Optimal investment timing under financing constraint

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Related papers

Investment decision problem for

- unconstrained all-equity financed firm (standard model)
  - McDonald and Siegel (1986, *QJE*, hereafter MS)
- constrained all-equity financed firm
  - Boyle and Guthrie (2003, *J. Finance*, hereafter, BG): Financial constraint makes the firm accelerate the investment
- unconstrained debt-equity financed firm
  - Sundaresan and Wang (2007, *AER*, hereafter SW): Debt financing makes the firm accelerate the investment
- constrained debt-equity financed firm (our model)
  - Based on SW (2007), we consider the debt capacity constraint

Contents

Model setup

- $(X_t)_{t\geq 0}$: price of the product

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}z_t, \; X_0 = x > 0, \tag{1}$$

  where $\mu > 0$, $\sigma > 0$, and $(z_t)_{t\geq 0}$: standard Brownian motion.
- Πx: after-tax present value of the project

$$\Pi x := \mathbb{E}\Big[\int_0^{+\infty} \mathrm{e}^{-rt}(1-\tau)QX_t \mathrm{d}t\Big] = \frac{1-\tau}{r-\mu}Qx$$

  - $r > 0$: risk-neutral discount factor
  - $\tau > 0$: tax rate
  - $Q > 0$: quantity of the product

- Investment timing:

$$T_{\mathrm{U}}^{\mathrm{i}} := \inf\{t \geq 0; X_t \geq x_{\mathrm{U}}^{\mathrm{i}}\},$$

where $x_{\mathrm{U}}^{\mathrm{i}}$: investment trigger.
  - superscript "i" stands for investment strategy.
  - subscript "U" stands for unlevered (all-equity financed) firm .
- Equity option value:

$$E_{\mathrm{U}}^{\mathrm{o}}(x) := \sup_{T_{\mathrm{U}}^{\mathrm{i}}} \mathbb{E}\left[\mathrm{e}^{-rT_{\mathrm{U}}^{\mathrm{i}}}(\Pi X_{T_{\mathrm{U}}^{\mathrm{i}}} - I)\right] \tag{2}$$

  - superscript "o" stands for the option value before investment
  - $I > 0$: one-time cost expenditure at investment

[P1] All-equity financing problem

Optimization problem under all-equity financing

$$\max_{x_{\mathrm{U}}^{\mathrm{i}}} \left(\frac{x}{x_{\mathrm{U}}^{\mathrm{i}}}\right)^{\beta}(\Pi x_{\mathrm{U}}^{\mathrm{i}} - I), \tag{3}$$

where $x < x_{\mathrm{U}}^{\mathrm{i}}$ and

$$\beta := 1/2 - \mu/\sigma^2 + \sqrt{(\mu/\sigma^2 - 1/2)^2 + 2r/\sigma^2} > 1. \tag{4}$$

- subscript "U" represents the unlevered firm

Proof: $E_{\mathrm{U}}^{\mathrm{o}}(x)$ satisfies

$$\frac{1}{2}\sigma^2 x^2 E_{\mathrm{U}}^{\mathrm{o}\,\prime\prime}(x) + \mu x E_{\mathrm{U}}^{\mathrm{o}\,\prime}(x) - rE_{\mathrm{U}}^{\mathrm{o}}(x) + x = 0 \tag{5}$$

subject to

$$E_{\mathrm{U}}^{\mathrm{o}}(0) = 0, \quad E_{\mathrm{U}}^{\mathrm{o}}(x_{\mathrm{U}}^{\mathrm{i}}) = \Pi x_{\mathrm{U}}^{\mathrm{i}} - I \tag{6}$$

Solution and optimal value

$$x_{\mathrm{U}}^{\mathrm{i}*} = \frac{\beta}{\beta - 1}\frac{1}{\Pi}I, \qquad E_{\mathrm{U}}^{\mathrm{o}}(x) = \left(\frac{x}{x_{\mathrm{U}}^{\mathrm{i}*}}\right)^{\beta}\frac{1}{\beta - 1}I \tag{7}$$

