Asymptotic approximations for American options

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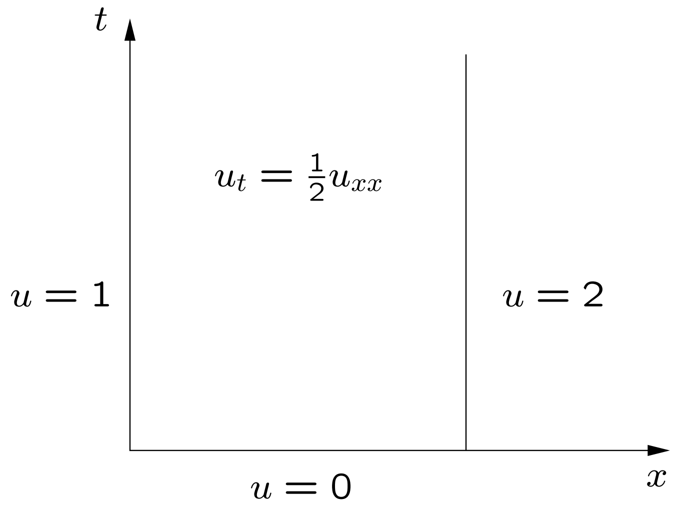
Quiz



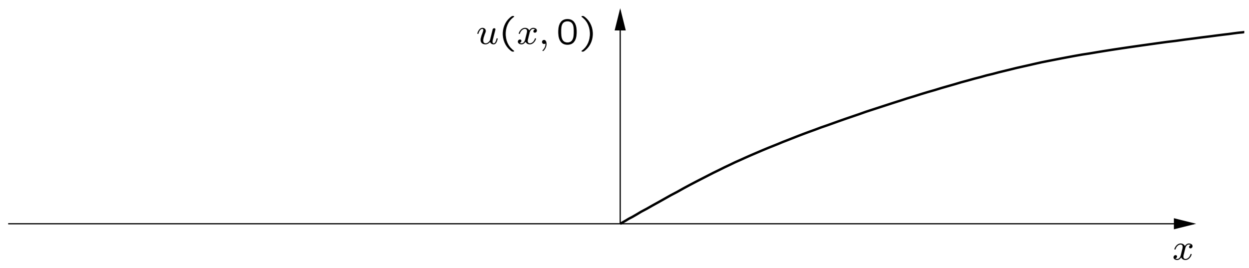
Where is the *initial* minimum of u (at $t = 0^+$)?

Short-time asymptotics of the heat equation

consider

$$u_t = \tfrac{1}{2} u_{xx}, \qquad t > 0,$$

with initial data vanishing for $x < 0$:



Initial behaviour as $x \to -\infty$, $t = O(1)$? (Or as $t \downarrow 0$, x fixed.)

Various ways:

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty u_0(s) \mathrm{e}^{-(x-s)^2/2t} \mathrm{d}s \\
&= \frac{\mathrm{e}^{-x^2/2t}}{\sqrt{2\pi t}} \int_0^\infty u_0(s) \mathrm{e}^{xs/t - s^2/2t} \mathrm{d}s \qquad (*) \\
&= \frac{\mathrm{e}^{-x^2/2t}}{\sqrt{2\pi t}} \left(\frac{-x}{t} \right) \int_0^\infty u_0(-t\xi/x) \mathrm{e}^{-\xi - t\xi^2/2x^2} \mathrm{d}\xi \\
&\sim \frac{\mathrm{e}^{-x^2/2t}}{\sqrt{2\pi t}} \times F(x/t)
\end{aligned}
$$

as $x \to -\infty$;

or do Laplace on $(*)$ above,

$$\frac{\mathrm{e}^{-x^2/2t}}{\sqrt{2\pi t}} \int_0^\infty u_0(s) \mathrm{e}^{xs/t - s^2/2t} \mathrm{d}s;$$

as $x/t \to -\infty$;

or expand $u_0(s) = \sum c_n s^n$ and get the answer as a sum of similarity solutions;

(both these conclude that the behaviour of u_0 at the origin is paramount) or

$$u = \frac{\mathrm{e}^{-x^2/2t}}{\sqrt{2\pi t}} v(x,t)$$

gives

$$t v_t + x v_x = \tfrac{1}{2} v_{xx}$$

so put RHS = 0 and say Euler; or more systematically, put $x = X/\epsilon$ (or $t = \epsilon^2 T$) and use the WKB ansatz

$$u \sim A \mathrm{e}^{v/\epsilon^2}$$

to get the same result via $v_T = \frac{1}{2} v_X^2$ etc.

The Stefan problem for small latent heat

Melting of a solid with small latent heat ϵ:

$$u_t = \tfrac{1}{2}u_{xx}, \quad 0 < x < s(t), \qquad u(0,t) = 1,$$

with free boundary conditions

$$u(s(t),t) = 0, \quad u_x(s(t),t) = -\epsilon\dot{s}.$$

There is a similarity solution $u = U(x/\sqrt{t})$, $s = \alpha\sqrt{t}$ from which, as $\epsilon \to 0$, the relevant timescale is

$$t = \delta T, \qquad \delta = 1/|\log \epsilon|.$$

Then there is a 3-layer structure:

Boundary layer near $x = 0$, $x = \delta^{\frac{1}{2}} X$, $t = \delta T$, giving the usual error function solution.

'Outer region' $x = O(1)$, $t = \delta T$, WKB solution (as above) of

$$\frac{1}{\delta} u_T = u_{xx}$$

Solution is $u \sim A \exp(v/\delta)$ with $v = -x^2/2T$, $A = (1/\sqrt{2\pi T})(T/x)$.

