von Neumann path.*

The main results for autonomous systems (stationary models)

Assume (**A1**) - (**A4**).

Theorem 2. *A von Neumann path exists and is rapid.*

A triplet of functions

$$\lambda(\bullet), x(\bullet), p(\bullet)$$

is called a *von Neumann equilibrium* if

$$x_t := \lambda(T\omega)\lambda(T^2\omega)\ldots\lambda(T^t\omega)x(T^t\omega),$$

is a von Neumann path and

$$p_t := \frac{p(T^t\omega)}{\lambda(T\omega)\lambda(T^2\omega)\ldots\lambda(T^t\omega)}$$

is a dual path supporting it.

Theorem 3. *A von Neumann equilibrium exists.*

Dynkin's problem. The above results give the positive answer to the existence problem for a stochastic equilibrium posed by Eugene Dynkin in the early 1970s. In the deterministic case: von Neumann (1937), Gale (1956).

The strategy of the proof is based on the idea of ”elimination of randomization” (Dvoretzky, Wald and Wolfowitz, 1950). First an appropriate extension of the original dynamical system constructed, using an additional source of randomness. Then, based on some subtle properties of convexity, the randomization is eliminated and the existence of a von Neumann path in the original system is established.

The existence theorem for a von Neumann equilibrium was obtained in the paper:

I.V. Evstigneev and K. R. Schenk-Hoppé, Stochastic equilibria in von Neumann-Gale dynamical systems, 2008, *Transactions of the American Mathematical Society*, 360, 3345-3364.

The paper relied substantially on the previous work:

I.V. Evstigneev and K. R. Schenk-Hoppé, Pure and randomized equilibria in the stochastic von Neumann-Gale model, 2007, *Journal of Mathematical Economics*, 43, 871-887.

L. Arnold, I.V. Evstigneev and V. M. Gundlach, Convex-valued random dynamical systems: A variational principle for equilibrium states, 1999, *Random Operators and Stoch. Equations*, 7, 23-38.

CAPITAL GROWTH THEORY

How to invest in order to maximize the asymptotic growth rate of wealth?

Pioneers: Shannon, Kelly, Breiman (1950s and 1960s). Most general results without transaction costs: Algoet and Cover (Ann. Probability, 1988).

Survey: Hakansson and Ziemba, Capital growth theory, in: Handbook in Finance, Elsevier, 1995.

Central concepts:

numeraire portfolios (Long 1990) or

benchmark strategies – Platen and others (e.g. Platen and Heath, A benchmark approach to quantitative finance, Springer, 2006).

Rapid paths generalize these concepts to the transaction cost framework. These paths turn out to be *asymptotically optimal*.

We will show how von Neumann-Gale systems can be applied to extend capital growth theory to the case of proportional transaction costs.

Asymptotic optimality

For a vector $b = (b^1, \ldots, b^n)$, put $|b| = |b^1| + \ldots + |b^n|$. If $b \geq 0$, then $|b| = b^1 + \ldots + b^n$, and if $b \geq 0$ represents a portfolio whose positions are measured in monetary units, then $|b|$ is the *value* of this portfolio—the total amount of money invested in all its assets.

Definition. Let $y_0, y_1, \ldots$ be an investment strategy. It is called *asymptotically optimal* if for any other strategy $y'_0, y'_1, \ldots$, we have

$$\sup \frac{|y'_t|}{|y_t|} < \infty \text{ (a.s.)},$$

i.e. the ratios of the portfolio values $|y'_t|/|y_t|$ are bounded (a.s.).

According to this definition, for no strategy wealth can grow asymptotically faster than for $y_0, y_1, \ldots$ (a.s.).

Note that the above property remains valid if $|\cdot|$ is replaced by any (possibly random) function $\psi(\cdot)$, where $c|a| \leq \psi(a) \leq C|a|$, where $0 < c < C$ are non-random constants.

It is immediate from the above definition that the strategy $y_0, y_1, \ldots$ maximizes a.s. the exponential growth rate of wealth

$$\liminf_{t \to \infty} \frac{1}{t} \ln |y_t|.$$

The main results on asymptotic optimality

Under assumptions (**A.1**)-(**A.4**) the following theorems hold.

Theorem 4. *For every $w > 0$, there exists an asymptotically optimal strategy $y_0, y_1, \ldots$ with initial wealth $|y_0| = w$.*

Theorem 5. *In a stationary model, there exists an asymptotically optimal **balanced** strategy.*

The strategy constructed in Theorem 5 is the von Neumann path in the corresponding von Neumann-Gale model. It is important to emphasize that this balanced strategy (having a relatively simple structure) is asymptotically optimal in the class of *all, not necessarily balanced,* strategies.