- superscript "$*$" denotes the optimum
- The investment trigger $x_{\mathrm{U}}^{\mathrm{i}*}$ is a linear function of I
- $T_{\mathrm{U}}^{\mathrm{i}*} = \inf\{t \geq 0 : X_t \geq x_{\mathrm{U}}^{\mathrm{i}*}\}$
- This problem is the simple version of McDonald and Siegel (1986, *QJE*)

[P1]′ All-equity financing problem with constraint

- P1: unconstrained (simplified) standard problem:

$$\sup_{T^{\mathrm{i}}} \mathbb{E}[\mathrm{e}^{-rT^{\mathrm{i}}}(X_{T^{\mathrm{i}}} - I)], \text{ s.t. } \mathrm{d}X_t/X_t = \mu \mathrm{d}t + \sigma \mathrm{d}z_t \tag{8}$$

- P1′: constrained problem:
  Simplified version in Boyle and Guthrie, (2003, *J. Finance*)

$$\sup_{T^{\mathrm{i}}} \quad \mathbb{E}[\mathrm{e}^{-rT^{\mathrm{i}}}(X_{T^{\mathrm{i}}} - I)] \tag{9}$$

$$\text{s.t} \quad \mathrm{d}X_t/X_t = \mu \mathrm{d}t + \sigma \mathrm{d}z_t^{(1)} \tag{10}$$

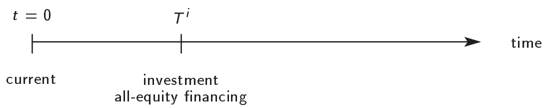
$$\mathrm{d}Y_t = rY_t\mathrm{d}t + \phi \mathrm{d}z_t^{(2)}, \ \mathrm{d}z^{(1)}\mathrm{d}z^{(2)} = \rho \mathrm{d}t \tag{11}$$

$$I \leq Y_t + \alpha X_t, \quad \alpha \in [0, 1) \tag{12}$$

where Y denotes cash stock.

Timeline under debt-equity financing

- Unlevered firm (all-equity financed firm):
  $T^{\mathrm{i}} = \inf\{t \geq 0; X_t \geq x^{\mathrm{i}}\}$



- Levered firm (debt-equity financed firm):
  $T^{\mathrm{i}} = \inf\{t \geq 0; X_t \geq x^{\mathrm{i}}\}, \; c, \; T^{\mathrm{d}} = \inf\{t \geq T^{\mathrm{i}}; X_t \leq x^{\mathrm{d}}\}$,

  - superscript "i" stands for investment strategy.
  - superscript "d" stands for default strategy.
  - c denotes the coupon payment

Equity value after investment

$E^{\mathrm{a}}(x, c)$: Equity value after investment:

- superscript "a" stands for the value after investment

$$E^{\mathrm{a}}(X_t, c) = \sup_{T^{\mathrm{d}}} \mathbb{E}_t \Big[\int_t^{T^{\mathrm{d}}} \mathrm{e}^{-r(u-t)} (1-\tau)(QX_u - c) \mathrm{d}u \Big] \quad (13)$$

where c: coupon payment. Here, $E^{\mathrm{a}}(X_t, c)$ is rewritten as

$$E^{\mathrm{a}}(X_t, c) = \max_{x^{\mathrm{d}}} \Pi X_t - (1-\tau)\frac{c}{r} - \{\Pi x^{\mathrm{d}} - (1-\tau)\frac{c}{r}\}\Big(\frac{X_t}{x^{\mathrm{d}}}\Big)^{\gamma} \quad (14)$$

where

$$\gamma := 1/2 - \mu/\sigma^2 - \sqrt{(\mu/\sigma^2 - 1/2)^2 + 2r/\sigma^2} < 0. \quad (15)$$

Optimal default trigger: Maximizing (14) with x^{d} gives

$$x^{\mathrm{d}}(c) = \kappa^{-1} c, \quad (16)$$

where

$$\kappa = \frac{\gamma - 1}{\gamma} \frac{\Pi}{1-\tau} r \quad (17)$$

- Default trigger $x^{\mathrm{d}}(c)$ is a linear function of the coupon payment c
- $\lim_{c \downarrow 0} x^{\mathrm{d}}(c) = 0$
- This result is given by Black and Cox (1976, *J. Finance*).

