See the Disclaimer Appendix

Trading CVA: A New Development in Correlation Modelling

Youssef Elouerkhaoui
Managing Director, Markets Quantitative Analysis

youssef.elouerkhaoui@citi.com

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Citigroup Global Markets Limited

Outline

- Motivation: CVA for CDO Tranches
- Review of CVA Concepts and Definitions
- The Pricing Problem: CVA as a New Exotic Credit Derivative
- CVA for CDS
- Pricing under the Conditional Forward Annuity Measure
- CVA for CDO
- Markovian Dynamics
- Applications

Introduction

- Since the beginning of the credit crisis, modelling counterparty risk and the correct pricing and hedging of CVA has become a critical issue for financial institutions.

- This is due to a number of reasons:

  – Given the unprecedented level of credit spreads, the CVA contribution to the quarterly earnings of banks are not negligible;

  – CVA numbers are very volatile and have a large impact on reported earnings;

  – Banks have large exposures to Monolines (and re-insurers) which experienced high default rates and have very high credit spreads.

- This has led to a renewed focus on trading and hedging the CVA P&L reported by the various businesses.

CVA for Correlation Desks

CVA is treated like any other Derivatives Book with Gamma and Cross-Gamma exposures.

This is even more relevant for Correlation desks since this is the primary risk that they trade.

The payoff of a CDO tranche with counterparty risk is similar to the payoff of a CDO-Squared structure where one of the underlyings is the CDO and the other one is the counterparty reference CDS.

CVA is another "Correlation" Credit Derivative Payoff which can be traded and risk-managed similarly to the rest of the Exotic book.

The same machinery developed for pricing, modelling and hedging structured credit risk can be used for the CVA book.

Correlation vs Other Asset Classes

The difference with other asset classes (Rates or LMFX who have exposures through unmargined swaps to Corporates) is that most of the correlation desks counterparties (or collaterals) have liquidly traded CDSs (monolines, re-insurers,...) and can be hedged.

The key issue is the modelling of the correlation between the CDO portfolio and the counterparty reference entity.

The counterparty, in some cases, can also be present in the CDO underlying portfolio which raises similar issues to modelling the portfolio overlap in CDO-Squared structures.

CDO tranches are usually sold to investors in a Funded format where a note is issued by the SPV, which holds some collateral bought by the proceeds of the note issuance. The counterparty risk to the bank is when there is a default of the collateral.

CVA Literature

- There has been many publications recently on CVA modelling for Credit Default Swaps. This includes:
  - Brigo and Chourdakis (2008) - Unilateral CVA with Correlation and Volatility
  - Brigo and Capponi (2009) - Bilateral CVA with Stochastic Dynamic Model
  - Crepey, Jeanblanc and Zagari (2009) - CDS CVA with Markov Chain Model

This Presentation

- In this presentation, we shall develop a general methodology for pricing and hedging CVA for CDO tranches.

- We shall see that given the "Exotic" nature of the CVA derivative payoff, we have to use a variety of modelling techniques that were developed over the last few year.

- This includes:
  - Default Correlation Modelling
  - Pricing of Credit Options
  - Dynamic Credit Modelling
  - CDO-Squared Pricing

- In the following, we will:
  - Review the main CVA concepts and definitions
  - Derive Generic model-independent CVA formulas
  - Construct the Building Blocks to evaluate CVA for CDO Tranches
  - Introduce the Conditional Forward Annuity Measure
  - Derive the Unfunded and Funded CVA for a CDS contract
  - Use a CDO-Squared model to price an Extinguishing CDO
  - Apply a Tranche Option model to derive CVA for CDOs
  - Finally, Use a Markovian Model for Forward Tranches to price CVA

CVA Definitions

We give some general definitions and derive generic CVA results that do not depend on a particular choice of dynamics or payoff.

Set-up. We work on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, where we have a set of default times $(\tau_1, ..., \tau_n)$, representing the defaults of a reference portfolio on n obigors. We denote by τ_c, the default time of the counterparty, and we denote by τ_b, the default time of the bank.

Their recovery rates are $(R_1, .., R_n)$, R_c and R_b respectively. And their default indicators are denoted by $D_t^i = \mathbf{1}_{\{\tau_i \leq t\}}$, $D_t^c = \mathbf{1}_{\{\tau_c \leq t\}}$, $D_t^b = \mathbf{1}_{\{\tau_b \leq t\}}$ respectively.

The enlarged filtration $\{\mathcal{G}_t\}$ that we work with contains both the defaults filtration $\{\mathcal{H}_t\} = \left(\bigvee_{i=1}^n \mathcal{H}_t^i\right) \vee \mathcal{H}_t^b \vee \mathcal{H}_t^c$ and the background filtration $\{\mathcal{F}_t\}$.

Set-up

We define a generic derivative contract by its cumulative dividend process C_t. In general, the dividend process is considered to be of the form $C_t = A_t - B_t$, where A and B are bounded increasing adapted right-contnuous left-limit (cadlag) processes.

The value of the derivative security C without counterparty risk is given by:

$$V_t = \mathbb{E}\left[\int_t^T \frac{B_s}{B_t} dC_s \,|\mathcal{G}_t\right],$$

where B_t is money-market discount factor $B_t = \exp\left(-\int_0^t r_u du\right)$. For example, for a CDS contract, the dividend process is:

$$C_s = (1-R)\,D_s - \sum_{T_i} \mathbf{1}_{\{T_i \le s\}} S^0 \delta_i \left(1 - D_{T_i}\right).$$

For a CDO tranche, the dividend process is

$$C_s = \widehat{L}_s - \sum_{T_i} \mathbf{1}_{\{T_i \le s\}} S^0 \delta_i \left(N - \widehat{L}_{T_i}\right).$$

Terminology

When two counterparties enter into a derivatives transaction, they are implicitly long each other's default risk. The bank is short default protection on the counterparty, and is long protection on its own default risk. **Counterparty Credit Risk** is the risk of losses (or gains) due the default of the counterparty (or one's own default), and the PV adjustment to be added to the value of a default-free derivative is referred to as **Counterparty Valuation Adjustment**.

The **Unilateral CVA** is the PV adjustment when we consider the default of the counterparty only and we assume that the bank is default-free.

The **Bilateral CVA** considers the symmetric effect of both the counterparty default and the bank's own default as well.

The **Asset CVA** is the credit charge due to the counterparty default.

The **Liability CVA** is the credit benefit due to the bank's own default.

Unilateral CVA

In general, upon termination of the contract in the case of default, the Close-Out value is considered to be the value of the contract without counterparty risk $\chi_{(\tau_c)} = V_{\tau_c}$.

Unilateral CVA. When the counterparty defaults, there are two cases: if the MtM of the swap is in-the-money for us, we can only recover a fraction of this P&L; if the MtM of the swap is in favour of the counterparty then we have to pay back the entire P&L to the liquidators:

$$\widehat{V}^R_{\tau_c} = \begin{cases} R_c V_{\tau_c} \text{ if } V_{\tau_c} \geq 0, \\ V_{\tau_c} \text{ if } V_{\tau_c} < 0. \end{cases}$$

Then, by taking the counterparty default into account, the value of the contract becomes:

$$\widehat{V}_t = \mathbb{E}\left[\left[\int_t^T \mathbf{1}_{\{\tau_c > s\}} \frac{B_s}{B_t} dC_s\right] + \left[\frac{B_{\tau_c}}{B_t}\left(R_c V^+_{\tau_c} + V^-_{\tau_c}\right)\mathbf{1}_{\{\tau_c \leq T\}}\right] |\mathcal{G}_t\right],$$

which has two legs, the value of the extinguishing contract and the value of the recovery leg in the case of early termination,

$$
\begin{aligned}
\widehat{V}_t^E &= \mathbb{E}\left[\int_t^T \mathbf{1}_{\{\tau_c > s\}} \frac{B_s}{B_t} dC_s \,|\mathcal{G}_t\right], \\
\widehat{V}_t^R &= \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\left(R_c V_{\tau_c}^+ + V_{\tau_c}^-\right) \mathbf{1}_{\{\tau_c \le T\}} \,|\mathcal{G}_t\right],
\end{aligned}
$$

where we use the notations $x^+ = \max(x, 0)$ and $x^- = \min(x, 0)$.

Let $\xi_{(\tau_c)}$ denote the loss incurred by the firm at time τ_c due the counterparty default. In the case of a non-margined swap, we have:

$$
\xi_{(\tau_c)} = (1 - R_c)\, V_{\tau_c}^+.
$$

The CVA and the EPE are defined as follows.

Definition 1 *The Credit Valuation Adjustment (CVA) is the $\mathcal{G}$-adapted process defined by, for $t \in [0, T]$,*

$$
CVA_t = \mathbb{E}\left[\frac{B_{\tau_c}}{B_t} \xi_{(\tau_c)} \mathbf{1}_{\{\tau_c \le T\}} \,|\mathcal{G}_t\right].
$$

The Expected Positive Exposure (EPE) is the function of time defined by, for $t \in [0, T]$,

$$EPE(t) = \mathbb{E}\left[\xi_{(\tau_c)} \left| \tau_c = t \right.\right].$$

The CVA process is given by the difference of the risky and riskless PVs.

Proposition 2 *For all times before the default of the counterparty, the CVA can be obtained as*

$$CVA_t = V_t - \widehat{V}_t, \text{ for all } t < \tau_c.$$

Proof. We can write the value of the extinguishing leg in terms of the risk-free PV as:

$$\begin{aligned}
\widehat{V}_t^E &= \mathbb{E}\left[\int_t^T \mathbf{1}_{\{\tau_c > s\}} \frac{B_s}{B_t} dC_s \,|\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\tau_c > T\}}\left[\int_t^T \frac{B_s}{B_t} dC_s\right] + \mathbf{1}_{\{\tau_c \le T\}}\left[\int_t^{\tau_c} \frac{B_s}{B_t} dC_s\right] |\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\tau_c > T\}} V_t + \mathbf{1}_{\{\tau_c \le T\}}\left[V_t - \int_{\tau_c}^T \frac{B_s}{B_t} dC_s\right] |\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\tau_c > T\}} V_t + \mathbf{1}_{\{\tau_c \le T\}}\left[V_t - \frac{B_{\tau_c}}{B_t}\mathbb{E}\left[\int_{\tau_c}^T \frac{B_s}{B_{\tau_c}} dC_s \,|\mathcal{G}_{\tau_c}\right]\right] |\mathcal{G}_t\right] \\
&= V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_c \le T\}} \frac{B_{\tau_c}}{B_t} V_{\tau_c} \,|\mathcal{G}_t\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\widehat{V}_t &= V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_c \le T\}} \frac{B_{\tau_c}}{B_t} V_{\tau_c} - \left[\frac{B_{\tau_c}}{B_t}\left(R_c V_{\tau_c}^+ + V_{\tau_c}^-\right) \mathbf{1}_{\{\tau_c \le T\}}\right] |\mathcal{G}_t\right] \\
&= V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_c \le T\}} \frac{B_{\tau_c}}{B_t} (1 - R_c)\, V_{\tau_c}^+ \,|\mathcal{G}_t\right] = V_t - CVA_t.
\end{aligned}$$

■

Netting

If we have a portfolio of transactions with the same counterparty, then when a default occurs, the exposures are netted against each other and the recovery value is applied to the net position.