Inner layer $x = s(t) + \delta\xi$, $u = \epsilon U$, with a travelling wave solution of the heat equation satisfying the free boundary conditions, $U = \frac{1}{2}(1 - \exp(-2\xi\dot{s}))$.

These all match and the scale δ follows from matching the outer region to the inner layer.

Generalises to more than one dimension and the free boundary is close to the isotherm $u = \epsilon$ of the corresponding pure heat conduction problem.

Can also be done via an integral equation (Grinberg/Chekhmareva) but doesn't work in 2 or more dimensions.

(Addison, SDH, King, QAM 2005.)

American options in the Black-Scholes model

The BS model is the standard description of normal (?!) financial markets.

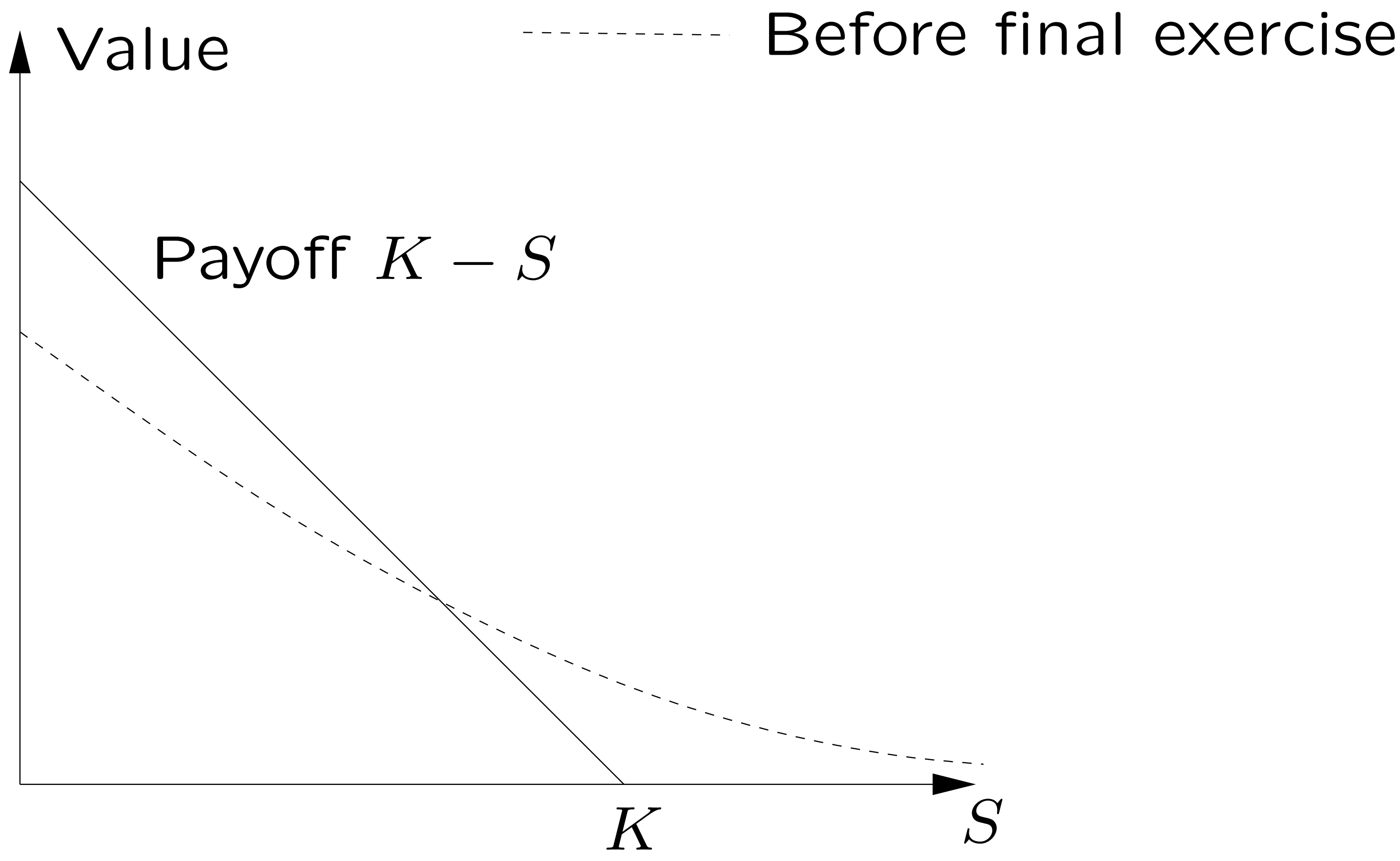
- Asset prices follow diffusions (SDEs driven by Wiener processes).
- Options are contracts paying a given function $P(S_T)$, the *payoff*, of the asset price S_T on a final date $t = T$.
- Options are valued as expectations; thus...

- … by Feynman-Kac, option prices satisfy a backward parabolic equation in S, t, with final data $P(S)$: the BS PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0.$$

Here r is the interest rate, q is the dividend rate and σ is the volatility.

Example: a put option is the right to sell the asset for K at time T and its payoff is $\max(K - S, 0)$.



