

Positive Stochastic Volatility Simulation

Simon J.A. Malham & Anke Wiese

Heriot-Watt University, Edinburgh

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Introduction: The Heston Model

$$\begin{aligned}dS_t &= \mu S_t dt + \sqrt{V_t} S_t (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2), \\dV_t &= \kappa(\theta - V_t) dt + \varepsilon \sqrt{V_t} dW_t^1,\end{aligned}$$

where

- ▶ W_t^1 and W_t^2 independent scalar Wiener processes
- ▶ μ , κ , θ and ε are positive constants
- ▶ $\rho \in (-1, 1)$
- ▶ S price process of underlying variable (e.g. stock index, exchange rate)
- ▶ V variance process.

Heston (1993), Cox, Ingersoll & Ross (1985), Feller (1951)

Properties of V

- ▶ $V_t \geq 0$ (assuming $V_0 \geq 0$).

- ▶ Let

$$\nu := 4\kappa\theta/\varepsilon^2,$$

Then

- ▶ If $\nu \geq 2$, then V is strictly positive.
- ▶ If $\nu < 2$, then the zero boundary is attainable and instantaneously reflecting.
- ▶ Attainability of zero boundary and reflection property are major obstacle in computational treatment.
- ▶ $\nu < 2$ is relevant for foreign exchange and long-dated interest-rate markets (Andersen 2008).

Here: focus on attainable zero boundary case, in particular $\nu < 1$.

Euler-Maruyama Scheme

Across the computation interval $[t_n, t_{n+1}]$ with $h = t_{n+1} - t_n$:

$$\hat{V}_{t_{n+1}} = \hat{V}_{t_n} + \kappa(\theta - \hat{V}_{t_n})h + \sqrt{\hat{V}_{t_n}}\Delta W_{t_n}^1.$$

Problem:

$$P\{\hat{V}_{t_{n+1}} < 0 | \hat{V}_{t_n}\} > 0.$$

Modifications of Euler-Maruyama Scheme

Extend vector fields to negative domain. e.g.

- ▶ Partial truncation (Delbaen and Deelstra 1998)

$$\hat{V}_{t_{n+1}} = \hat{V}_{t_n} + h\kappa(\theta - \hat{V}_{t_n}) + \varepsilon\Delta W_{t_n}^1 \sqrt{\hat{V}_{t_n}^+}.$$

- ▶ Reflection (Bossy and Diop 2007)

$$\hat{V}_{t_{n+1}} = |\hat{V}_{t_n}| + h\kappa(\theta - |\hat{V}_{t_n}|) + \varepsilon\Delta W_{t_n}^1 \sqrt{|\hat{V}_{t_n}|}.$$

- ▶ Full truncation (Lord, Koekoek & Van Dijk 2006)

$$\hat{V}_{t_{n+1}} = \hat{V}_{t_n} + h\kappa(\theta - \hat{V}_{t_n}^+) + \varepsilon\Delta W_{t_n}^1 \sqrt{\hat{V}_{t_n}^+}.$$

Full truncation works well, but properties (e.g. error) are difficult to derive.

Transition Probability for V

Scaled non-central chi-squared distribution:

$$\mathbb{P}(V_{t_{n+1}} < x \mid V_{t_n}) = F_{\chi_{\nu}^2(\lambda)}(x \cdot \eta(h) / \exp(-\kappa h)),$$

where

- ▶ $F_{\chi_{\nu}^2(\lambda)}$ non-central chi-squared distribution function with ν degrees of freedom and non-centrality parameter λ

$$F_{\chi_{\nu}^2(\lambda)}(z) = \frac{e^{-\lambda/2}}{2^{\nu/2}} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j! 2^j \Gamma(\nu/2 + j)} \int_0^z \xi^{\nu/2+j-1} e^{-\xi/2} d\xi,$$