Definition 3 *If we have a portfolio of trades* $\left(V_t^i\right)_{1\leq i\leq n}$*, with a given counterparty where a netting agreement is in place, then the CVA is given by*

$$CVA_t = \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}(1-R_c)\left(\sum_{i=1}^{n} V_{\tau_c}^i\right)^+ \mathbf{1}_{\{\tau_c\leq T\}}\,|\mathcal{G}_t\right].$$

Bilateral CVA

Bilateral CVA. We consider the bank's own default probability and the possibility of the bank defaulting before the counterparty. The loss incurred when the counterparty defaults after the bank is referred to as **Asset CVA**. The benefit gained when the bank defaults before the counterparty is called **Liability CVA**.

Let τ_f denote the first-to-default time of the counterparty and the bank $\tau_f = \min(\tau_b, \tau_c)$.

The value of the risky security then becomes

$$\widehat{V}_t = \mathbb{E}\left[\left[\int_t^T \mathbf{1}_{\{\tau_f > s\}} \frac{B_s}{B_t} dC_s\right] + \mathbf{1}_{\{\tau_f \le T\}} \left[\begin{array}{c} \mathbf{1}_{\{\tau_f = \tau_c\}} \frac{B_{\tau_f}}{B_t}\left(R_c V_{\tau_f}^{+} + V_{\tau_f}^{-}\right) \\ +\mathbf{1}_{\{\tau_f = \tau_b\}} \frac{B_{\tau_f}}{B_t}\left(V_{\tau_f}^{+} + R_b V_{\tau_f}^{-}\right) \end{array}\right] | \mathcal{G}_t\right].$$

The loss $\xi^c_{(\tau_f)}$ incurred by the firm at time τ_f due the counterparty default (conditional on the firm's survival) is

$$\xi^c_{(\tau_f)} = (1 - R_c)\, V_{\tau_f}^{+},$$

and the loss $\xi^b_{(\tau_f)}$ incurred by the counterparty due to the firm's default (conditional on the counterparty surviving) is

$$\xi^b_{(\tau_f)} = (1 - R_b)\, V^-_{\tau_f}.$$

To simplify the formulas, we have assumed that we don't have simultaneous defaults.

Definition 4 *The (Asset) Credit Valuation Adjustment (CVA) is the $\mathcal{G}$-adapted process defined by, for $t \in [0, T]$,*

$$CVA^{Asset}_t = \mathbb{E}\left[\frac{B_{\tau_f}}{B_t}\xi^c_{(\tau_f)}\mathbf{1}_{\{\tau_f=\tau_c\}}\mathbf{1}_{\{\tau_f\leq T\}}\,|\mathcal{G}_t\right].$$

The (Liability) CVA, also referred to as Debit Valuation Adjustment (DVA), is the $\mathcal{G}$-adapted process defined by, for $t \in [0, T]$,

$$CVA^{Liability}_t = DVA_t = \mathbb{E}\left[\frac{B_{\tau_f}}{B_t}\xi^b_{(\tau_f)}\mathbf{1}_{\{\tau_f=\tau_b\}}\mathbf{1}_{\{\tau_f\leq T\}}\,|\mathcal{G}_t\right].$$

The Expected Positive Exposure (EPE) is the function of time defined by, for $t \in [0, T]$,

$$EPE\,(t) = \mathbb{E}\left[\xi^c_{(\tau_f)}\,\middle|\,\tau_f = t, \tau_f = \tau_c\right].$$

The Expected Negative Exposure (ENE) is the function of time defined as, for $t \in [0, T]$,

$$ENE(t) = \mathbb{E}\left[\xi^b_{(\tau_f)} \middle| \tau_f = t, \tau_f = \tau_b\right].$$

The Bilateral CVA is defined as the sum of the Asset CVA and the Liability CVA

$$CVA_t = CVA_t^{Asset} + CVA_t^{Liability}$$

The Bilateral CVA process will be given by the difference between the risky and the risk-free PVs.

Proposition 5 *For all times before the default of either the counterparty or the firm, the Bilateral CVA can be obtained as*

$$CVA_t = V_t - \widehat{V}_t, \text{ for all } t < \tau_f.$$

Proof. The value of the extinguishing leg can be written in terms of the risk-free PV as:

$$\widehat{V}_t^E = \mathbb{E}\left[\int_t^T \mathbf{1}_{\{\tau_f > s\}} \frac{B_s}{B_t} dC_s \,|\mathcal{G}_t\right] = V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_f \leq T\}} \frac{B_{\tau_f}}{B_t} V_{\tau_f} \,|\mathcal{G}_t\right].$$

Hence,

$$\begin{aligned}
\widehat{V}_t &= V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_f \le T\}} \frac{B_{\tau_f}}{B_t} V_{\tau_f} - \mathbf{1}_{\{\tau_f \le T\}} \left[\begin{array}{c} \mathbf{1}_{\{\tau_f = \tau_c\}} \frac{B_{\tau_f}}{B_t} \left(R_c V_{\tau_f}^{+} + V_{\tau_f}^{-} \right) \\ + \mathbf{1}_{\{\tau_f = \tau_b\}} \frac{B_{\tau_f}}{B_t} \left(V_{\tau_f}^{+} + R_b V_{\tau_f}^{-} \right) \end{array} \right] | \mathcal{G}_t \right] \\
&= V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_f \le T\}} \mathbf{1}_{\{\tau_f = \tau_c\}} \frac{B_{\tau_f}}{B_t} (1 - R_c) V_{\tau_c}^{+} | \mathcal{G}_t \right] \\
&\quad - \mathbb{E}\left[\mathbf{1}_{\{\tau_f \le T\}} \mathbf{1}_{\{\tau_f = \tau_b\}} \frac{B_{\tau_f}}{B_t} (1 - R_b) V_{\tau_f}^{-} | \mathcal{G}_t \right] \\
&= V_t - CVA_t.
\end{aligned}$$

■

In the rest of this presentation, we will focus on the unilateral asset-side CVA and develop the modelling framework for this payoff. Similar methods and techniques can also be applied to the bilateral payoffs.

Funded CVA

Many CDO tranches are usually sold to investors in a Funded format: a note is issued by an SPV, which holds some collateral bought at inception by the proceeds of the note issuance, and faces the bank by exchanging the cashflows of the swap.

The counterparty risk to the bank is when there is a default of the collateral.

Typical questions around ISDAs/CSAs, netting agreements, margin posting, and collateral management are not relevant in this case. Usually, there is only one swap in the SPV transaction and the payment at default and the SPV unwind process are documented in the SPV termsheet.

SPVs

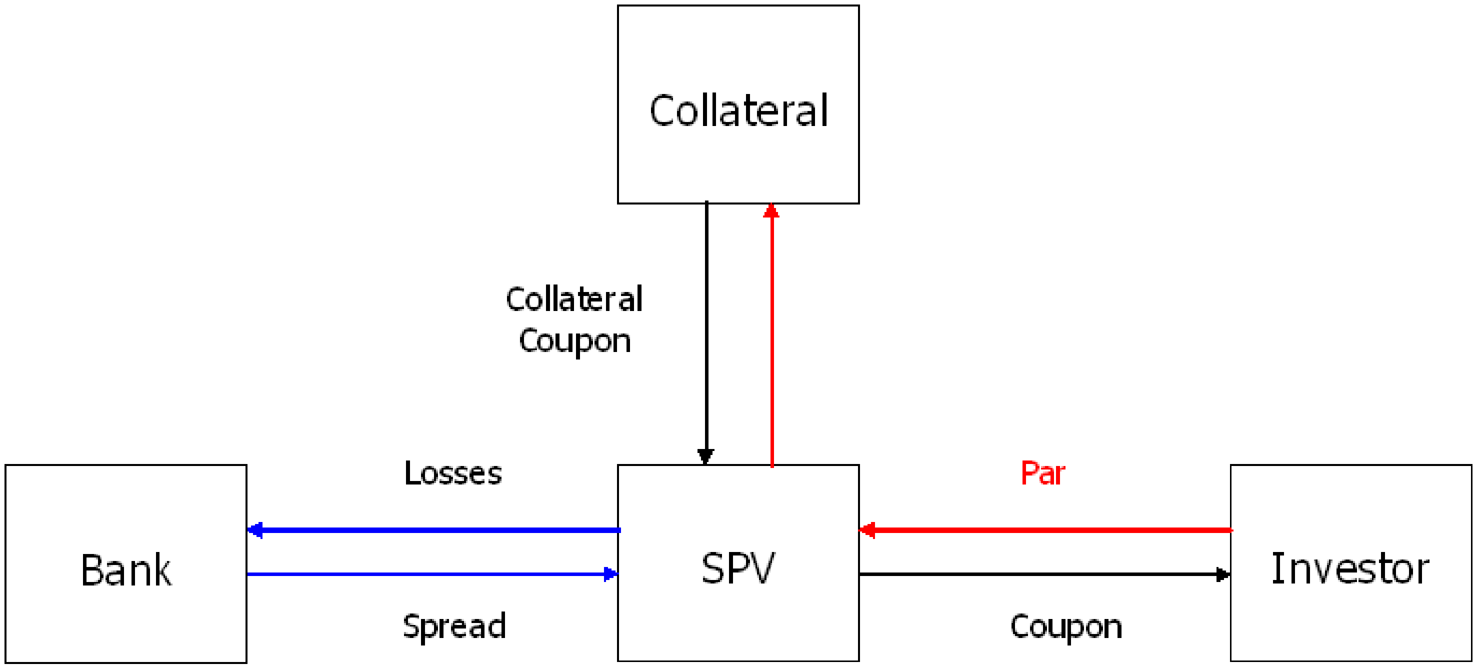
When the counterparty of the swap is a collateralized SPV, then the default of the SPV can be triggered by two events: i) default of the collateral; ii) insufficient collateral to cover the losses incurred on the underlying contract.

The credit risk in this structure is driven by the SPV collateral, and the counterparty in this case is the underlying reference of the collateral.

When the SPV is unwound because of the erosion of collateral this is deemed to be a market risk event and is not included in CVA, but this option is valued as part of the deal.

We can also have a recovery overide on the collateral if additional guarantees are added to the structure.

SPVs



Source: Citi. Structure of a Note issued by an SPV with Collateral

CVA for SPVs

The default time τ_c in this case refers to the default of the collateral. And the value of the CVA recovery leg is given by

$$\widehat{V}_t^R = \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\min\left(R_c\widehat{N}_{\tau_c}, V_{\tau_c}\right)\mathbf{1}_{\{\tau_c \leq T\}}|\mathcal{G}_t\right].$$

$\widehat{N}_{\tau_c}$ is the remaining notional at the time of default.

The payoff upon default will be different depending on whether we are long or short the note. Typically, we are issuing the note and we pay the coupons to the client.

If we have a funded note where the counterparty is an SPV with collateral, then the loss incurred by the firm at time τ_c due the counterparty default is

$$\xi_{(\tau_c)} = \left(V_{\tau_c} - R_c\widehat{N}_{\tau_c}\right)^+.$$

The payoff upon default can be different between various SPVs depending on the agreed termsheet. We could have for example: Par Put, Deferred Loss Payment,...