A simple scaling and time-reversal

$$t' = \sigma^2 (T - t)$$

(so t' is dimensionless) turns

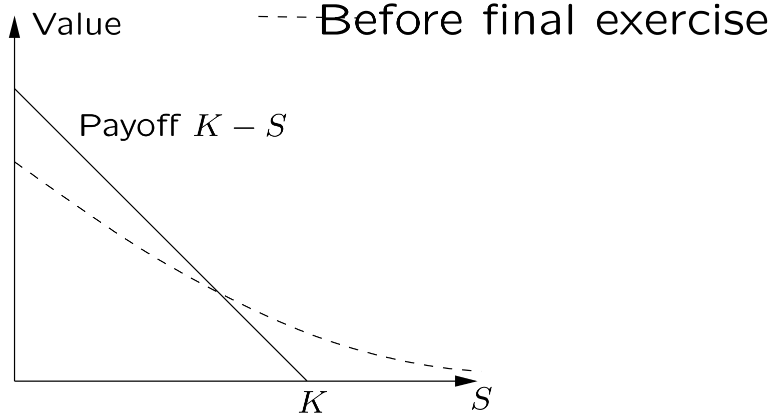
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - rV = 0.$$

into

$$\frac{\partial V}{\partial t'} = \tfrac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (\rho - \gamma) S \frac{\partial V}{\partial S} - \rho V, \qquad \rho = \frac{r}{\sigma^2}, \quad \gamma = \frac{q}{\sigma^2},$$

with the payoff as *initial* data.

$$\frac{\partial V}{\partial t'} = \underbrace{\frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}}_{\text{convexity increases } V} + \underbrace{(\rho-\gamma)S\frac{\partial V}{\partial S} - \rho V}_{\text{both decrease } V}$$

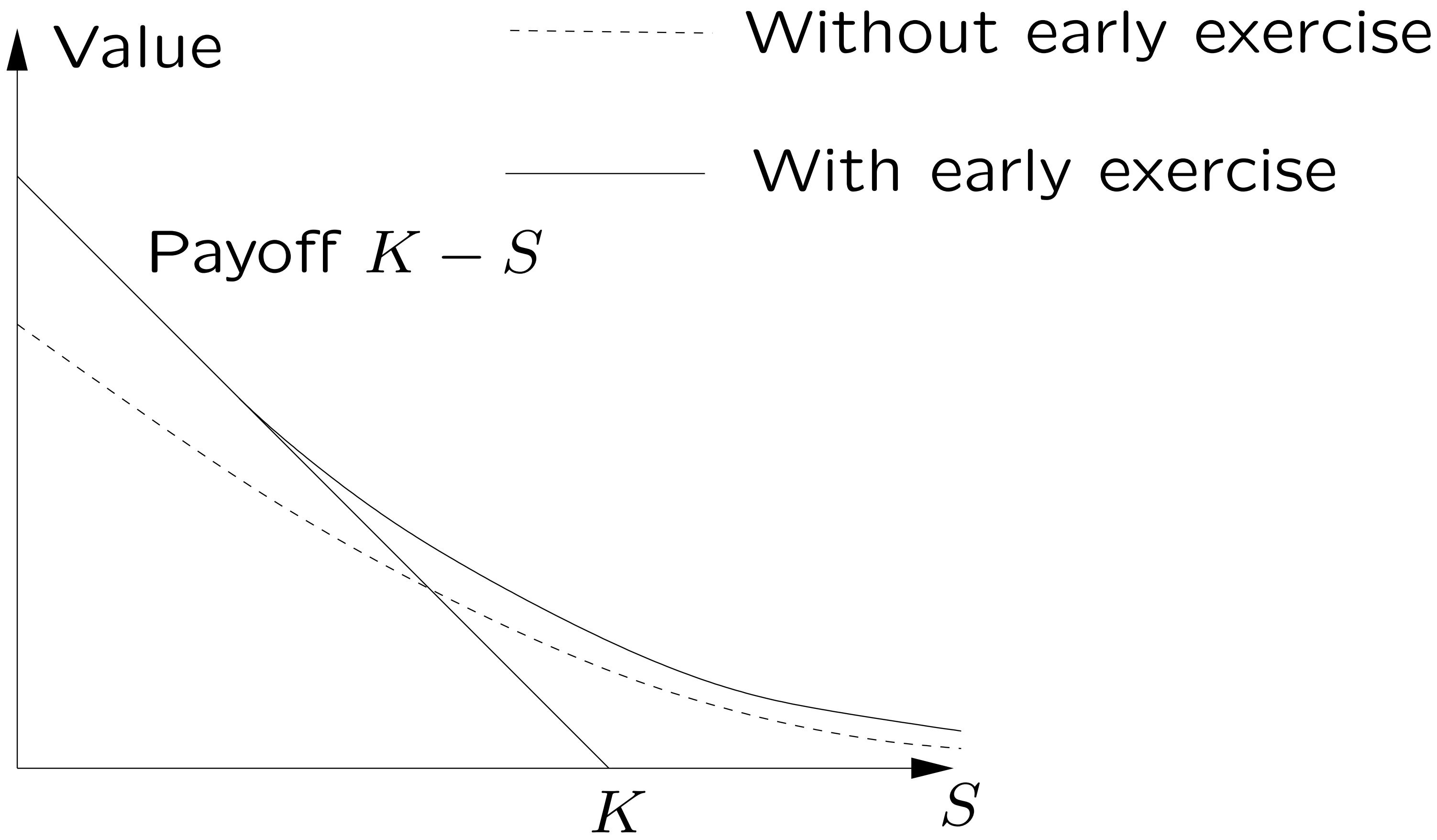


An American option can be exercised at any time (not just at the final date).