Debt value after investment

$D^{\mathrm{a}}(x, c)$: Debt value after investment:

$$D^{\mathrm{a}}(X_t, c) = \mathbb{E}_t \Big[\int_t^{T^{\mathrm{d}}} \mathrm{e}^{-r(u-t)} c \mathrm{d}u + \mathrm{e}^{-r(T^{\mathrm{d}}-t)} (1-\alpha)\Pi x^{\mathrm{d}}(c) \Big] \quad (18)$$

where $\alpha \in (0,1)$: bankruptcy cost. Here, $D^{\mathrm{a}}(X_t, c)$ is written as

$$D^{\mathrm{a}}(X_t, c) = \frac{c}{r} - \{\frac{c}{r} - (1-\alpha)\Pi x^{\mathrm{d}}(c)\}\Big(\frac{X_t}{x^{\mathrm{d}}(c)}\Big)^{\gamma}, \quad (19)$$

$V^{\mathrm{a}}(X_t, c)$: Total firm value after investment:

$$\begin{aligned} V^{\mathrm{a}}(X_t, c) &= E^{\mathrm{a}}(X_t, c) + D^{\mathrm{a}}(X_t, c) \\ \underbrace{}_{\text{Levered value}} &= \underbrace{\Pi X_t}_{\text{Unlevered value}} + \underbrace{\tau \frac{c}{r}\Big(1 - \Big(\frac{X_t}{x^d(c)}\Big)^{\gamma}\Big)}_{\text{Tax benefit}} - \underbrace{\alpha \Pi x^{\mathrm{d}}(c)\Big(\frac{X_t}{x^{\mathrm{d}}(c)}\Big)^{\gamma}}_{\text{Bankruptcy cost}}, \quad (20) \end{aligned}$$

- $\lim_{c \downarrow 0} D^{\mathrm{a}}(X_t, c) = 0$ and $\lim_{c \downarrow 0} V^{\mathrm{a}}(X_t, c) = \Pi X_t$

[P2] Debt-equity financing problem

Optimization problem of unconstrained debt-equity financed firm

$$\max_{x_N^i, c_N} \quad \left(\frac{x}{x_N^i}\right)^\beta \left(E^a(x_N^i, c_N) - (I - D^a(x_N^i, c_N)) \right), \quad (21)$$

$$\text{subject to} \quad x_N^i \geq 0, c_N \geq 0, \quad (22)$$

- subscript "N" represents the non-constrained problem
- This problem is the same as the simple version in Sundaresan and Wang (2007, *AER*).
- The problem by removing the optimization with x_N^i is the problem in Leland (1994, *J.Finance*).

Simplified optimization problem:

$$\max_{x_N^i} \quad \left(\frac{x}{x_N^i}\right)^\beta (\psi^{-1}\Pi x_N^i - I), \quad (23)$$

where $X_0 = x$ and

$$\psi = \left(1 + \frac{1}{h}\frac{\tau}{1-\tau}\right)^{-1} \leq 1 \quad (24)$$

$$h = \left(1 - \gamma\left(1 - \alpha + \frac{\alpha}{\tau}\right)\right)^{-1/\gamma} \geq 1. \quad (25)$$