- ▶ $\nu := 4\kappa\theta/\varepsilon^2,$
- ▶ $\eta(h) := \frac{4\kappa \exp(-\kappa h)}{\varepsilon^2 (1 - \exp(-\kappa h))},$
- ▶ $\lambda = V_{t_n} \eta(h).$

Properties of Chi-Square Distribution

▶ *Dealing with Non-Centrality*

$$\chi_{\nu}^2(\lambda) = \chi_{\nu+2N}^2,$$

where N is Poisson distributed with parameter $\lambda/2$. (Johnson 1959, Glasserman 2003)

▶ *Divisibility of Chi-Squared Distribution: Assume*

- ▶ $Y_1, Y_2, \dots, Y_{2N}, Z$ independent,
- ▶ Y_i standard Normally distributed, $i = 1, \dots, 2N$,
- ▶ Z χ_{ν}^2 -distributed.

Then

$$\sum_{i=1}^{2N} Y_i^2 + Z \sim \chi_{\nu+2N}^2.$$

Questions:

- ▶ How to simulate a χ_{ν}^2 random variable for non-integer $\nu < 1$?
- ▶ Is there a representation for a χ_{ν}^2 variable with non-integer $\nu < 1$ similar to the integer degrees of freedom case?

Generalized Gaussian Distribution

Definition: A generalized $N(0, 1, q)$ random variable, for $q \geq 1$, has density

$$f_{N(0,1,q)}(x) := \frac{q}{2^{1/q+1}\Gamma(1/q)} \cdot \exp\left(-\frac{1}{2}|x|^q\right),$$

where $x \in \mathbb{R}$ and $\Gamma(\cdot)$ is the standard gamma function.

Note that for $q = 2$, we recover the Normal distribution.

(Gupta & Song 1997, Song & Gupta 1997, Sinz, Gerwinn & Bethge 2009, Sinz & Bethge 2008)

Representation of Chi-Square by Generalized Gaussian

Theorem:

Suppose $X_i \sim N(0, 1, 2q)$ are independent identically distributed random variables for $i = 1, \dots, p$, where $q \geq 1$ and $p \in \mathbb{N}$. Then we have

$$\sum_{i=1}^p |X_i|^{2q} \sim \chi_{p/q}^2.$$

Idea of Proof

If $X \sim N(0, 1, 2q)$, then

$$\begin{aligned}\mathbb{P}(|X|^{2q} < x) &= \frac{2q}{2^{1/2q}\Gamma(1/2q)} \int_0^{|x|^{1/2q}} \exp\left(-\frac{1}{2}|\tau|^{2q}\right) d\tau \\ &= \frac{1}{2^{1/2q}\Gamma(1/2q)} \int_0^x \xi^{1/2q-1} \exp\left(-\frac{1}{2}\xi\right) d\xi\end{aligned}$$

where $\xi = |\tau|^{2q}$. Hence

$$|X|^{2q} \sim \chi_{1/q}^2.$$

Now using that the sum of p independent identically distributed $\chi_{1/q}^2$ random variables have a $\chi_{p/q}^2$ distribution establishes the result.

Generalized Marsaglia Polar Method

Theorem:

Suppose for some $q \in \mathbb{N}$ that U_1, \dots, U_q are independent identically distributed uniform random variables over $[-1, 1]$. Condition this sample set to satisfy the requirement $\|U\|_q < 1$, where $\|U\|_q$ is the q -norm of $U = (U_1, \dots, U_q)$. Then the q random variables generated by $U \cdot (-2 \log \|U\|_q^q)^{1/q} / \|U\|_q$ are independent $N(0, 1, q)$ distributed random variables.

Proof: Calculate density.

Remark: $q = 2$: Marsaglia's Polar Method for Normal distribution.