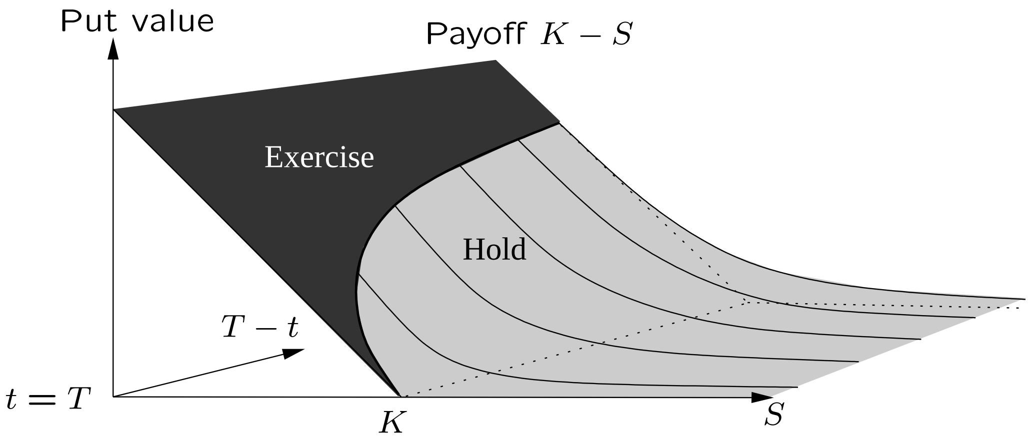
Hence **option value** $\geq$ **payoff**.

Example: an *American put*; the option to sell the asset at *any* time for K. The lower the asset price S falls, the more you get on exercise, but if S goes up you get less. Choosing when to exercise is a balance between the potential reward (if the asset falls) and the risk if it rises.

For each t (calendar time), there is an optimal exercise price $S = S^*(t)$ at which to exercise the option. Below this price, the risk-reward trade-off favours exercise, above it favours waiting.


Value
Without early exercise
With early exercise
Payoff $K - S$
K
S

The American option is like a continuous series of obstacle-type problems (a parabolic variational inequality).



The optimality translates into 'smooth pasting' free boundary conditions: V and $\partial V/\partial S$ are continuous at the interface $S = S^*(t)$:

$$V = K - S, \quad \frac{\partial V}{\partial S} = -1, \quad S = S^*(t).$$

Short time behaviour of American options ($q = 0$)

At $t = T$, the optimal exercise boundary starts from $S = K$ and there is clearly a singularity. How does $S^*(t)$ behave as $t \to T$?

Has been analysed by Keller, Kuske, Evans; Chadam, Chen, Stamicar; Knessl; ... Most approach via an integral equation reformulation.

Roughly equivalent to solving Stefan problem with delta-function initial data. Expect $S^*(t) \sim$ const. $\times \sqrt{(T-t)\log(T-t)}$, NOT $\sqrt{T-t}$ which is for step function initial data.

Can be analysed in ray framework for the eikonal equation with details complicated by more complicated PDE. In particular there are 2 families of important rays and the amplitude function is proportional to

$$\frac{1}{\sqrt{2\pi t}}\frac{1}{\zeta(\zeta+2)}, \qquad \zeta = \frac{S/K - 1}{\sigma^2(T-t)}$$

(ζ is a local 'space' variable) rather than just

$$\frac{1}{\sqrt{2\pi t}}\frac{1}{\zeta}$$

as for the Stefan problem.

Again, generalises to more dimensions.

Discrete dividend payments

When dividends are paid the asset price falls (in calendar time t):

$$S_{\text{before}} = S_{\text{after}} + \text{dividend}$$

The model above has dividends paid continuously at rate q, asset price process

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

The corresponding scaled and forwardised BS PDE is

$$\frac{\partial V}{\partial t'} = \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + (\rho - \gamma)S\frac{\partial V}{\partial S} - \rho V, \qquad \rho = \frac{r}{\sigma^2}, \quad \gamma = \frac{q}{\sigma^2}.$$

For *discrete* dividends, paying $qS_{t_n^-}\,\delta t$ at (equal) time intervals t_n separated by δt,

$$S_{t_n^+} = (1 - q\,\delta t)S_{t_n^-},$$

or in scaled time $T - t = \sigma^2 t'$,

$$S_{t_n'^-} = (1 - \gamma\epsilon^2)S_{t_n'^+}, \qquad \boxed{\epsilon^2 = \sigma^2\delta t.}$$

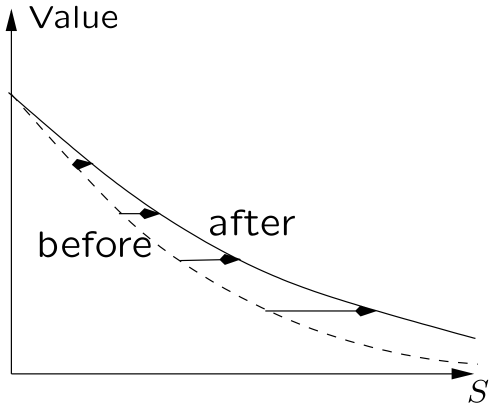
Between these dates, zero-dividend forwardised BS PDE holds:

$$\frac{\partial V}{\partial t'} = \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \rho S\frac{\partial V}{\partial S} - \rho V.$$

At dividend dates, **option value is continuous** for each realisation of S_t, so $V(S_{t'_n{}^+}, t'_n{}^+) = V(S_{t'_n{}^-}, t'_n{}^-)$ which is

$$V(S, t'_n{}^+) = V((1 - \gamma\epsilon^2)S, t'_n{}^-)$$

for all $0 < S < \infty$. That is, the option values are **shifted to the right** across a dividend date (in backwards time).



Discrete PDE + jump cond's to cont's PDE

Multiple scale ansatz $V(S, t', \tau)$ where

$$t' = t'_n + \epsilon^2 \tau$$

so discrete problem is

$$\frac{\partial V}{\partial t'} + \frac{1}{\epsilon^2}\frac{\partial V}{\partial \tau} = \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \rho S\frac{\partial V}{\partial S} - \rho V, \qquad 0 < \tau < 1$$

with

$$V(S, t', 1^+) = V((1 - \gamma\epsilon^2)S, t', 1^-)$$

and **periodic in** τ to eliminate secular terms, so

$$V(S, t', 1^+) = V(S, t', 0^+).$$

Expand

$$V \sim V_0 + \epsilon^2 V_1 + \cdots$$

and find $V_0 = V_0(S, t')$ only; then

$$\frac{\partial V_1}{\partial \tau} = \mathcal{L} V_0, \qquad \mathcal{L} = \text{zero-div BS operator.}$$

So

$$V_1 = \tau \mathcal{L} V_0 + F(S, t')$$

and then periodicity plus expanding jump cond'n to $O(\epsilon^2)$ gives

$$\mathcal{L} V_0 = \gamma S \frac{\partial V_0}{\partial S}$$

as required.