Proof:

$$c_N(X_t) := \operatorname*{argmax}_c \{(x/x_N^i)^\beta (V^a(X_t, c) - I)\} = \frac{\kappa}{h} X_t, \quad (26)$$

where κ is defined by (17). Substituting (26) into the objective function (21) gives

$$\underbrace{E^a(x, c_N(x)) + D^a(x, c_N(x))}_{=V^a(x, c_N(x))} - I = \psi^{-1}\Pi x - I \quad (27)$$

Optimal solution:

$$x_N^{i*} = \psi x_U^{i*} \leq x_U^{i*}, \quad c_N^* = \frac{\kappa}{h}\psi x_U^{i*}, \quad x_N^{d*} = \frac{1}{h}\psi x_U^{i*}$$

- All the solutions are linear functions of I

Optimal value:

$$E_N^o(x) := \left(\frac{x}{x_N^{i*}}\right)^\beta (V^a(x_N^{i*}, c_N^*) - I) = \left(\frac{x}{\psi x_U^{i*}}\right)^\beta \frac{1}{\beta - 1} I$$

$$\geq \left(\frac{x}{x_U^{i*}}\right)^\beta \frac{1}{\beta - 1} I = E_U^o(x) \quad (28)$$

- Note that $\psi \leq 1$ and $\beta > 1$.
- These results are obtained by SW (2007)

[P3] Debt-equity financing problem with constraint

Optimization problem of constrained debt-equity financed firm

$$\max_{x_C^i, c_C} \quad \left(\frac{x}{x_C^i}\right)^\beta \left(E^a(x_C^i, c_C) - (I - D^a(x_C^i, c_C)) \right), \quad (29)$$

$$\text{subject to} \quad D^a(x_C^i, c_C) \leq qI, x_C^i \geq 0, c_C \geq 0, \quad (30)$$

where $q \in [0, 1]$.

- subscript "C" represents the constrained problem
- This financing constraint represents the limited debt issuance.
  - When $q = 0$, [P3] becomes [P1].
  - When q is a sufficient large, [P3] turns out to be [P2].

  Thus, our problem [P3] includes two problems [P1] and [P2].

Our conjecture for [P3]

- We have already obtained in the standard investment theory:

$$x_{\rm N}^{\rm i*} \leq x_{\rm U}^{\rm i*}, \quad E_{\rm N}^{\rm o}(x) \geq E_{\rm U}^{\rm o}(x)$$

- In our intuitive conjecture based the above results, the constrained firm has the following properties:

$$x_{\rm N}^{\rm i*} \leq x_{\rm C}^{\rm i*} \leq x_{\rm U}^{\rm i*}, \quad E_{\rm N}^{\rm o}(x) \geq E_{\rm C}^{\rm o}(x) \geq E_{\rm U}^{\rm o}(x).$$

The results are...

We define a critical value x_1 by

$$D^{\rm a}(x_1, c_{\rm N}(x_1)) = qI. \tag{31}$$

Since $x^{\rm d}(c_{\rm N}(x)) = (1/h)x$ in (16) and $c_{\rm N}(x) = (\kappa/h)x$ in (26),

$$D^{\rm a}(x, c_{\rm N}(x)) = \frac{c_{\rm N}(x)}{r} - \Big\{\frac{c_{\rm N}(x)}{r} - (1-\alpha)\Pi x^{\rm d}(c_{\rm N}(x))\Big\}\Big(\frac{x}{x^{\rm d}(c_{\rm N}(x))}\Big)^{\gamma}$$

is strictly monotonically increasing continuous function of x with $\lim_{x\to 0} D^{\rm a}(x, c_{\rm N}(x)) = 0$ and $\lim_{x\to +\infty} D^{\rm a}(x, c_N(x)) = +\infty$. So, there exists an unique x_1.
Then, using x_1, we can examine whether or not the debt capacity constraint is binding:

- If $x_{\rm N}^{\rm i*} > x_1$, the firm is financially constrained
- Otherwise ($x_{\rm N}^{\rm i*} \leq x_1$), the firm is not financially constrained