The CVA process in this case is defined as follows.

Definition 6 *The Credit Valuation Adjustment (CVA) is the* $\mathcal{G}$*-adapted process defined by, for* $t \in [0, T]$,

$$CVA_t = \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\left(V_{\tau_c} - R_c \widehat{N}_{\tau_c}\right)^+ \mathbf{1}_{\{\tau_c \le T\}} |\mathcal{G}_t\right].$$

The Expected Positive Exposure (EPE) is the function of time defined by, for $t \in [0, T]$,

$$EPE(t) = \mathbb{E}\left[\left(V_{\tau_c} - R_c \widehat{N}_{\tau_c}\right)^+ |\tau_c = t\right].$$

And the CVA will be equal to the difference between the risky and riskless PVs.

Proposition 7 *For all times before the default of the counterparty, the CVA can be obtained as*

$$CVA_t = V_t - \widehat{V}_t, \text{ for all } t < \tau_c.$$

Proof. We have

$$\widehat{V}_t^E = V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_c \le T\}} \frac{B_{\tau_c}}{B_t} V_{\tau_c} | \mathcal{G}_t\right].$$

Hence,

$$\begin{aligned}
\widehat{V}_t &= V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_c \le T\}} \frac{B_{\tau_c}}{B_t} V_{\tau_c} - \left[\frac{B_{\tau_c}}{B_t} \min\left(R_c \widehat{N}_{\tau_c}, V_{\tau_c}\right) \mathbf{1}_{\{\tau_c \le T\}}\right] | \mathcal{G}_t\right] \\
&= V_t - \mathbb{E}\left[\mathbf{1}_{\{\tau_c \le T\}} \frac{B_{\tau_c}}{B_t} \left[V_{\tau_c} - \min\left(R_c \widehat{N}_{\tau_c}, V_{\tau_c}\right)\right] | \mathcal{G}_t\right] \\
&= V_t - \mathbb{E}\left[\frac{B_{\tau_c}}{B_t} \left(V_{\tau_c} - R_c \widehat{N}_{\tau_c}\right)^+ \mathbf{1}_{\{\tau_c \le T\}} | \mathcal{G}_t\right] \\
&= V_t - CVA_t.
\end{aligned}$$

■

CVA Lower Bound

In general, to compute the CVA, we need to do an integral of swaption prices (i.e. options on the forward value of the trade) weighted by the counterparty default probabilities.

Using Jensen's inequality, we can simplify the pricing and get a lower bound on CVA which gives a good approximation when the underlying trade is deeply in-the-money.

For a non-margined swap, we have:

$$
\begin{aligned}
CVA_t &= \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}(1-R_c)\,\mathbf{1}_{\{\tau_c \le T\}}\,\mathrm{max}(V_{\tau_c},0)\,|\mathcal{G}_t\right] \\
&\geq \mathrm{max}\left((1-R_c)\,\mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\mathbf{1}_{\{\tau_c \le T\}}V_{\tau_c}\,|\mathcal{G}_t\right],0\right).
\end{aligned}
$$

For a funded note, we have:

$$
\begin{aligned}
CVA_t &= \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\max\left(V_{\tau_c} - R_c\widehat{N}_{\tau_c}, 0\right)\mathbf{1}_{\{\tau_c \le T\}}\,|\mathcal{G}_t\right] \\
&\ge \max\left(\mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\left(V_{\tau_c} - R_c\widehat{N}_{\tau_c}\right)\mathbf{1}_{\{\tau_c \le T\}}\,|\mathcal{G}_t\right], 0\right) \\
&\ge \max\left(\mathbb{E}\left[\frac{B_{\tau_c}}{B_t}V_{\tau_c}\mathbf{1}_{\{\tau_c \le T\}}\,|\mathcal{G}_t\right] - R_c\mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\widehat{N}_{\tau_c}\mathbf{1}_{\{\tau_c \le T\}}\,|\mathcal{G}_t\right], 0\right).
\end{aligned}
$$

The first term can be computed easily as

$$
\begin{aligned}
\mathbb{E}\left[\frac{B_{\tau_c}}{B_t}V_{\tau_c}\mathbf{1}_{\{\tau_c \le T\}}\,|\mathcal{G}_t\right] &= \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}V_{\tau_c}\,|\mathcal{G}_t\right] = \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\mathbb{E}\left[\int_{\tau_c}^{T}\frac{B_s}{B_{\tau_c}}dC_s\,|\mathcal{G}_{\tau_c}\right]|\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\int_{\tau_c}^{T}\frac{B_s}{B_t}dC_s\,|\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\int_{t}^{T}\frac{B_s}{B_t}dC_s\,|\mathcal{G}_t\right] - \mathbb{E}\left[\int_{t}^{\tau_c}\frac{B_s}{B_t}dC_s\,|\mathcal{G}_t\right] \\
&= V_t - \widehat{V}_t^E.
\end{aligned}
$$

The second term is given by

$$
\begin{aligned}
\mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\widehat{N}_{\tau_c}\mathbf{1}_{\{\tau_c\le T\}}\,|\mathcal{G}_t\right] &= \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}N\mathbf{1}_{\{\tau_c\le T\}}\,|\mathcal{G}_t\right] - \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}\widehat{L}_{\tau_c}\mathbf{1}_{\{\tau_c\le T\}}\,|\mathcal{G}_t\right] \\
&= N\left[\int_t^T \frac{B_s}{B_t}\mathbb{P}_t\left(\tau_c\in ds\right)\right] - \left[\int_t^T \frac{B_s}{B_t}\mathbb{E}_t\left[\widehat{L}_{s-}dD_s^c\right]\right].
\end{aligned}
$$

To simplify the formula, we have assumed that the collateral trades at Par during the life of the trade.

To compute the value of the extinguishing leg of the trade, we need a correlation model that links the default of the counterparty to the losses on the CDO.

To value the CVA payoff and/or the recovery leg of the trade with counterparty risk, we need an option model since the pricing boils down to the integral of swaption prices weighted by the counterparty default probability.

Next, we show how this is done for CDSs, then for CDOs.

CVA for CDS

The filtration in this case contains the information on the CDS reference default and the default of the counterparty $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^i \vee \mathcal{H}_t^c$.

In this section, we omit the index i to lighten up the notations.

We denote by $p_{t,T}$ the risk-free discount factor, at time t, $p_{t,T} \triangleq \frac{B_t}{B_T}$.

We consider a CDS maturing at time T, with payment schedule $(T_0 = 0, T_1, ..., T_n)$; δ_i is the accrual fraction between T_i and T_{i-1}, and S^0 is the CDS running spread.

The value, at time t, of a CDS contract where we buy protection is given by

$$\begin{aligned} V_{t,t,T}^{CDS} &= \mathbb{E}\left[(1-R)\int_t^T p_{t,s} dD_s - \sum_{T_i>t} p_{t,T_i} S^0 \delta_i \left(1 - D_{T_i}\right) |\mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}\left[(1-R)\int_t^T p_{t,s} d\mathbb{E}\left[D_s\,|\mathcal{G}_t\right] - \sum_{T_i>t} p_{t,T_i}\mathbb{E}\left[\left(1 - D_{T_i}\right)|\mathcal{G}_t\right] S^0 \delta_i\right]. \end{aligned}$$

The unfunded CVA of this contract is

$$CVA_0 = (1 - R_c) \int_0^T p_{0,t} \mathbb{E} \left[\left(V_{t,t,T}^{CDS} \right)^+ | \tau_c = t \right] \mathbb{P} \left(\tau_c \in dt \right).$$

To evaluate the CVA for a CDS contract, we need two components: a CDS Option model and the correct modelling of the forward value which captures spread widening effects due the default correlation between the underlying reference credit and the counterparty.

Pricing CDS Options

The Filtration in this case is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$.

We denote by $Q_{t,T}$ the $\{\mathcal{F}_t\}$-adapted process representing the conditional survival probability of the reference entity:

$$Q_{t,T} \triangleq \frac{\mathbb{P}\left(\tau > T \,|\mathcal{F}_t\right)}{\mathbb{P}\left(\tau > t \,|\mathcal{F}_t\right)}.$$

We have on the set $\mathbf{1}_{\{\tau>t\}}$:

$$\mathbf{1}_{\{\tau>t\}}\mathbb{P}\left(\tau > T \,|\mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\frac{\mathbb{P}\left(\tau > T \,|\mathcal{F}_t\right)}{\mathbb{P}\left(\tau > t \,|\mathcal{F}_t\right)} = \mathbf{1}_{\{\tau>t\}}Q_{t,T}.$$

The Break-even spread, at time t, of the forward swap starting at U and maturing at T is given by

$$S_{t,U,T} = \frac{-\left(1-R\right)\int_U^T p_{t,U,s}dQ_{t,U,s}}{\sum_{T_i>U} p_{t,U,T_i}Q_{t,U,T_i}\delta_i},$$

where $p_{t,U,T}$ and $Q_{t,U,T}$ are the forward discount factor and foward survival probabilities:

$$p_{t,U,T} \triangleq \frac{p_{t,T}}{p_{t,U}},$$
$$Q_{t,U,T} \triangleq \frac{Q_{t,T}}{Q_{t,U}}.$$

We denote by $A_{t,U,T}$ the value of the forward annuity at time t:

$$A_{t,U,T} \triangleq \sum_{T_i > U} p_{t,U,T_i} Q_{t,U,T_i} \delta_i.$$

Then, the value of the CDS can be expressed as

$$V_{t,t,T}^{CDS} = \mathbf{1}_{\{\tau > t\}} A_{t,t,T} \left(S_{t,t,T} - S^0 \right).$$

And the value of the option to enter into the CDS swap starting at time t and maturing at T is given by

$$O_{0,t,T} = p_{0,t} \mathbb{E} \left[\left(V_{t,t,T}^{CDS} \right)^+ \right] = p_{0,t} Q_{0,t} \mathbb{E} \left[A_{t,t,T} \left(S_{t,t,T} - S^0 \right)^+ \right].$$

The annuity defined here is strictly positive a.s. (and $\{\mathcal{F}_t\}$-adapted), and can be used to define a forward annuity measure.

The Break-even spread is a martingale under this measure, and we can use the standard Black formula to compute this expectation (e.g., see Brigo and Morini (2005) for technical details) :

$$\begin{aligned} O_{0,t,T} &= p_{0,t} Q_{0,t} A_{0,t,T} \mathbb{E}^{Q_A}\left[\left(S_{t,t,T} - S^0\right)^+\right] \\ &= p_{0,t} Q_{0,t} A_{0,t,T} Black\left(S_{0,t,T}, S^0, \sigma_{BS}\sqrt{T-t}\right) \\ &= \left[\sum_{T_i > U} p_{0,T_i} Q_{0,T_i} \delta_i\right] Black\left(S_{0,t,T}, S^0, \sigma_{BS}\sqrt{T-t}\right). \end{aligned}$$

The Conditional Forward Annuity Measure

Pricing a Conditional CDS Option. The Filtration in this case is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \vee \mathcal{H}_t^c$. We use similar techniques to price the option payoff required in CVA.