Asian options

Asian options can be sampled discretely and depend on a running average

$$A_t = \frac{1}{N}\sum_{1}^{n} S_{t_i}$$

(can also do any function of S_t). Between the sample dates A_t is constant and the option satisfies the BSPDE.

At sampling dates the average is updated by

$$A_{t_i^+} = A_{t_i^-} + \frac{1}{N} S_{t_i}.$$

So the option value is updated by

$$V(S, t_i^+, A) = V(S, t_i^-, A - S_{t_i}/N)$$

(whatever A was before). Evidently the same machinery as for dividends can be used to derive the continuously sampled BS PDE

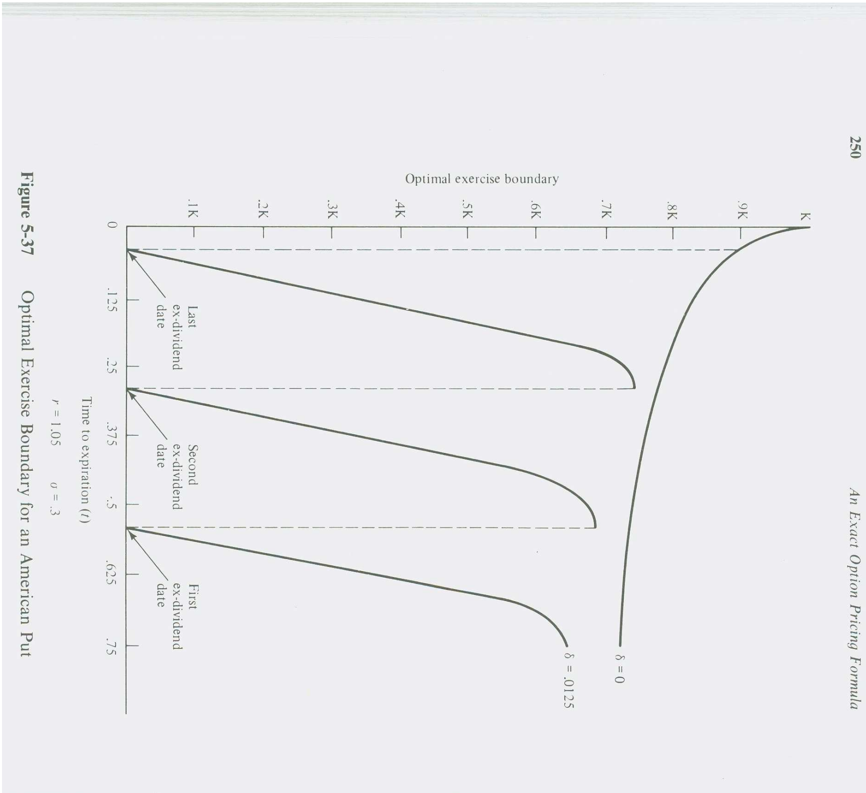
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + S\frac{\partial V}{\partial A} = 0.$$

This is particularly clear when the average is arithmetic and the payoff is affine in S and A, say $\max(S - A - K, 0)$: there is a similarity reduction

$$V(S, A, t) = SW((A - K)/S, t)$$

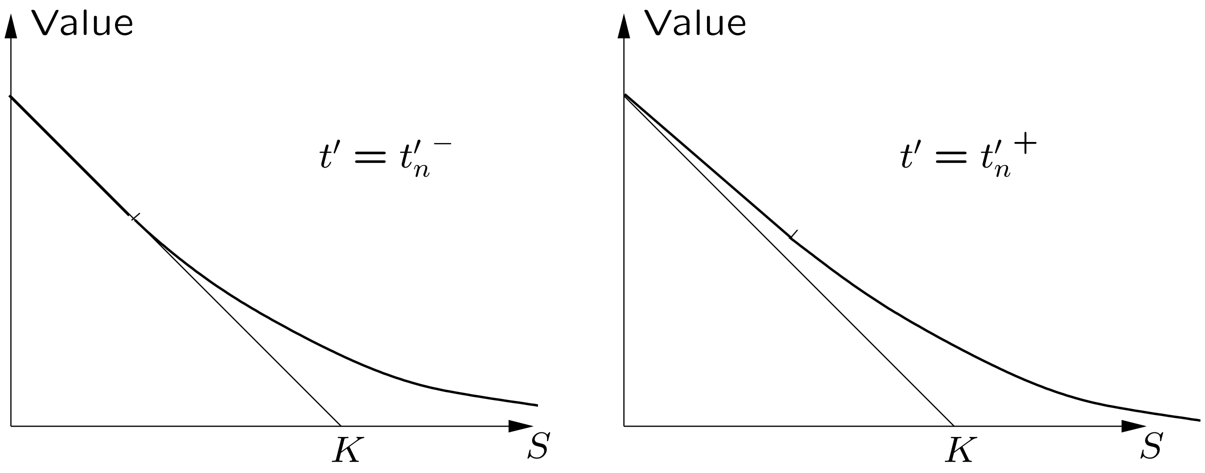
where the average looks like a dividend payment in the equation for W.