Summary: Algorithm

Assume $\nu = \frac{p}{q}$ with p and q natural numbers (no loss of generality). To produce an exact $\chi_{p/q}^2$ sample:

1. Generate $2q$ independent uniform random variables over $[-1, 1]$: $U = (U_1, \dots, U_{2q})$.
2. If $\|U\|_{2q} < 1$ continue, otherwise repeat Step 1.
3. Compute $Z = U \cdot (-2 \log \|U\|_{2q}^{2q})^{1/2q} / \|U\|_{2q}$. This gives $2q$ independent $N(0, 1, 2q)$ random variables $Z = (Z_1, \dots, Z_{2q})$.
4. Compute $Z_1^{2q} + \dots + Z_p^{2q} \sim \chi_{p/q}^2$.

Probability of Acceptance

Probability of acceptance in each attempt:

$$p_M = \left(\frac{\Gamma(1/2q)}{2q} \right)^{2q}$$

- ▶ $q = 1$: probability of acceptance is 0.7854 (sample from Gaussian distribution)
- ▶ As $q \rightarrow \infty$, we have $p_M \rightarrow e^{-\gamma} \approx 0.5615$, where γ is the Euler-Mascheroni constant.
- ▶ In each accepted attempt, $2q$ independent generalized Gaussian variables are generated, of which $p < q$ are used to generate one $\chi_{p/q}^2$ variable.
- ▶ Expected number of attempts to generate $2q/p$ independent $\chi_{p/q}^2$ variables is

$$1/p_M \in [1.2732, 1.7809].$$

Acceptance-Rejection Method

Acceptance-Rejection methods:

- ▶ Density f , we wish to sample from.
- ▶ Assume, there exist a density g and a constant c such that

$$f(x) \leq c g(x) \quad \forall x.$$

- ▶ Method:
 - ▶ Sample from g : X .
 - ▶ Accept X with probability $f(X)/c g(X)$.
 - ▶ Probability of acceptance = $1/c$.
 - ▶ If accepted, then density of accepted sample is f .

(Glasserman 2003)

Comparison with Acceptance-Rejection Method

Ahrens & Dieter: acceptance-rejection method with mixture of prior densities

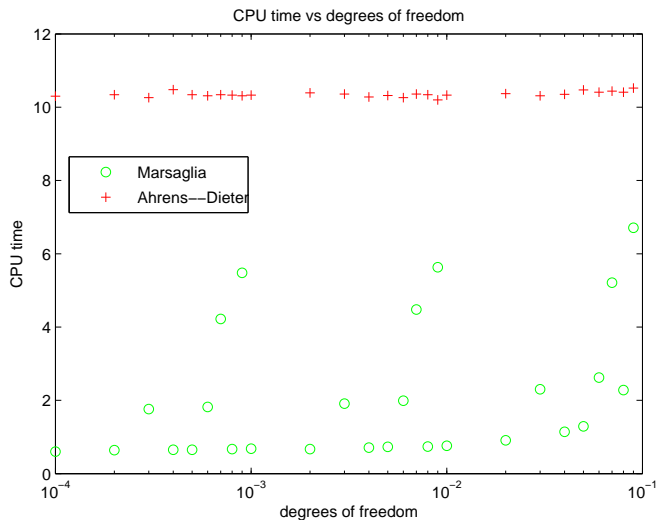
- ▶ $(p/2q)x^{p/2q-1}$ on $[0, 1]$ with weight $e/(e + p/2q)$,
- ▶ $\exp(1 - x)$ on $[1, \infty)$ with weight $(p/2q)(e + p/2q)$.
- ▶ Expected number of steps to generate $2q/p$ independent $\chi_{p/q}^2$ variables is