Optimal solution:
If $x_{\rm N}^{\rm i*} > x_1$, $(x_{\rm C}^{\rm i*},\ c_{\rm C}^{*})$ are decided by

$$\begin{cases} \Big\{(1-\beta)\Pi x_{\rm C}^{\rm i} - \beta\frac{\tau c_{\rm C}}{r} + (\beta-\gamma)\Big(\frac{x_{\rm C}^{\rm i}}{x^{\rm d}(c_{\rm C})}\Big)^{\gamma}\Big(\frac{\tau c_{\rm C}}{r} + \alpha\Pi x^{\rm d}(c_{\rm C}) + \beta I\Big)\Big\} \\ \qquad \times\Big\{\gamma\Big(\frac{x_{\rm C}^{\rm i}}{x^{\rm d}(c_{\rm C})}\Big)^{\gamma}\Big(\frac{c_{\rm C}}{r} - (1-\alpha)\Pi x^{\rm d}(c_{\rm C})\Big\}^{-1} \\ \qquad -\Big\{\frac{\tau}{c} - \Big(\frac{x_{\rm C}^{\rm i}}{x^{\rm d}(c_{\rm C})}\Big)^{\gamma}\Big(\frac{\tau}{r} + \alpha\Pi x^{\rm d}(c_{\rm C})c_{\rm C}^{-1}\Big)(1-\gamma)\Big\} \\ \qquad \times\Big\{\frac{1}{r} + \Big(\frac{x_{\rm C}^{\rm i}}{x^{\rm d}(c_{\rm C})}\Big)^{\gamma}\Big(\frac{1}{r} - (1-\alpha)\Pi x^{\rm d}(c_{\rm C})c_{\rm C}^{-1}\Big)(1-\gamma)\Big\}^{-1} = 0 \\ \frac{c_{\rm C}}{r} - \Big\{\frac{c_{\rm C}}{r} - (1-\alpha)\Pi x^{\rm d}(c_{\rm C})\Big\}\Big(\frac{x_{\rm C}^{\rm i}}{x^{\rm d}(c_{\rm C})}\Big)^{\gamma} - qI = 0 \end{cases}$$

and $x_{\rm C}^{\rm d*} = x^{\rm d}(c_{\rm C}^{*})$ is obtained by substituting $c_{\rm C}^{*}$ into (16).
Otherwise, we have $(x_{\rm C}^{\rm i*},\ c_{\rm C}^{*},\ x_{\rm C}^{\rm d*}) = (x_{\rm N}^{\rm i*}, c_{\rm N}^{*}, x_{\rm N}^{\rm d*})$.

Optimal values:

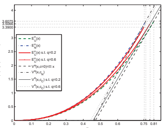
$$E_{\rm C}^{\rm o}(x) := \Big(\frac{x}{x_{\rm C}^{\rm i*}}\Big)^{\beta}\big(V^{\rm a}(x_{\rm C}^{\rm i*}, c_{\rm C}^{*}) - I\big) \leq E_{\rm N}^{\rm o}(x) \tag{32}$$