American option with discrete dividends



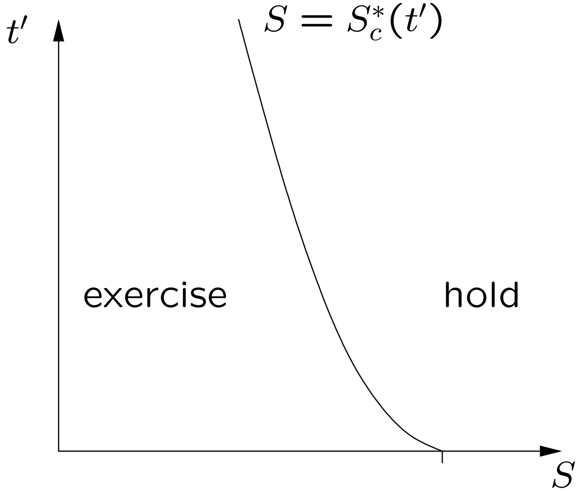
Cox & Rubinstein 1985.

The discrete dividend payment lifts the value function off the payoff:

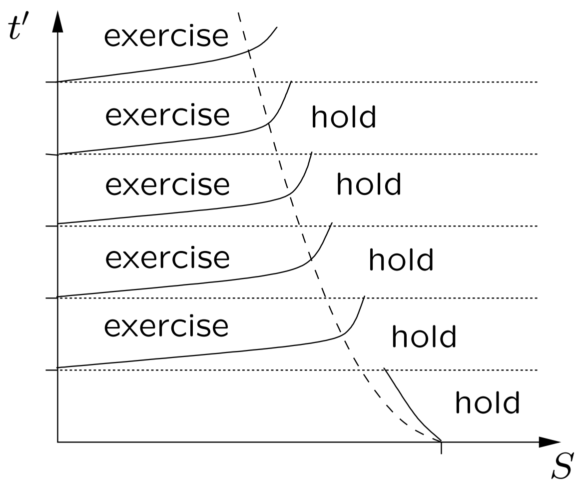


So the exercise boundary falls to $S = 0$ just after (in backwards time) a dividend date.

With multiple dividend dates (cf Cox & Rubinstein 1985):



continuous dividend

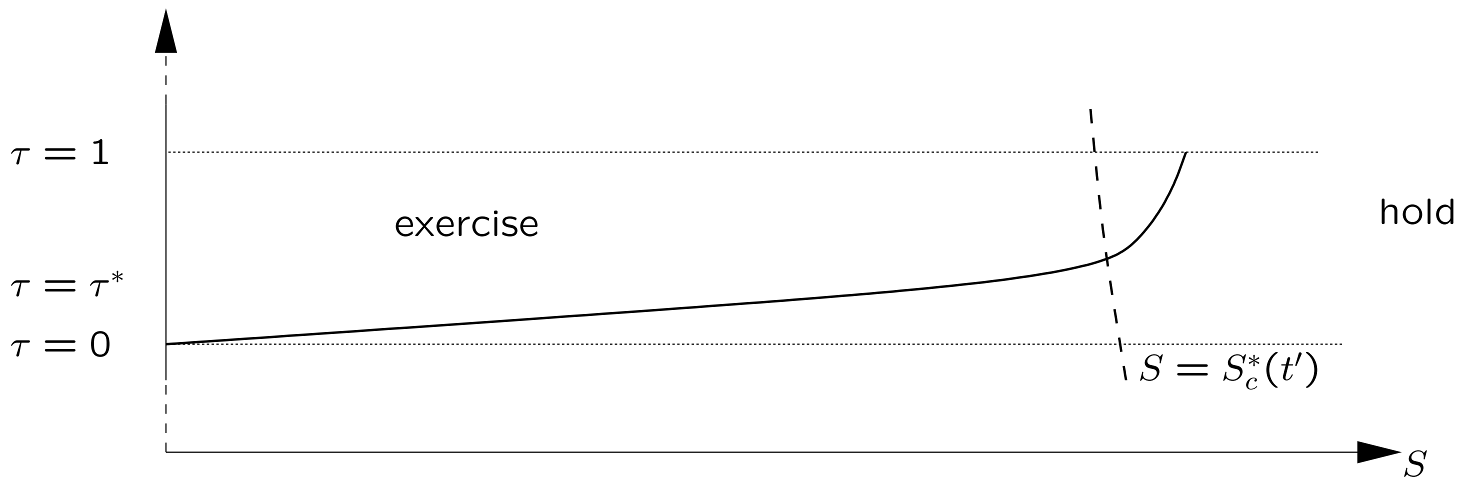


discrete dividend

A kind of ‘mushy region’ when we homogenise it.

Asymptotics of typical inter-dividend period

Put $t' = t'_n + \epsilon^2\tau$ as before.



Set $V = K - S + W(S, t', \tau)$ and then

$$\frac{\partial W}{\partial t'} + \frac{1}{\epsilon^2}\frac{\partial W}{\partial \tau} = \tfrac{1}{2}S^2\frac{\partial^2 W}{\partial S^2} + \rho S\frac{\partial W}{\partial S} - \rho W - \rho K.$$