When there is no correlation between the underlying credit and the counterparty, this is simply the integral of the standard CDS option prices given above

$$\begin{aligned} CVA_0 &= (1-R_c)\int_0^T p_{0,t}\mathbb{E}\left[\left(V_{t,t,T}^{CDS}\right)^+ |\tau_c = t\right]\mathbb{P}(\tau_c \in dt) \\ &= (1-R_c)\int_0^T p_{0,t}\mathbb{E}\left[\left(V_{t,t,T}^{CDS}\right)^+\right]\mathbb{P}(\tau_c \in dt) \\ &= (1-R_c)\int_0^T O_{0,t,T}\mathbb{P}(\tau_c \in dt)\,. \end{aligned}$$

If there is a correlation, then we need to work under the **Conditional Forward Annuity Measure**.

Using the Generalized Dellacherie Formula (see Jeanblanc and Rutkowski (2000)), we have, for all $T \geq t$,

$$\begin{aligned}\mathbb{P}(\tau > T \,|\mathcal{G}_t) &= \mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau_c>t\}}\frac{\mathbb{P}(\tau > T, \tau_c > t \,|\mathcal{F}_t)}{\mathbb{P}(\tau > t, \tau_c > t \,|\mathcal{F}_t)} \\ &\quad +\mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau_C\leq t\}}\frac{\mathbb{P}(\tau > T \,|\mathcal{F}_t \vee \sigma(\tau_c))}{\mathbb{P}(\tau > t \,|\mathcal{F}_t \vee \sigma(\tau_c))}.\end{aligned}$$

Conditional on the counterparty default, the reference entity survival probability becomes

$$\begin{aligned}&\mathbb{P}(\tau > T \,|\mathcal{G}_t \vee \{\tau > U\} \vee \{\tau_c = U\}) \\ =\;& \mathbf{1}_{\{\tau>U\}}\mathbf{1}_{\{\tau_C=U\}}\mathbb{P}(\tau > T \,|\mathcal{G}_t \vee \{\tau > U\} \vee \{\tau_c = U\}) \\ =\;& \mathbf{1}_{\{\tau>U\}}\mathbf{1}_{\{\tau_C=U\}}\mathbb{P}(\tau > T \,|\mathcal{F}_t \vee \{\tau > U\} \vee \{\tau_c = U\}) \\ =\;& \mathbf{1}_{\{\tau>U\}}\mathbf{1}_{\{\tau_C=U\}}\frac{\mathbb{P}(\tau > T \,|\mathcal{F}_t \vee \{\tau_c = U\})}{\mathbb{P}(\tau > U \,|\mathcal{F}_t \vee \{\tau_c = U\})} \\ =\;& \mathbf{1}_{\{\tau>U\}}\mathbf{1}_{\{\tau_C=U\}}\widetilde{Q_{t,U,T}}.\end{aligned}$$

We define the $\{\mathcal{F}_t\}$-adapted process $\widetilde{Q_{t,U,T}}$ as

$$\widetilde{Q_{t,U,T}} \triangleq \frac{\mathbb{P}(\tau > T \,|\mathcal{F}_t \vee \{\tau_c = U\})}{\mathbb{P}(\tau > U \,|\mathcal{F}_t \vee \{\tau_c = U\})}.$$

To compute the $\widetilde{Q_{t,U,T}}$ we can use a dynamic copula model (see Schonbucher and Schubert (2001)) where the intensities are $\{\mathcal{F}_t\}$-adapted dynamic processes and the default treshholds are linked via a copula function $\mathbb{C}^{\theta}(x_1, x_2)$:

$$\widetilde{Q_{t,U,T}} = \frac{\partial_{x_2}\mathbb{C}^{\theta}\left(\mathbb{P}\left(\tau > T\,|\mathcal{F}_t\right), \mathbb{P}\left(\tau_c > U\,|\mathcal{F}_t\right)\right)}{\partial_{x_2}\mathbb{C}^{\theta}\left(\mathbb{P}\left(\tau > U\,|\mathcal{F}_t\right), \mathbb{P}\left(\tau_c > U\,|\mathcal{F}_t\right)\right)}.$$

We consider the value of the forward swap (U, T) conditional on the default of the counterparty at time t:

$$\begin{aligned}\widetilde{V_{t,U,T}^{CDS}} &= \mathbb{E}\left[V_{t,U,T}^{CDS}\,|\tau_c = U\right] = \mathbf{1}_{\{\tau > U\}}\mathbb{E}\left[V_{t,U,T}^{CDS}\,|\tau > U, \tau_c = U\right] \\ &= \mathbf{1}_{\{\tau > U\}}\mathbf{1}_{\{\tau_C = U\}}\left[-(1-R)\int_U^T p_{t,s}d\widetilde{Q_{t,U,s}} - \sum_{T_i > U} p_{t,T_i}S^0\delta_i\widetilde{Q_{t,U,T_i}}\right].\end{aligned}$$

The Break-even spread of the forward swap conditional on default is

$$\widetilde{S_{t,U,T}} = \frac{-(1-R)\int_U^T p_{t,U,s}d\widetilde{Q_{t,U,s}}}{\sum_{T_i > U} p_{t,U,T_i}\widetilde{Q_{t,U,T_i}}\delta_i}.$$

The conditional forward annuity $\widetilde{A_{t,U,T}}$ is defined as:

$$\widetilde{A_{t,U,T}} \triangleq \sum_{T_i > U} p_{t,U,T_i}\widetilde{Q_{t,U,T_i}}\delta_i.$$

And the conditional value of the forward CDS is

$$\widetilde{V_{t,t,T}^{CDS}} = \mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau_C=t\}}\widetilde{A_{t,t,T}}\left(\widetilde{S_{t,t,T}} - S^0\right).$$

The conditional forward CDS option value is then given by

$$\widetilde{O_{0,t,T}} = p_{0,t}\mathbb{E}\left[\left(\widetilde{V_{t,t,T}^{CDS}}\right)^+\right] = p_{0,t}\mathbb{P}\left(\tau > t, \tau_C = t\right)\mathbb{E}\left[\widetilde{A_{t,t,T}}\left(\widetilde{S_{t,t,T}} - S^0\right)^+\right].$$

The conditional forward annuity is $\{\mathcal{F}_t\}$-adapted and strictly positive a.s., so we can do a change of measure and use the Black-Scholes derivations:

$$\begin{aligned}\widetilde{O_{0,t,T}} &= p_{0,t}\mathbb{P}\left(\tau > t, \tau_C = t\right)\widetilde{A_{0,t,T}}Black\left(\widetilde{S_{0,t,T}}, S^0, \widetilde{\sigma_{BS}}\sqrt{T-t}\right) \\ &= \left[\sum_{T_i>t} p_{0,T_i}\mathbb{P}\left(\tau > t, \tau_C = t\right)\widetilde{Q_{0,t,T_i}}\delta_i\right]Black\left(\widetilde{S_{0,t,T}}, S^0, \widetilde{\sigma_{BS}}\sqrt{T-t}\right).\end{aligned}$$

CVA Formula

Proposition 8 *The CVA for a CDS is given by the sum of conditional (Knock-in) CDS options weighted by the default probability of the counterparty,*

$$CVA_0 = (1 - R_c) \int_0^T \widetilde{O_{0,t,T}} \mathbb{P}\left(\tau_c \in dt\right),$$

where $\widetilde{O_{0,t,T}}$ *is the value of the conditional* (t,T)*-forward CDS option*

$$\widetilde{O_{0,t,T}} = \left[\sum_{T_i > t} p_{0,T_i} \widetilde{Q^*_{0,t,T_i}} \delta_i\right] Black\left(\widetilde{S_{0,t,T}}, S^0, \widetilde{\sigma_{BS}} \sqrt{T-t}\right),$$

with $\widetilde{Q^*_{0,t,T_i}}$ *and* $\widetilde{S_{0,t,T}}$ *are the conditional forward survival probabilities and conditional forward breakeven spread respectively*

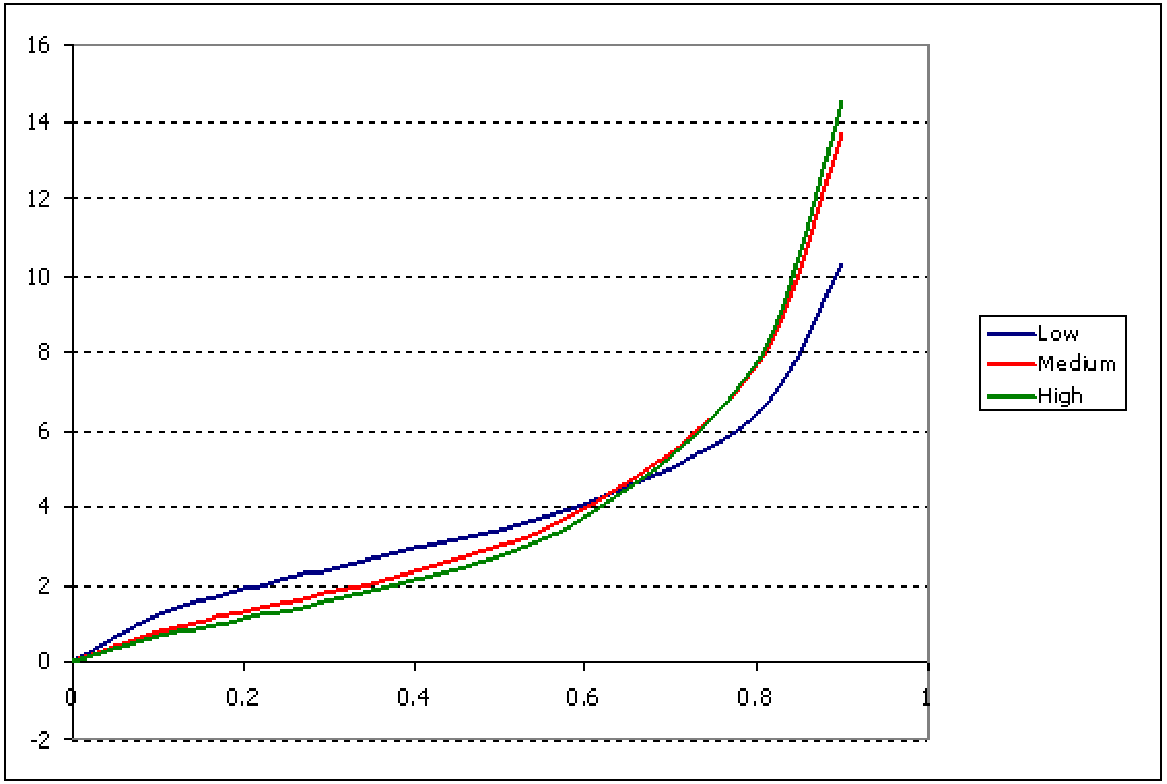
$$\begin{aligned} \widetilde{Q^*_{0,t,T}} &= \mathbb{P}\left(\tau > T \,|\tau_c = t\right), \\ \widetilde{S_{0,t,T}} &= \frac{-\left(1-R\right) \int_t^T p_{0,s} d\widetilde{Q^*_{0,t,s}}}{\sum_{T_i > t} p_{0,T_i} \widetilde{Q^*_{0,t,T_i}} \delta_i}. \end{aligned}$$

The main ingredients that are needed to value the CVA are the conditional forward curves and the conditional forward CDS break-even spreads.