Numerical examples

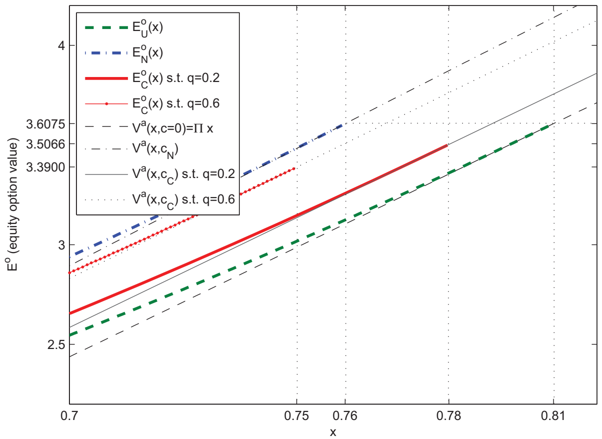
Basic parameters

$$r = 0.09; \mu = 0.01; I = 5; \tau = 0.15; \alpha = 0.4; X_0 = x = 0.5;$$

Equity option value and investment trigger

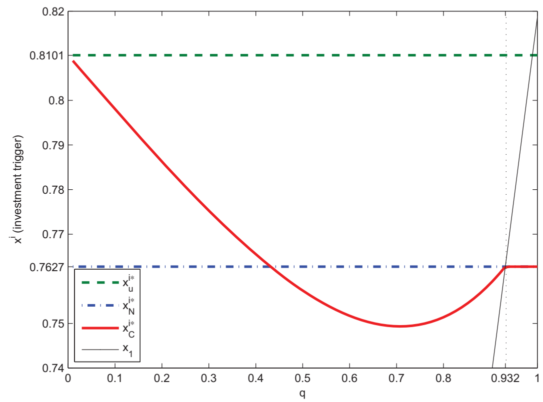


trigger
$x_{\mathrm{U}}^{i*} = 0.81$
$x_{\mathrm{N}}^{i*} = 0.76$
$x_{\mathrm{C}}^{i*} = 0.78$
under $q = 0.2$
$x_{\mathrm{C}}^{i*} = 0.75$
under $q = 0.6$



- x_{C}^{i*} $(q = 0.6) \leq x_{\mathrm{N}}^{i*} \leq x_{\mathrm{C}}^{i*}$ $(q = 0.2) \leq x_{\mathrm{U}}^{i*}$
- $E_{\mathrm{N}}^{o}(x) \geq E_{\mathrm{C}}^{o}(x) \geq E_{\mathrm{U}}^{o}(x)$

Investment trigger with financial constraint parameter q



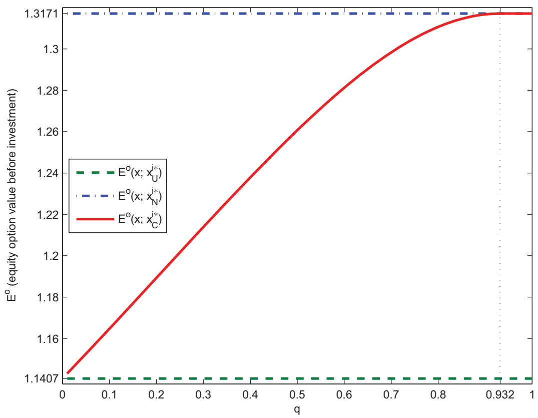
As $q \uparrow$,

(i) $x_{\mathrm{C}}^{i} \downarrow$ (via $D^a \uparrow$)

(ii) $x_{\mathrm{C}}^{i} \uparrow$ (via $c \uparrow$)

- If $q < 0.932$ (due to $x_1 < x_{\mathrm{N}}^{i*}$), debt financing is constrained.
- The investment trigger is non-monotonic (convex) with q. (this result is similar to in Boyle and Guthrie, 2003, *J.Finance*)

Equity option value with financial constraint parameter q



- $E_{\mathrm{N}}^{o}(x) \geq E_{\mathrm{C}}^{o}(x) \geq E_{\mathrm{U}}^{o}(x)$
- The equity option value is monotonically increasing with q

Concluding remarks

- We extend the optimal investment timing problem under debt-equity financing by incorporating the debt capacity constraint.
- Our conjecture 1: $x_{\mathrm{N}}^{i*} \leq x_{\mathrm{C}}^{i*} \leq x_{\mathrm{U}}^{i*}$:
  - The constrained investment trigger may be smaller than the unconstrained one.
  - The constrained investment trigger is non-monotonic in the financial friction parameter.
- Our conjecture 2: $E_{\mathrm{N}}^{o}(x) \geq E_{\mathrm{C}}^{o}(x) \geq E_{\mathrm{U}}^{o}(x)$