The free boundary cnditions are

$$W = 0, \qquad \frac{\partial W}{\partial S} = 0.$$

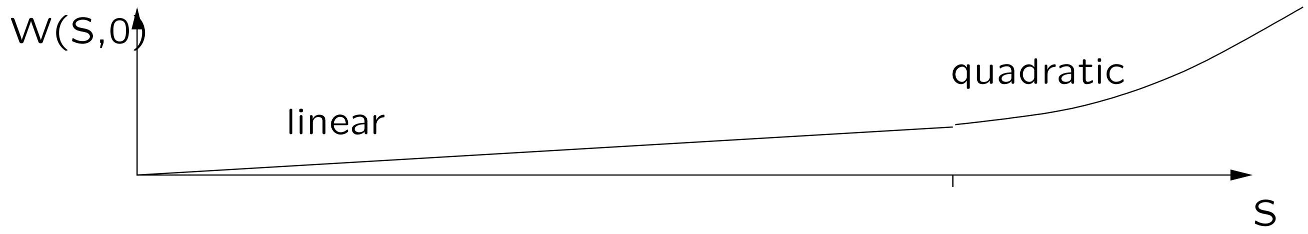
The payoff constraint is

$$W \geq 0.$$

The initial condition (for a periodic solution) is (at leading order)

$$W(S, t', 0) = \begin{cases} \epsilon^2 \gamma S, & 0 < S < S^*(t') \\ \epsilon^2 F(S), & S > S^*(t') \end{cases}$$

where F comes from the outer solution (as above) and it has a (known) second-derivative discontinuity at $S = S^*(t')$. Note it is *linear* for $S < S^*(t')$.



$$\frac{\partial W}{\partial t'} + \frac{1}{\epsilon^2}\frac{\partial W}{\partial \tau} = \tfrac{1}{2}S^2\frac{\partial^2 W}{\partial S^2} + \rho S\frac{\partial W}{\partial S} - \rho W - \rho K.$$

The term $-\rho K$ drags W down, but we have $W \geq 0$. Hence a free boundary $S = s^*(\tau)$, at which W vanishes.

Clearly $W \sim \epsilon^2 W_0 + \cdots$ and then we have

$$\frac{\partial W_0}{\partial \tau} = -\rho K.$$

Hence

$$W_0 = \gamma S - \rho K \tau$$

for $s^*(\tau) = \rho K \tau / \gamma < S < S^*(t')$, where W_0 vanishes, but only for

$$0 < \tau < \tau^* = \frac{\gamma S^*(t')}{\rho K}.$$

Near $S = s^*(\tau)$ is a small travelling-wave region (as in Stefan) to allow both free boundary conditions to apply.

Meanwhile the curvature jump near $S = S^*(t')$ evolves: put

$$S = S^*(t')(1 + \epsilon x), \qquad W = \epsilon^2 w(x, \tau)$$

for an inner region $-\infty < x < \infty$, to get

$$\frac{\partial w}{\partial \tau} = \tfrac{1}{2}\frac{\partial^2 w}{\partial \tau^2} - \rho K,$$

with initial data having a curvature jump. Solution is in similarity form.

Transition

This solution only lasts until the free boundary ‘sees’ the far-field effect of the inner solution. There is a short transition time

$$\tau = \tau^* + O(\epsilon\sqrt{|\log\epsilon|})$$

(determined by 2-term matching. . .) in which the free boundary behaviour (still the location of $W_0 = 0$) goes from

$$s^*(\tau) \sim \rho K\tau/\gamma \qquad \text{for} \quad \tau < \tau^* = \gamma S^*(t')/\rho K$$

to

$$s^*(\tau) \sim S^*(t')\left(1 - \epsilon\sqrt{2\tau^*}\left[\sqrt{-\log(\tau-\tau^*)} - \frac{3}{2}\frac{\log\sqrt{-\log(\tau-\tau^*)}}{\sqrt{-\log(\tau-\tau^*)}}\right]\right)$$

as τ increases away from τ^*.

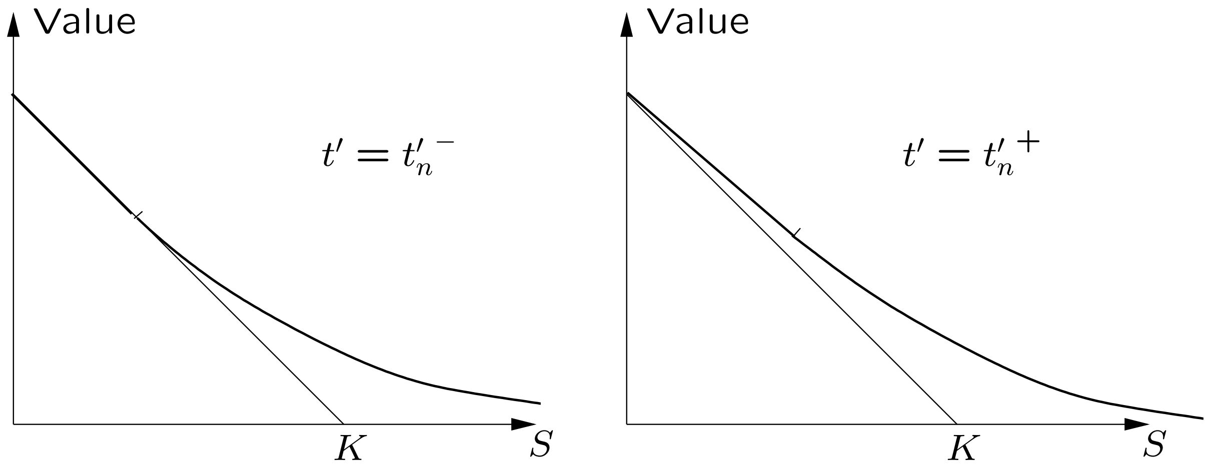
This is the 'initial' condition for a free boundary problem on $x^*(\tau) < x < \infty$, $\tau^* < \tau < 1$. It is the Oxygen Consumption Problem