The term-structure of BS volatilities can be provided as an input or can be based on a term-structure model for the hazard rates.

Conditional Forward Spreads

We compute the (average) CDS spread widening for different correlations. The counterparty CDS is 1000 bps. The base CDS is: low = 100 bps, medium = 1000 bps, high = 2000 bps.



Source: Citi. Spread widening for different values of correlation

CVA for CLN

Funded CVA Formula. When we have a CDS facing a collateralised SPV, the CVA is given by

$$CVA_0 = \mathbb{E}\left[\frac{B_{\tau_c}}{B_t}(V_{\tau_c} - R_c N)^+ \mathbf{1}_{\{\tau_c \le T\}}\right].$$

For a CDS, there is no erosion of collateral because of defaults, if the reference entity defaults the losses on the CDS will be covered by the collateral, but this is market risk and not credit risk included in CVA.

We need to price the following option payoff

$$\widetilde{O_{0,t,T}} = p_{0,t}\mathbb{E}\left[\mathbf{1}_{\{\tau_c = t\}}\left(V_{t,t,T}^{CDS} - R_c N\right)^+ |\tau_c = t\right].$$

The strike of the option will change because we have the additional cash buffer provided by the collateral at the time of default. The recovery value of the collateral is a cash-dollar amount that needs to be converted to a running spread strike.

We can re-write the option payoff as

$$\begin{aligned}
\mathbf{1}_{\{\tau_c=t\}}\left(V^{CDS}_{t,t,T}-R_cN\right)^+ &= \mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau_c=t\}}\left(\widetilde{A_{t,t,T}}\left(S_{t,t,T}-S^0\right)-R_cN\right)^+\\
&= \mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau_c=t\}}\left(\widetilde{A_{t,t,T}}\left(S_{t,t,T}-S^*\right)\right)^+,
\end{aligned}$$

where S^* is the option strike adjusted with the cash-settled component

$$\begin{aligned}
S^* &= S^0+S^c,\\
S^c &\triangleq \frac{R_cN}{\widetilde{A_{t,t,T}}}\simeq\frac{R_cN}{\widetilde{A_{0,t,T}}},
\end{aligned}$$

which can be simplified further by approximating the equivalent running spread by its forward value.

This is a reasonable approximation, to first-order, since typically the volatility of the annuity process is much lower than the volatility of the break-even spread. The correct modelling of this term would add a second-order convexity adjustment term which can be neglected for our purposes.

Funded CVA Formula

Proposition 9 *The CVA for a CDS facing a collateralised SPV is given by the sum of conditional CDS options, with collateral-adjusted strikes, weighted by the default probability of the collateral,*

$$CVA_0 = (1 - R_c) \int_0^T \widetilde{O_{0,t,T}} \mathbb{P}\left(\tau_c \in dt\right),$$

where $\widetilde{O_{0,t,T}}$ *is the value of the conditional* (t, T)*-forward CDS option*

$$\widetilde{O_{0,t,T}} = \left[\sum_{T_i > t} p_{0,T_i} \widetilde{Q^*_{0,t,T_i}} \delta_i\right] Black\left(\widetilde{S_{0,t,T}}, S^*, \widetilde{\sigma_{BS}} \sqrt{T - t}\right),$$

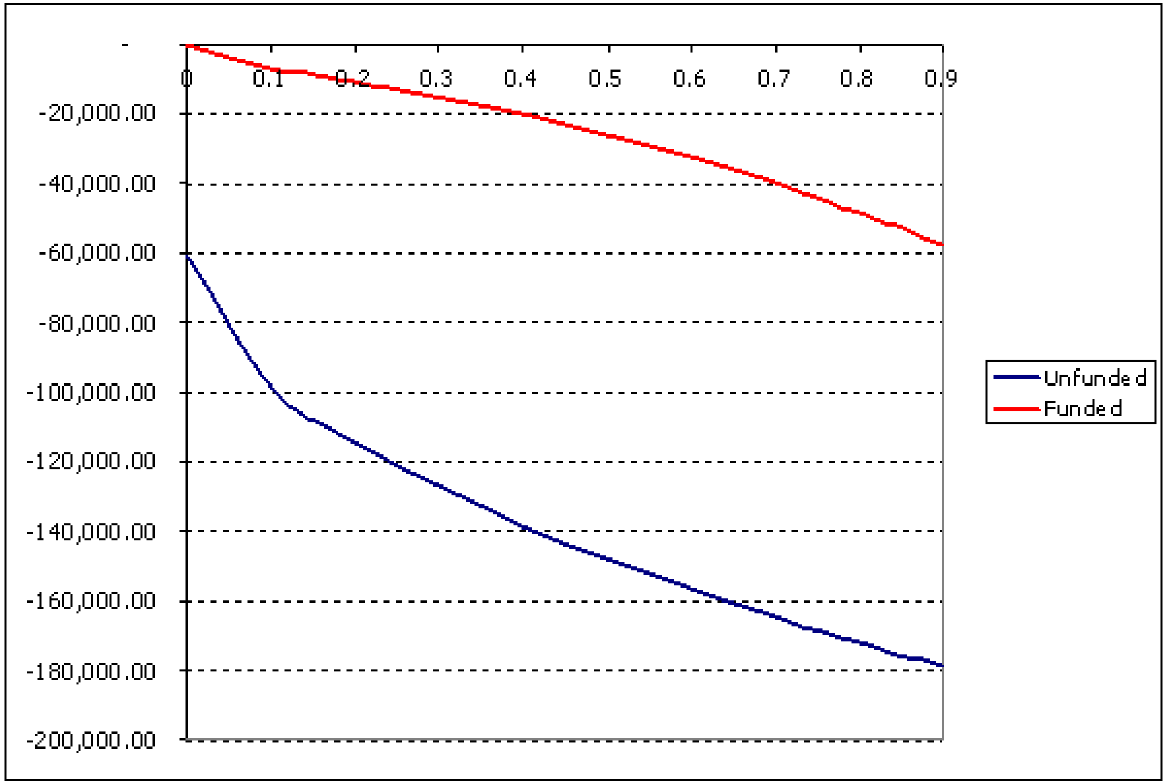
with $\widetilde{Q^*_{0,t,T_i}}$, $\widetilde{S_{0,t,T}}$ *and* S^* *are the conditional forward survival probability, the conditional forward breakeven spread and the collateral-*

adjusted strike respectively

$$\begin{aligned} \widetilde{Q^*_{0,t,T}} &= \mathbb{P}\left(\tau > T \,|\tau_c = t\right), \\ \widetilde{S_{0,t,T}} &= \frac{-\left(1-R\right)\int_t^T p_{0,s} d\widetilde{Q^*_{0,t,s}}}{\sum_{T_i>t} p_{0,T_i}\widetilde{Q^*_{0,t,T_i}}\delta_i}, \\ S^* &= S^0 + \frac{R_c N}{\sum_{T_i>t} \frac{p_{0,T_i}}{p_{0,t}}\frac{\widetilde{Q^*_{0,t,T_i}}}{\widetilde{Q^*_{0,t,t}}}\delta_i}. \end{aligned}$$

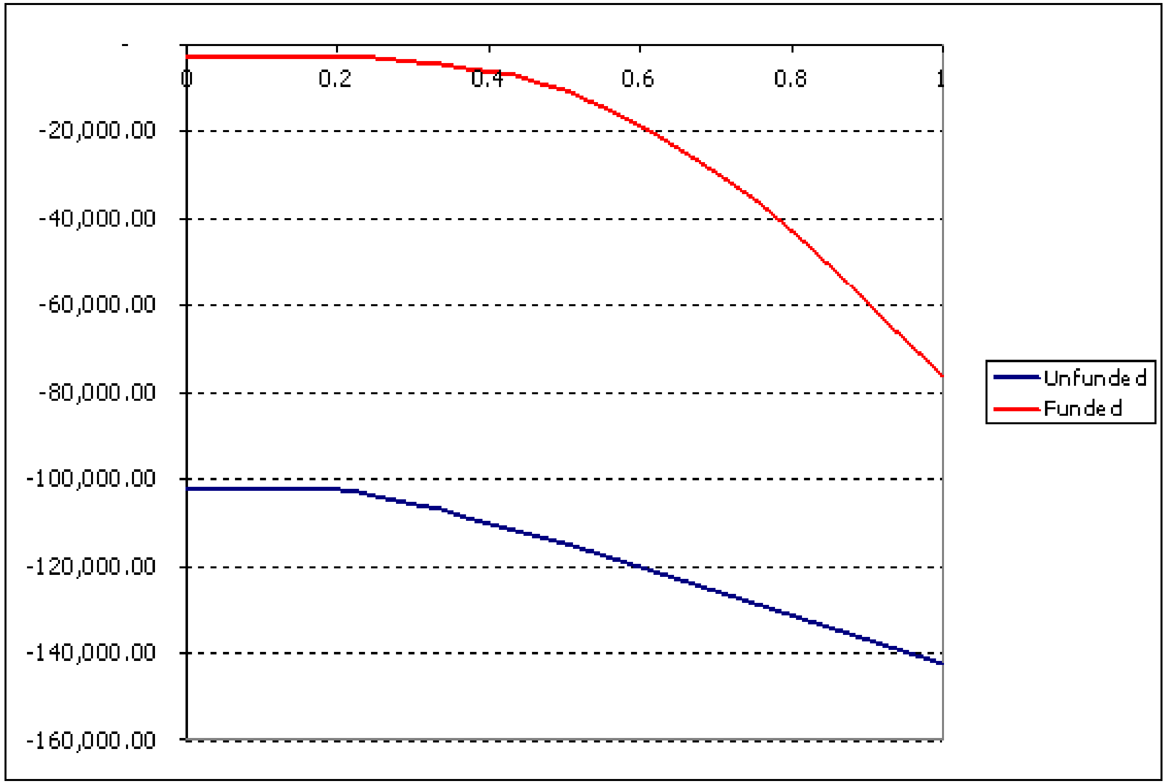
Funded vs Unfunded CVA (1)

We compare the Funded and Unfunded CVA for a 5Y CDS with a 100 bps running spread, marked at 0.921 upfront (SNAC level 132bps). The counterparty curve is marked at 2557 bps.



Source: Citi. Funded vs Unfunded CVA for a non-margined CDS as a function of correlation

Funded vs Unfunded CVA (2)



Source: Citi. Funded vs Unfunded CVA for a non-margined CDS as a function of volatility

CVA for CDO

The filtration in this case contains the information on the underlying credits in the portfolio and the default of the counterparty (or the collateral).

We denote by L_t and $\widehat{L}_t$ the (normalised) portfolio loss and tranche loss variables respectively

$$\begin{aligned} L_t &= \frac{1}{n}\sum_{i=1}^{n}(1-R_i)\,D_t^i, \\ \widehat{L}_t &= \frac{1}{\beta-\alpha}\left[(L_t-\alpha)^+ - (L_t-\beta)^+\right]. \end{aligned}$$

The value of a CDO tranche where we are buyer of protection is given by

$$V_{t,t,T}^{CDO} = \mathbb{E}\left[\int_t^T p_{t,s}d\widehat{L}_s - \sum_{T_i>t} p_{t,T_i}S^0\delta_i\left(1-\widehat{L}_{T_i}\right)|\mathcal{G}_t\right].$$

To evaluate the CVA for a CDO tranche, we need three components.