$$(2q/p) \cdot \frac{p/2q + e}{p/2q\Gamma(p/2q)e}.$$

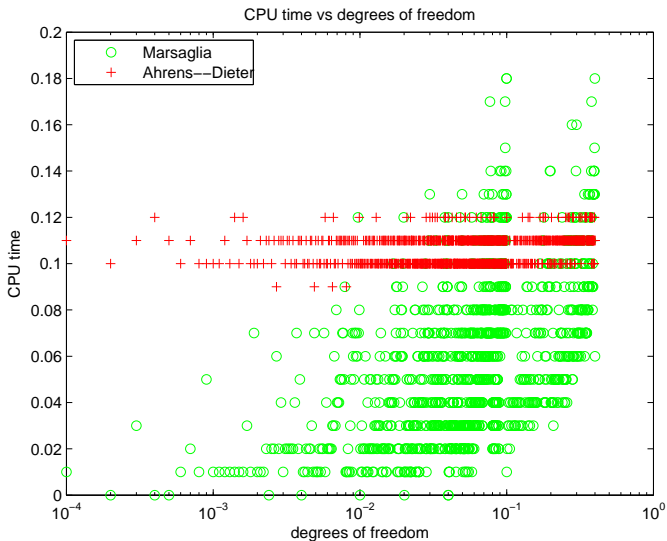
This is

- ▶ larger than expected number of steps in generalized Marsaglia method for all $p/q < 1$.
- ▶ unbounded.
- ▶ Computational effort (CPU time): generalized Marsaglia method compares very favourably with acceptance-rejection method (see overleaf).

CPU Time vs Degrees of Freedom – 1 Digit



CPU Time vs Degrees of Freedom – 3 Significant Digits



Andersen's Distribution Approximation

- ▶ If \hat{V}_{t_n} is large:

$$\hat{V}_{t_{n+1}} = (a + bZ)^2,$$

where $Z \sim N(0, 1)$.

- ▶ If \hat{V}_{t_n} is small: replace true density with mixture of Dirac delta function and exponential density

$$p\delta(0) + (1 - p)\beta \exp(-\beta x),$$

where $\delta(0)$ is the Dirac delta function and p and β are constants.

Parameters are chosen to match expected value and variance.
(Andersen 2008)

Simulating S

Recall

$$\begin{aligned}dS_t &= \mu S_t dt + \sqrt{V_t} S_t (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2), \\dV_t &= \kappa(\theta - V_t) dt + \varepsilon \sqrt{V_t} dW_t^1.\end{aligned}$$

Solving for S :

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2} \int_0^t V_s ds + \rho \int_0^t \sqrt{V_s} dW_s^1 + \sqrt{1 - \rho^2} \int_0^t \sqrt{V_s} dW_s^2\right).$$

Conditional Distribution of S

Given $\int_0^t \sqrt{V_s} dW_s^1$ and $\int_0^t V_s ds$

$$\log(S_t/S_0) \sim N\left(\mu t - \frac{1}{2} \int_0^t V_s ds + \rho \int_0^t \sqrt{V_s} dW_s^1, (1-\rho^2) \int_0^t V_s ds\right).$$

Note that

$$dV_t = \kappa(\theta - V_t) dt + \varepsilon \sqrt{V_t} dW_t^1$$

implies

$$\int_0^t \sqrt{V_s} dW_s^1 = \frac{1}{\varepsilon} (V_t - V_0 - \kappa\theta t + \kappa \int_0^t V_s ds).$$

(Broadie & Kaya 2006)

Trapezoidal Rule

Task: Simulate $(V_{t_{n+1}} - V_{t_n}, \int_{t_n}^{t_{n+1}} V_s ds)$

- ▶ Laplace transform of $\int_{t_n}^{t_{n+1}} V_s ds$ given $V_{t_{n+1}}$ and V_{t_n} (Pitman & Yor 1982, Broadie & Kaya 2006)
- ▶ Representation as infinite sums and mixtures of independent Gamma-distributed random variables (Glasserman & Kim 2009)
- ▶ Trapezoidal rule (Andersen 2007): approximation of time integral by

$$\frac{1}{2}(V_{t_{n+1}} + V_{t_n})(t_{n+1} - t_n).$$

Require martingale: $e^{-\mu(t_{n+1}-t_n)} E[S_{t_{n+1}} | (V_{t_n}, S_{t_n})] = S_{t_n}$
 \rightsquigarrow adjustment by multiplicative factor

$$\exp(K_0(t_{n+1} - t_n) + K_1 V_{t_n}).$$

Test Cases

	Case I	Case II
ϵ	1	0.9
κ	0.5	0.3
ρ	-0.9	-0.5
T	10	15
$V(0), \theta$	0.04	0.04
$4\kappa\theta/\epsilon^2$	8/100	48/810

Table: Test cases from Andersen. In all cases $S(0) = 100$.