These results are derived from those presented above (pertaining to the von Neumann-Gale model) and the following simple fact holding under assumptions (**A.1**)-(**A.4**):

Proposition. *Any rapid path is asymptotically optimal.*

Rapid paths and numeraire portfolios. Why rapid paths may be regarded as a generalization of numeraire portfolios (benchmark strategies)? Consider the standard model without transaction costs in which self-financing strategies $y_0, y_1, \ldots$ are defined by

$$S_t(\omega)y_t(\omega) \leq S_t(\omega)y_{t-1}(\omega),$$

where $S_t(\omega) = (S_t^1(\omega), \ldots, S_t^n(\omega))$ is the vector of asset prices at date t (here, portfolio positions are specified in terms of physical units of assets). The von Neumann-Gale transition cone is

$$Z_t(\omega) = \{(a,b) \in R_+^n \times R_+^n : \ S_t(\omega)b \leq S_t(\omega)a\}.$$

Definition. A *numeraire portfolio (benchmark strategy)* is a strategy (x_t) such that $S_t x_t > 0$ and $S_t^i / S_t x_t$ is a supermartingale for each i.

It is easily seen that *a numeraire portfolio* (x_t) *defines a rapid path.* To show this put

$$p_t = S_t / S_t x_t.$$

Then (p_t) is a dual path. Indeed, if $(x,y) \in Z_t$, i.e. $S_t y \leq S_t x$ (a.s.), and $x \in \mathrm{F}_{t-1},\ y \in \mathrm{F}_t$,, then

$$E(p_t y | \mathrm{F}_{t-1}) = E(S_t y / S_t x_t | \mathrm{F}_{t-1}) \leq E(S_t x / S_t x_t | \mathrm{F}_{t-1}) =$$

$$x E(S_t / S_t x_t | \mathrm{F}_{t-1}) \leq x S_{t-1} / S_{t-1} x_{t-1} = p_{t-1} x.$$

Clearly, $p_t x_t = 1$, and so the dual path (p_t) supports (x_t). Hence (x_t) is a rapid path.

Under transaction costs, different asset prices S_t^i are discounted by different factors (rather than by $S_t x_t$).

On the history of capital growth theory

The first publication on capital growth theory was that by Kelly (1956), who considered the case of Arrow securities, interpreted as a “horse race model”. It was shown that the growth optimal investment strategy could be found by the maximization of the expected logarithm of the portfolio return. In the case of the horse race model, this has led to the famous Kelly portfolio rule – “betting your beliefs” – allocating wealth in the proportions equal to the probabilities of winning. Kelly arrived at his results from information theory, and his paper was entitled “A new interpretation of information rate”.

Claude Shannon’s role. Although Claude Shannon – the famous founder of the mathematical theory of information – did not publish on investment-related issues, his ideas, expressed in his lectures on investment problems, strongly influenced his collaborators: Kelly, Breiman, Cover and others, whose publications initiated the strand of literature on growth optimal investments. For the history of these ideas and a related discussion (Shannon vs Samuelson) see Cover:

T. M. Cover, Shannon and investment, IEEE Information Theory Society Newsletter, Summer 1998, Special Golden Jubilee Issue, pp. 10–11.

An important special case: **Stochastic Perron-Frobenius theory**

Given: probability space (Ω, F, P),

filtration $\ldots \subseteq \mathrm{F}_0 \subseteq \mathrm{F}_1 \subseteq \ldots \subseteq \mathrm{F}_t \subseteq \ldots \mathrm{F}$

automorphism $T : \Omega \to \Omega$:

$$TP = T^{-1}P = P, \ \mathrm{F}_{t+1} = T^{-1}\mathrm{F}_t,$$

random matrix $0 \leq A(\omega) : R^n \to R^n$, F_0-measurable.

Perron-Frobenius operators – von Neumann-Gale operators of the form:

$$G_t(\omega, a) = \{b \in R^n_+ : \ \ 0 \leq b \leq A(T^t\omega)a\}.$$

Random Perron-Frobenius eigenvalue and eigenvector

Suppose $A(\omega)A(T\omega)\ldots A(T^l\omega) > 0$ for some l.

Theorem 6. *There is a random scalar $\lambda(\omega) > 0$ (F_0-measurable) and a random vector $x(\omega) > 0$ (F_0-measurable) such that*

$$\lambda(T\omega)x(T\omega) = A(T\omega)x(\omega), \ |x(\omega)| = 1.$$

The pair $(\lambda, x) \geq 0$ satisfying the above two equations is unique.

Von Neumann equilibrium in this system: (x, λ, p),

λ and x are the random P.-F. eigenvalue and eigenvector for $A(\bullet)$ and T, and

p can be expressed (by taking a conditional expectation and normalizing) through the random P.-F. eigenvector of $A^*(\bullet)$ and T^{-1}.

Further references

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