1. A **CDO-Squared model** that correlates the counterparty with the underlying names in the CDO portfolio. The CVA Payoff can be viewed as as a single-name CDO-Squared with two underlyings: the CDO tranche and the counterparty. Modelling the overlap is key as in some cases the counterparty (or the collateral) can also be one of the names in the CDO.

2. We need a **CDO Option model** to price the optionality in the CVA payoff.

3. We need a **Dynamic Credit model** to capture the evolution of the CDS spreads and the forward skew. In the case of deterministic intensities, the CVA payoff boils down to pricing a series of forward-starting tranches.

Forward Tranches

The unfunded CVA is

$$CVA_0 = (1 - R_c) \int_0^T p_{0,t} \mathbb{E}\left[\left(V_{t,t,T}^{CDO}\right)^+ | \tau_c = t\right] \mathbb{P}\left(\tau_c \in dt\right).$$

First, we need to price the **Conditional Forward Tranche** profile correctly

$$\begin{aligned} FV_{0,t,T}^{CDO} &\triangleq \mathbb{E}\left[V_{t,t,T}^{CDO} | \tau_c = t\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\int_t^T p_{t,s} d\widehat{L}_s - \sum_{T_i > t} p_{t,T_i} S^0 \delta_i \left(1 - \widehat{L}_{T_i}\right) | \mathcal{G}_t\right] | \tau_c = t\right] \\ &= \mathbb{E}\left[\int_t^T p_{t,s} d\widehat{L}_s - \sum_{T_i > t} p_{t,T_i} S^0 \delta_i \left(1 - \widehat{L}_{T_i}\right) | \tau_c = t\right]. \end{aligned}$$

Basically, for each default time t, we need to compute the following forward tranche loss curve

$$\mathbb{E}\left[\widehat{L}_T | \tau_c = t\right], \text{ for all } T \geq t.$$

We need to use a CDO-Squared approach.

CDO-Squared Approach

In a nutshell, the CDO-Squared Model defines a family of copula functions of the loss variables, parametrised by ρ, which is, then, combined with the marginal (skewed) distributions of the loss variables to get the multivariate joint loss distribution.

To create the correlation dependence between the underlying CDO loss variables, we use a Gaussian copula with a flat correlation parameter ρ for all the names in the CDO-Squared master portfolio.

We have a portfolio of $(n+1)$ credits (including the counterparty reference) linked through a Gaussian copula

$$\begin{aligned} X_i &= \sqrt{\rho}Y + \sqrt{1-\rho}\varepsilon_i, \text{ for } 1 \leq i \leq n+1, \\ \{\tau_i \leq t\} &\Leftrightarrow \left\{X_i \leq \Phi^{-1}\left(\mathbb{P}\left(\tau_i \leq t\right)\right)\right\}, \\ \tau_{n+1} &= \tau_c. \end{aligned}$$

Step 1. Using this default dependence of the single-names, we can generate the (unskewed) univariate loss distributions and the joint loss distribution

$$\begin{aligned}\phi(K) &= \mathbb{P}(L_T \leq K), \\ \phi_c(t) &= \mathbb{P}(\tau_c \leq t), \\ \Phi(K,t) &= \mathbb{P}(L_T \leq K, \tau_c \leq t).\end{aligned}$$

Step 2. Using the base correlation skew, we generate the skewed marginal loss distribution

$$\phi^*(K) = \mathbb{P}(L_T^* \leq K).$$

Step 3. The skewed and unskewed loss variables are then mapped via this functional-mapping

$$L_T^* = (\phi^*)^{-1}(\phi(L_T)).$$

The joint loss distribution is given by

$$\mathbb{P}(L_T^* \leq K, \tau_c \leq t) = \Phi\left((\phi^*)^{-1}(\phi(K)), t\right).$$

The skewed and unskewed marginal distributions are pre-computed and stored.

Conditional on Y, we can generate the distribution of L_T using standard methods (such as recursion, FFT, proxy,...), and the expected tranche loss is obtained by using the functional-mapping and integrating over the values of L_T:

$$\begin{aligned}\mathbb{E}\left[\widehat{L}_T \,|Y\right] &= \mathbb{E}\left[f\left(L_T^*\right)|Y\right] = \mathbb{E}\left[f\left(\left(\phi^*\right)^{-1}\left(\phi\left(L_T\right)\right)\right)|Y\right] \\ &= \int f\left(\left(\phi^*\right)^{-1}\left(\phi\left(x\right)\right)\right)\mathbb{P}\left(L_T \in dx\,|Y\right).\end{aligned}$$

Conditional Forward Tranche Loss Curve

Now, having defined the correlation structure, we can compute the conditional forward tranche loss curve: for all $T \geq t$,

$$\mathbb{E}\left[\widehat{L}_T \,|\tau_c = t\right] = \frac{\mathbb{E}\left[\widehat{L}_T \mathbf{1}_{\{\tau_c = t\}}\right]}{\mathbb{P}\left(\tau_c = t\right)}.$$

We have two cases.

1. If the counterparty reference is not included in the CDO portfolio (i.e., there is no overlap), we can compute this expectation by conditioning on the common factor

$$\mathbb{E}\left[\widehat{L}_T \mathbf{1}_{\{\tau_c = t\}}\right] = \int \mathbb{E}\left[\widehat{L}_T \,|Y\right] \mathbb{P}\left(\tau_c = t\,|Y\right) \phi\left(Y\right) dY.$$

The expected tranche loss $\mathbb{E}\left[\widehat{L}_T \,|Y\right]$ is given by functional-mapping formula, and the probability $\mathbb{P}\left(\tau_c = t\,|Y\right)$ is given by the Gaussian copula.

2. If there is overlap, i.e., the counterpary belongs also to the underlying portfolio, then this will have an impact on the expectation and needs to be included in the conditioning, we get

$$\mathbb{E}\left[\widehat{L}_T \mathbf{1}_{\{\tau_c = t\}}\right] = \int \mathbb{E}\left[\widehat{L}_T \,|Y, \tau_c = t\right] \mathbb{P}\left(\tau_c = t \,|Y\right) \phi\left(Y\right) dY \,.$$

Note. To compute the conditional expectation, we need to set the counterparty default probability to 1, regenerate the distribution of L_T and use the mapping to get the tranche loss

$$\mathbb{E}\left[\widehat{L}_T \,|Y, \tau_c = t\right] = \int f\left(\left(\phi^*\right)^{-1}\left(\phi\left(x\right)\right)\right) \mathbb{P}\left(L_T \in dx \,|Y, \tau_c = t\right).$$

Extinguishing CDOs

Similar calculations can be used to price an Extinguishing CDO. This is useful, for example, for computing the CVA lower bound or for checking the call/put parity.

The value of an extinguishing CDO is given by

$$V_0^E = \mathbb{E}\left[\int_0^T p_{0,s}\left(1 - D^c_{s-}\right) d\widehat{L}_s - \sum_{T_i > 0} p_{0,T_i} S^0 \delta_i \left(1 - D^c_{T_i}\right)\left(1 - \widehat{L}_{T_i}\right)\right].$$

Using the integration by parts, we can re-write the Loss leg as

$$\begin{aligned}\mathbb{E}\left[\int_0^T p_{0,s}\left(1 - D^c_{s-}\right) d\widehat{L}_s\right] &= \mathbb{E}\left[p_{0,T}\left(1 - D^c_T\right)\widehat{L}_T\right] \\ &\quad - \mathbb{E}\left[\int_0^T \left(1 - D^c_s\right)\widehat{L}_s dp_{0,s} - \int_0^T p_{0,s}\widehat{L}_{s-} dD^c_s\right].\end{aligned}$$

Similarly, the premium leg is given by

$$\begin{aligned}
&\mathbb{E}\left[\sum_{T_i>0} p_{0,T_i} S^0 \delta_i \left(1 - D^c_{T_i}\right)\left(1 - \widehat{L}_{T_i}\right)\right] \\
= &\sum_{T_i>0} p_{0,T_i} S^0 \delta_i \left[\mathbb{P}\left(\tau_c > T_i\right) - \mathbb{E}\left[\left(1 - D^c_{T_i}\right)\widehat{L}_{T_i}\right]\right].
\end{aligned}$$

Conditional Tranche Loss

So, we need to compute two terms: the tranche loss conditional on survival and the tranche loss conditional on default

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_c>t\}}\widehat{L}_t\right], \text{ for all } t \geq 0,$$
$$\mathbb{E}\left[\widehat{L}_{t-}\mathbf{1}_{\{\tau_c=t\}}\right], \text{ for all } t \geq 0.$$

1. If the counterparty is not included in the portfolio, we condition on the common factor, and we get

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_c>t\}}\widehat{L}_t\right] = \int \mathbb{E}\left[\widehat{L}_t\,|Y\right]\mathbb{P}\left(\tau_c > t\,|Y\right)\phi\left(Y\right)dY,$$
$$\mathbb{E}\left[\widehat{L}_{t-}\mathbf{1}_{\{\tau_c=t\}}\right] = \int \mathbb{E}\left[\widehat{L}_{t-}\,|Y\right]\mathbb{P}\left(\tau_c = t\,|Y\right)\phi\left(Y\right)dY.$$

2. If the counterparty belongs to the underlying portfolio, then we have to include it in the conditioning as well

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_c>t\}}\widehat{L}_t\right] = \int \mathbb{E}\left[\widehat{L}_t\,|Y,\tau_c > t\right]\mathbb{P}\left(\tau_c > t\,|Y\right)\phi\left(Y\right)dY,$$
$$\mathbb{E}\left[\widehat{L}_{t-}\mathbf{1}_{\{\tau_c=t\}}\right] = \int \mathbb{E}\left[\widehat{L}_{t-}\,|Y,\tau_c = t\right]\mathbb{P}\left(\tau_c = t\,|Y\right)\phi\left(Y\right)dY.$$

Note. In this case, when computing the expected tranche loss, we need to set the counterparty default probability to 0.

Options on Tranches

To price a **Tranche Option**, we can use one of the following methods:

a) Use a simple **Black formula** conditional on the survival of the tranche. In this case, we would typically have an implied vol skew since there will be jumps induced by the conditioning on the $\{\mathcal{G}_t\}$-filtration and the default of the other names.

b) Use a full **Bottom-up model** where the default of the indvidual names and their dynamics are modelled. This would be numerically very intensive.

c) Use a **Top-down approach** without random-thinning, and assume that $\mathcal{G}_t = \sigma(L_t)$. This leads to a combination (or a mixture) of Black formulas by conditioning on the realizations of the portfolio loss variable L_t.

For the purposes of valuing CVA for a correlation book, the main factor that needs to be modelled properly is the correlation between the portfolio and the collateral. Getting the Forward values correctly priced is key, the volatility modelling aspect is a second step. And the impact of volatility is typically very limited when trades are deeply-in-the-money.

Black Formula

Trick. We need to condition on the survival of the tranche (i.e., the tranche has not been wiped out before the option maturity date). This is necessary to be able to define an (a.s.) strictly positive annuity.