Test cases are “challenging and practically relevant”

- ▶ Case I representative of long-dated FX option market,
- ▶ Case II representative of long-dated interest-rate option market.

(Andersen 2008, p. 26.)

Numerical Results Case I: Error and Sdev

h	Andersen		Marsaglia
		Strike 100	
1	0.2211 (0.012)		-0.2374 (0.013)
1/2	0.1164 (0.013)		-0.0707 (0.013)
1/4	0.0143* (0.013)		-0.0440 (0.013)
1/8	-0.0277* (0.013)		-0.0050* (0.013)
1/16	0.0162* (0.013)		0.0019* (0.013)
		Strike 140	
1	-0.0883 (0.002)		-0.0283 (0.002)
1/2	-0.0274 (0.003)		-0.0121 (0.002)
1/4	-0.0013 (0.003)		-0.0048 (0.003)
1/8	0.0047 (0.003)		-0.0011 (0.003)
1/16	0.0018 (0.003)		0.0015 (0.003)
		Strike 60	
1	0.0317* (0.025)		-0.1234 (0.025)
1/2	0.0345* (0.025)		-0.0556* (0.025)
1/4	0.0111* (0.025)		-0.0388* (0.025)
1/8	0.0407* (0.025)		0.0120* (0.025)
1/16	0.0284* (0.025)		0.0003* (0.025)

Numerical Results Case II: Error and Sdev

h	Andersen	Marsaglia
Strike 100		
1	-0.4833 (0.042)	-0.1404 (0.042)
1/2	-0.0400* (0.046)	-0.0264* (0.044)
1/4	-0.0231* (0.044)	0.0217* (0.048)
1/8	0.0807* (0.045)	-0.0553* (0.052)
1/16	-0.0026* (0.042)	0.0521* (0.046)
Strike 140		
1	-0.3082 (0.036)	-0.0926* (0.036)
1/2	0.0515* (0.040)	0.0029* (0.037)
1/4	-0.0016* (0.038)	0.0207* (0.043)
1/8	0.0740* (0.039)	-0.0327* (0.047)
1/16	0.0069* (0.035)	0.0509* (0.040)
Strike 60		
1	0.1180 (0.048)	-0.0379* (0.049)
1/2	0.1349 (0.052)	-0.0036* (0.050)
1/4	-0.0066* (0.050)	0.0290* (0.054)
1/8	0.0809* (0.052)	-0.0650* (0.058)
1/16	-0.0170* (0.049)	0.0492* (0.052)

Numerical Results

- ▶ Generalized Marsaglia method compares very favourably with Andersen's method in terms of efficiency (average CPU time over all steps): it is two times faster than Andersen's method in case 1 and uses 20% less CPU time in case 2.
- ▶ Convergence rate in Andersen's method unknown.
- ▶ Generalized Marsaglia method has advantage of simulating the chi-square distribution *exactly*.

Conclusion

- ▶ Derive representation of a chi-square random variable as sum of powers of independent generalized Gaussian random variables.
- ▶ Prove a new method – the generalized Marsaglia method – for sampling generalized Gaussian random variables.
- ▶ Establish a new method to sample a chi-square distributed random variable, and thus to simulate the Cox–Ingersoll–Ross model exactly and efficiently.
- ▶ Establish a new method to simulate the Heston volatility model in cases that are “challenging and practically relevant” (Andersen 2008 p. 26).
- ▶ Method is efficient and easy to implement.