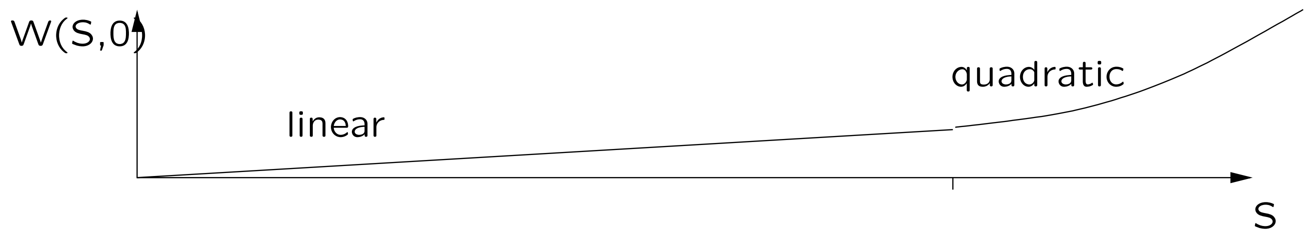
$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} - \rho K, \quad w \geq 0, \qquad \tau > \tau^*, \quad x^*(\tau) < x < \infty$$

with initial data

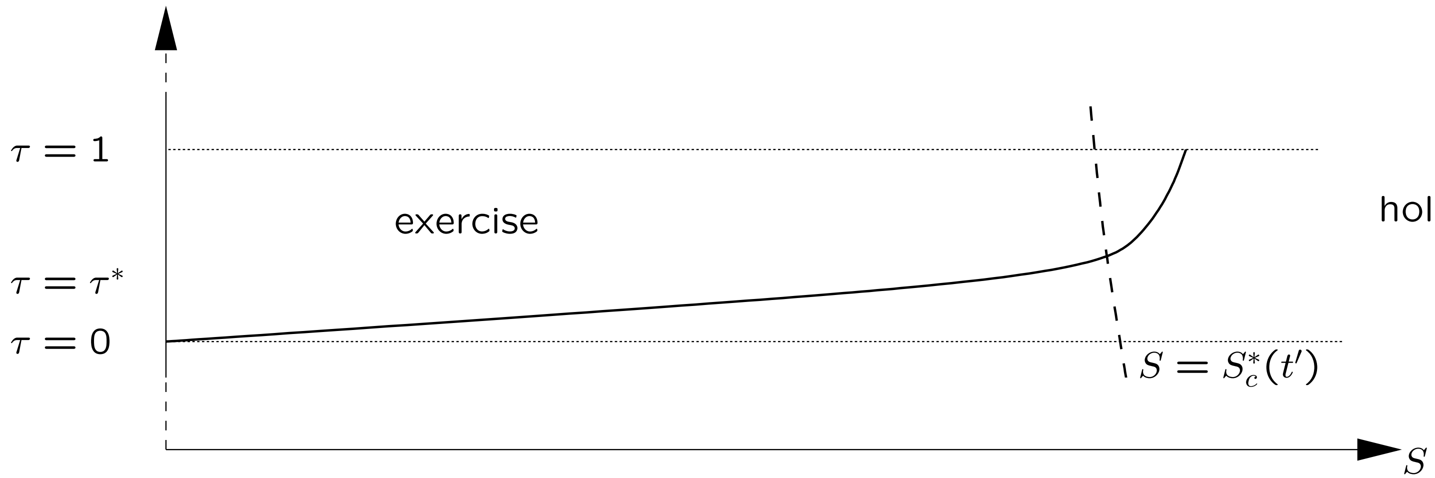
$$w(x, \tau^*) \sim \begin{cases} \text{constant} \times x^{-3} e^{-x^2/2\tau^*} & x \to -\infty \\ \text{constant} \times x^2 & x \to +\infty. \end{cases}$$

At $\tau = 1$ it all starts again...

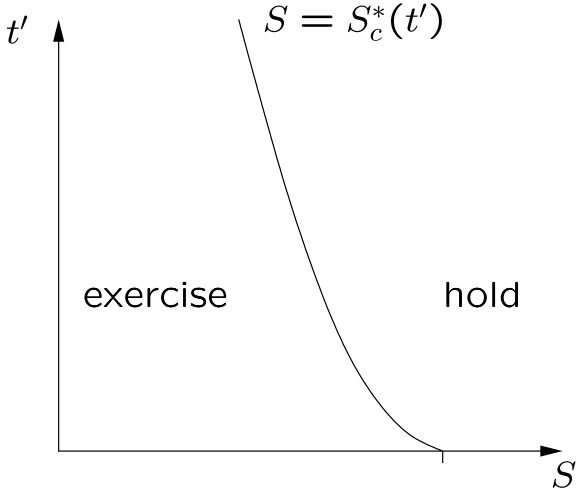

Value
$t' = t_n'^{\,-}$
K
S
Value
$t' = t_n'^{\,+}$
K
S



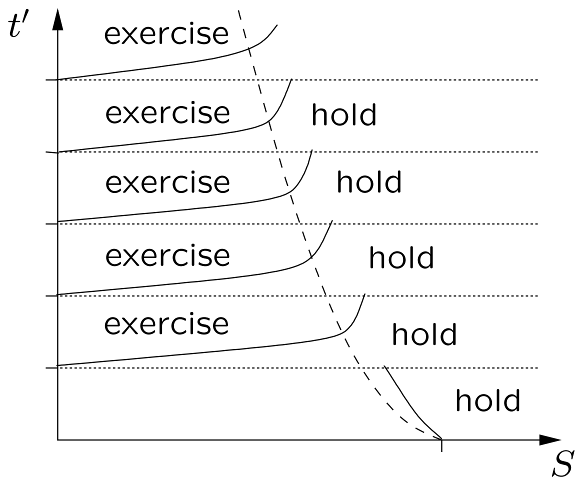
W(S,0)
quadratic
linear
S



$\tau = 1$
exercise
hold
$\tau = \tau^*$
$\tau = 0$
$S = S_c^*(t')$
S



continuous dividend



discrete dividend