The value of a (U,T)-tranche at time at t is

$$
\begin{aligned}
V_{t,U,T}^{CDO} &= \mathbb{E}\left[\int_U^T p_{t,U,s} d\widehat{L}_s - \sum_{T_i>U} p_{t,U,T_i} S^0 \delta_i \left(1-\widehat{L}_{T_i}\right) | \mathcal{G}_t\right] \\
&= \mathbf{1}_{\left\{\widehat{N}_U>0\right\}} \mathbb{E}\left[\int_U^T p_{t,U,s} d\widehat{L}_s - \sum_{T_i>U} p_{t,U,T_i} S^0 \delta_i \left(1-\widehat{L}_{T_i}\right) \middle| \mathcal{G}_t \vee \left\{\widehat{N}_U>0\right\}\right].
\end{aligned}
$$

The Break-even spread is given by

$$
S_{t,U,T} = \frac{\mathbb{E}\left[\int_U^T p_{t,U,s} d\widehat{L}_s \middle| \mathcal{G}_t \vee \left\{\widehat{N}_U>0\right\}\right]}{\mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i} \delta_i \left(1-\widehat{L}_{T_i}\right) \middle| \mathcal{G}_t \vee \left\{\widehat{N}_U>0\right\}\right]}.
$$

And the Forward Annuity can be defined as

$$A_{t,U,T} = \mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\Big|\mathcal{G}_t \vee \left\{\widehat{N}_U > 0\right\}\right].$$

The value of the tranche can be written as

$$V^{CDO}_{t,U,T} = \mathbf{1}_{\left\{\widehat{N}_t>0\right\}} A_{t,U,T}\left(S_{t,U,T} - S^0\right),$$

and the CDO option is given by

$$\begin{aligned}
O_0 &= p_{0,t}\mathbb{E}\left[\left(V^{CDO}_{t,t,T}\right)^+\right] = p_{0,t}\mathbb{P}\left(\widehat{N}_t > 0\right)\mathbb{E}\left[A_{t,t,T}\left(S_{t,t,T} - S^0\right)^+\right] \\
&= p_{0,t}\mathbb{P}\left(\widehat{N}_t > 0\right)\mathbb{E}\left[\sum_{T_i>t} p_{0,t,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\Big|\left\{\widehat{N}_t > 0\right\}\right]\mathbb{E}^{Q_A}\left[\left(S_{t,t,T} - S^0\right)^+\right] \\
&= \mathbb{E}\left[\sum_{T_i>t} p_{0,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\mathbf{1}_{\left\{\widehat{N}_t>0\right\}}\right] Black\left(S_{0,t,T}, S^0, \sigma_{BS}\sqrt{T-t}\right) \\
&= \mathbb{E}\left[\sum_{T_i>t} p_{0,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\right] Black\left(S_{0,t,T}, S^0, \sigma_{BS}\sqrt{T-t}\right).
\end{aligned}$$

The last step is due to the fact that $\left\{\widehat{N}_t = 0\right\} \Longrightarrow \left\{\widehat{N}_T = 1 - \widehat{L}_T = 0\right\}$, for all $T \geq t$.

Black CVA

To compute the CVA, we need to compute the value of the forward tranche on the set $\mathbf{1}_{\left\{\widehat{N}_t>0\right\}}\mathbf{1}_{\{\tau_c=t\}}$:

$$\begin{aligned}
\widetilde{V^{CDO}_{t,U,T}} &= \mathbf{1}_{\{\tau_c=U\}}\mathbb{E}\left[\int_U^T p_{t,U,s}d\widehat{L}_s - \sum_{T_i>U} p_{t,U,T_i}S^0\delta_i\left(1-\widehat{L}_{T_i}\right)|\mathcal{G}_t\right] \\
&= \mathbf{1}_{\left\{\widehat{N}_U>0\right\}}\mathbf{1}_{\{\tau_c=U\}}\mathbb{E}\left[\int_U^T p_{t,U,s}d\widehat{L}_s - ...\middle|\mathcal{G}_t \vee \left\{\widehat{N}_U>0\right\} \vee \{\tau_c=U\}\right].
\end{aligned}$$

The Break-even spread is

$$\widetilde{S_{t,U,T}} = \frac{\mathbb{E}\left[\int_U^T p_{t,U,s}d\widehat{L}_s\middle|\mathcal{G}_t \vee \left\{\widehat{N}_U>0\right\} \vee \{\tau_c=U\}\right]}{\mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\middle|\mathcal{G}_t \vee \left\{\widehat{N}_U>0\right\} \vee \{\tau_c=U\}\right]},$$

and the Forward Annuity is given by

$$\widetilde{A_{t,U,T}} = \mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\middle|\mathcal{G}_t \vee \left\{\widehat{N}_U>0\right\} \vee \{\tau_c=U\}\right].$$

The CVA knock-in option price is

$$\begin{aligned}
\widetilde{O_0} &= p_{0,t}\mathbb{P}\left(\tau_c = t\right)\mathbb{E}\left[\left(V_{t,t,T}^{CDO}\right)^{+} |\{\tau_c = t\}\right] = p_{0,t}\mathbb{E}\left[\mathbf{1}_{\{\tau_c=t\}}\left(V_{t,t,T}^{CDO}\right)^{+}\right] \\
&= p_{0,t}\mathbb{P}\left(\widehat{N}_t > 0, \tau_c = t\right)\mathbb{E}\left[\widetilde{A_{t,t,T}}\left(\widetilde{S_{t,t,T}} - S^0\right)^{+}\right] \\
&= p_{0,t}\mathbb{P}\left(\widehat{N}_t > 0, \tau_c = t\right)\widetilde{A_{0,t,T}}\mathbb{E}^{Q_{\widetilde{A}}}\left[\left(\widetilde{S_{t,t,T}} - S^0\right)^{+}\right] \\
&= \mathbb{E}\left[\sum_{T_i > t} p_{0,T_i}\delta_i\left(1 - \widehat{L}_{T_i}\right)\mathbf{1}_{\left\{\widehat{N}_t > 0\right\}}\mathbf{1}_{\{\tau_c=t\}}\right] Black\left(\widetilde{S_{0,t,T}}, S^0, \widetilde{\sigma_{BS}}\sqrt{T-t}\right) \\
&= \mathbb{E}\left[\sum_{T_i > t} p_{0,T_i}\delta_i\left(1 - \widehat{L}_{T_i}\right)\mathbf{1}_{\{\tau_c=t\}}\right] Black\left(\widetilde{S_{0,t,T}}, S^0, \widetilde{\sigma_{BS}}\sqrt{T-t}\right).
\end{aligned}$$

Dynamic Loss Model

Assumption. We use a Top-Down approach (without random-thinning) and we assume that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L_t)$. So, we can condition on realizations of the portfolio loss variable L_t and we get a mixture of Black formulas.

The value of the forward tranche (U,T) is

$$\begin{aligned} V_{t,U,T}^{CDO} &= \mathbb{E}\left[\int_U^T p_{t,U,s} d\widehat{L}_s - \sum_{T_i>U} p_{t,U,T_i} S^0 \delta_i \left(1-\widehat{L}_{T_i}\right) | \mathcal{F}_t \vee \sigma(L_U)\right] \\ &= \mathbf{1}_{\left\{\widehat{N}_U>0\right\}} \mathbb{E}\left[\int_U^T p_{t,U,s} d\widehat{L}_s - \ldots \middle| \mathcal{F}_t \vee \sigma(L_U) \vee \left\{\widehat{N}_U > 0\right\}\right]. \end{aligned}$$

We define the Break-even spread conditional on $\{L_U = x\}$:

$$S_{t,U,T}^x = \frac{\mathbb{E}\left[\int_U^T p_{t,U,s} d\widehat{L}_s \middle| \mathcal{F}_t \vee \{L_U = x\} \vee \left\{\widehat{N}_U > 0\right\}\right]}{\mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i} \delta_i \left(1-\widehat{L}_{T_i}\right) \middle| \mathcal{F}_t \vee \{L_U = x\} \vee \left\{\widehat{N}_U > 0\right\}\right]}.$$

And the Forward Annuity conditional on $\{L_U = x\}$ can be defined as

$$A^x_{t,U,T} = \mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\Big|\mathcal{F}_t \vee \{L_U = x\} \vee \left\{\widehat{N}_U > 0\right\}\right].$$

We rewrite the value of the tranche as

$$V^{CDO}_{t,t,T} = \sum_x \mathbf{1}_{\left\{\widehat{N}_t>0\right\}}\mathbf{1}_{\{L_t=x\}}A^x_{t,t,T}\left(S^x_{t,t,T} - S^0\right).$$

The option value becomes

$$\begin{aligned} O_0 &= p_{0,t}\mathbb{E}\left[\left(V^{CDO}_{t,t,T}\right)^+\right] = p_{0,t}\mathbb{E}\left[\int_x \mathbf{1}_{\left\{\widehat{N}_t>0\right\}}\mathbf{1}_{\{L_t\in dx\}}A^x_{t,t,T}\left(S^x_{t,t,T}-S^0\right)^+\right] \\ &= p_{0,t}\left[\int_x \mathbb{P}\left(\widehat{N}_t > 0, L_t \in dx\right)A^x_{0,t,T}\mathbb{E}^{Q_{A^x}}\left[\left(S^x_{t,t,T}-S^0\right)^+\right]\right] \\ &= \int_x \mathbb{E}\left[\sum_{T_i>t} p_{0,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\mathbf{1}_{\left\{\widehat{N}_t>0\right\}}\mathbf{1}_{\{L_t\in dx\}}\right] Black\left(S^x_{0,t,T}, S^0, \sigma^x_{BS}\sqrt{T-t}\right) \\ &= \int_x \mathbb{E}\left[\sum_{T_i>t} p_{0,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\mathbf{1}_{\{L_t\in dx\}}\right] Black\left(S^x_{0,t,T}, S^0, \sigma^x_{BS}\sqrt{T-t}\right). \end{aligned}$$

To compute the terms used in the Annuity and the Break-even spread, we need a dynamic loss model (or a time-copula), to compute the conditional forward tranche loss:

$$\mathbb{E}\left[\widehat{L}_T \,|L_t = x\right], \text{ for all } T \geq t.$$

This will be a double-integral on (L_t, L_T), which can be computed numerically in a dynamic loss model (see, for example, Elouerkhaoui (2008)).

To construct a dynamic loss model, we typically have a low-dimensional Markov process z_t and a mapping function to convert to the loss variable at each time step by matching the marginals

$$\begin{aligned} dz_t &= ..., \\ L_t &= \phi_{L_t}^{-1}\left(\phi_{z_t}\left(z_t\right)\right). \end{aligned}$$

To compute the tranche loss $\mathbb{E}\left[\widehat{L}_T \,|L_t = x\right]$, we integrate over the conditional densities of the Markovian process

$$\mathbb{E}\left[\widehat{L}_T \,|L_t = x\right] = \mathbb{E}\left[f\left(L_T\right)|L_t = x\right] = \mathbb{E}\left[f\left(\phi_{L_T}^{-1}\left(\phi_{z_T}\left(z_T\right)\right)\right)\Big|z_t = \phi_{z_t}^{-1}\left(\phi_{L_t}\left(x\right)\right)\right].$$

Dynamic CVA

To compute the CVA, we need to compute the value of the tranche on the set $\mathbf{1}_{\left\{\widehat{N}_t>0\right\}}\mathbf{1}_{\{L_t=x\}}\mathbf{1}_{\{\tau_c=t\}}$:

$$\begin{aligned}\widetilde{V_{t,U,T}^{CDO}} &= \mathbf{1}_{\{\tau_c=U\}}\mathbb{E}\left[\int_U^T p_{t,U,s}d\widehat{L}_s - ...\,|\mathcal{F}_t \vee \sigma\left(L_U\right) \vee \{\tau_c = U\}\right] \\ &= \mathbf{1}_{\left\{\widehat{N}_U>0\right\}}\mathbf{1}_{\{\tau_c=U\}}\mathbb{E}\left[...\left|\mathcal{F}_t \vee \sigma\left(L_U\right) \vee \left\{\widehat{N}_U > 0\right\} \vee \{\tau_c = U\}\right.\right].\end{aligned}$$

The Break-even spread conditional on $\{L_U = x\}$ is

$$\widetilde{S_{t,U,T}^x} = \frac{\mathbb{E}\left[\int_U^T p_{t,U,s}d\widehat{L}_s\left|\mathcal{F}_t \vee \{L_U = x\} \vee \left\{\widehat{N}_U > 0\right\} \vee \{\tau_c = U\}\right.\right]}{\mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i}\delta_i\left(1 - \widehat{L}_{T_i}\right)\left|\mathcal{F}_t \vee \{L_U = x\} \vee \left\{\widehat{N}_U > 0\right\} \vee \{\tau_c = U\}\right.\right]},$$

and the Forward Annuity conditional on $\{L_U = x\}$ is given by

$$\widetilde{A_{t,U,T}^x} = \mathbb{E}\left[\sum_{T_i>U} p_{t,U,T_i}\delta_i\left(1 - \widehat{L}_{T_i}\right)\left|\mathcal{F}_t \vee \{L_U = x\} \vee \left\{\widehat{N}_U > 0\right\} \vee \{\tau_c = U\}\right.\right].$$

The CVA knock-in option price is

$$
\begin{aligned}
\widetilde{O_0} &= p_{0,t}\mathbb{P}\left(\tau_c = t\right)\mathbb{E}\left[\left(V^{CDO}_{t,t,T}\right)^{+}|\{\tau_c = t\}\right] = p_{0,t}\mathbb{E}\left[\mathbf{1}_{\{\tau_c=t\}}\left(V^{CDO}_{t,t,T}\right)^{+}\right] \\
&= p_{0,t}\left[\int \mathbb{P}\left(\widehat{N}_t > 0, L_t \in dx, \tau_c = t\right)\mathbb{E}\left[\widetilde{A^x_{t,t,T}}\left(\widetilde{S^x_{t,t,T}} - S^0\right)^{+}\right]\right] \\
&= p_{0,t}\left[\int \mathbb{P}\left(\widehat{N}_t > 0, L_t \in dx, \tau_c = t\right)\widetilde{A^x_{0,t,T}}\mathbb{E}^{Q_{\widetilde{A}^x}}\left[\left(\widetilde{S^x_{t,t,T}} - S^0\right)^{+}\right]\right] \\
&= \int \mathbb{E}\left[\sum_{T_i>t} p_{0,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\mathbf{1}_{\left\{\widehat{N}_t>0\right\}}\mathbf{1}_{\{L_t\in dx\}}\mathbf{1}_{\{\tau_c=t\}}\right] Black\left(\widetilde{S^x_{0,t,T}}, S^0, \dots\right) \\
&= \int \mathbb{E}\left[\sum_{T_i>t} p_{0,T_i}\delta_i\left(1-\widehat{L}_{T_i}\right)\mathbf{1}_{\{L_t\in dx\}}\mathbf{1}_{\{\tau_c=t\}}\right] Black\left(\widetilde{S^x_{0,t,T}}, S^0, \widetilde{\sigma^x_{BS}}\sqrt{T-t}\right)
\end{aligned}
$$

To compute the conditional expected tranche losses involved here,

$$\mathbb{E}\left[\widehat{L}_T\,|L_t = x, \tau_c = t\right], \text{ for all } T \geq t,$$

we need to combine both the CDO-Squared copula and the dynamic loss model.

This can be done in the following way.

Markovian Dynamics

To compute the tranche loss $\mathbb{E}\left[\widehat{L}_T \,|L_t = x, \tau_c = t\right]$, we need to condition on Y, and compute the loss densities conditional on $\{\tau_c = t\}$, then integrate using the z_t-densities

$$\mathbb{E}\left[\widehat{L}_T \,|L_t^* = x, \tau_c = t\right] = \frac{\mathbb{E}\left[f\left(L_T^*\right)\mathbf{1}_{\{L_t^*=x\}}\mathbf{1}_{\{\tau_c=t\}}\right]}{\mathbb{P}\left(L_t^* = x, \tau_c = t\right)}.$$

We pre-compute and store the skew mappings $g_T^* = \phi_{L_T^*}^{-1} \circ \phi_{L_T}$.

The denominator is given by

$$\begin{aligned}\mathbb{P}\left(L_t^* = x, \tau_c = t\right) &= \int_Y \mathbb{P}\left(L_t^* = x, \tau_c = t \,|Y\right)\phi\left(Y\right)dY \\ &= \int_Y \mathbb{P}\left(L_t = \left(g_t^*\right)^{-1}(x)\,|Y, \tau_c = t\right)\mathbb{P}\left(\tau_c = t\,|Y\right)\phi\left(Y\right)dY.\end{aligned}$$

The numerator can be computed as

$$
\begin{aligned}
&\mathbb{E}\left[f\left(L_T^*\right)\mathbf{1}_{\left\{L_t^*=x\right\}}\mathbf{1}_{\{\tau_c=t\}}\right] \\
=\ &\int_Y \mathbb{E}\left[f\left(L_T^*\right)\mathbf{1}_{\left\{L_t^*=x\right\}}\mathbf{1}_{\{\tau_c=t\}}\,|Y\right]\phi\left(Y\right)dY \\
=\ &\int_Y \mathbb{E}\left[f\left(g_T^*\left(L_T\right)\right)\mathbf{1}_{\left\{g_t^*\left(L_t\right)=x\right\}}\,|Y,\tau_c=t\right]\mathbb{P}\left(\tau_c=t\,|Y\right)\phi\left(Y\right)dY.
\end{aligned}
$$

Then, to calculate the conditional expectation in the integrand, we need to use the z_t-time-dependence and the conditional marginals of (L_t, L_T):

$$
\begin{aligned}
&\mathbb{E}\left[f\left(L_T^*\right)\mathbf{1}_{\left\{L_t^*=x\right\}}\,|Y,\tau_c=t\right] \\
=\ &\mathbb{E}\left[f\left(g_T^*\left(L_T\right)\right)\mathbf{1}_{\left\{g_t^*\left(L_t\right)=x\right\}}\,|Y,\tau_c=t\right] \\
=\ &\mathbb{E}\left[f\left(g_T^*\left(\phi^{-1}_{L_T|Y,\tau_c=t}\left(\phi_{z_T}\left(z_T\right)\right)\right)\right)\mathbf{1}_{\left\{g_t^*\left(\phi^{-1}_{L_t|Y,\tau_c=t}\left(\phi_{z_t}\left(z_t\right)\right)\right)=x\right\}}\right] \\
=\ &\mathbb{E}\left[f\left(g_T^*\left(\phi^{-1}_{L_T|Y,\tau_c=t}\left(\phi_{z_T}\left(z_T\right)\right)\right)\right)\Big|z_t=\phi_{z_t}^{-1}\left(\phi_{L_t|Y,\tau_c=t}\left(\left(g_t^*\right)^{-1}\left(x\right)\right)\right)\right] \\
&\times\mathbb{P}\left(L_t=\left(g_t^*\right)^{-1}\left(x\right)|Y,\tau_c=t\right).
\end{aligned}
$$

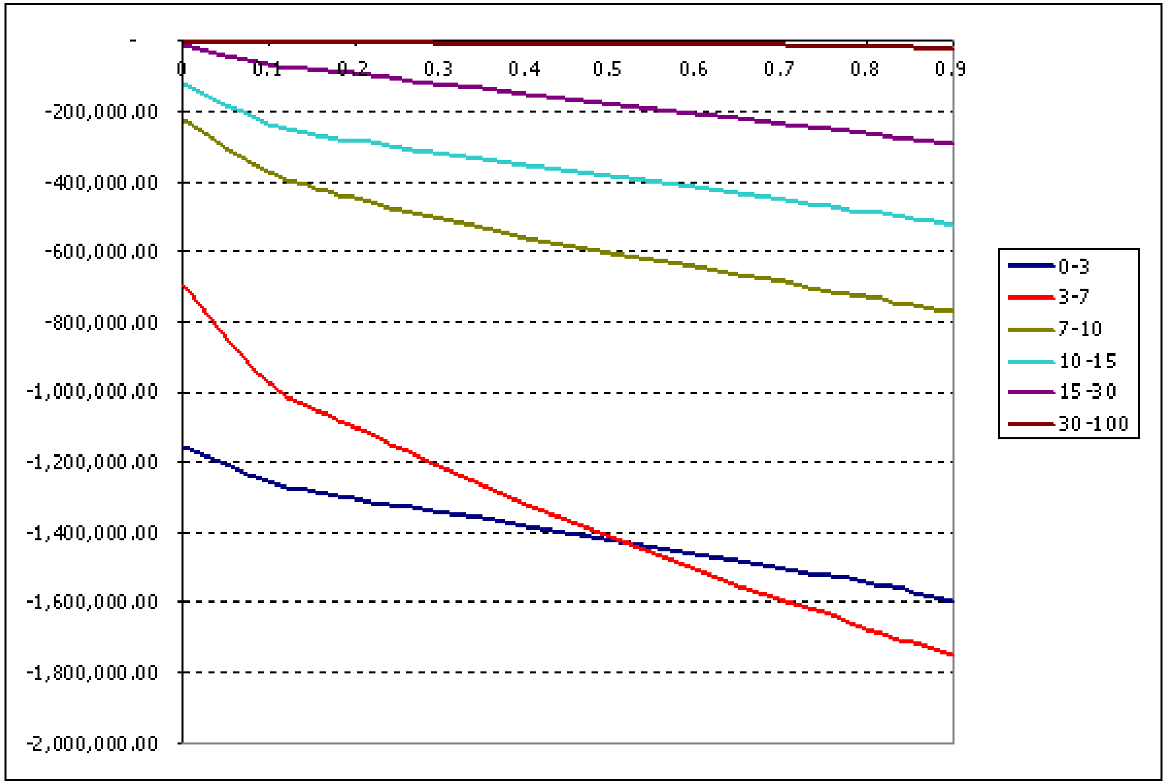
Applications

We run the CVA on the tranches of the CDX10 portfolio as of 7th May 2010. The June-13 index level is 132 bps, and the tranche upfronts and break-even spreads are given below.

0%-3%	44.182	3996 bps
3%-7%	15.936	1083 bps
7%-10%	-1.427	452 bps
10%-15%	1.946	164 bps
15%-30%	-1.675	46 bps
30%-100%	-2.567	13 bps

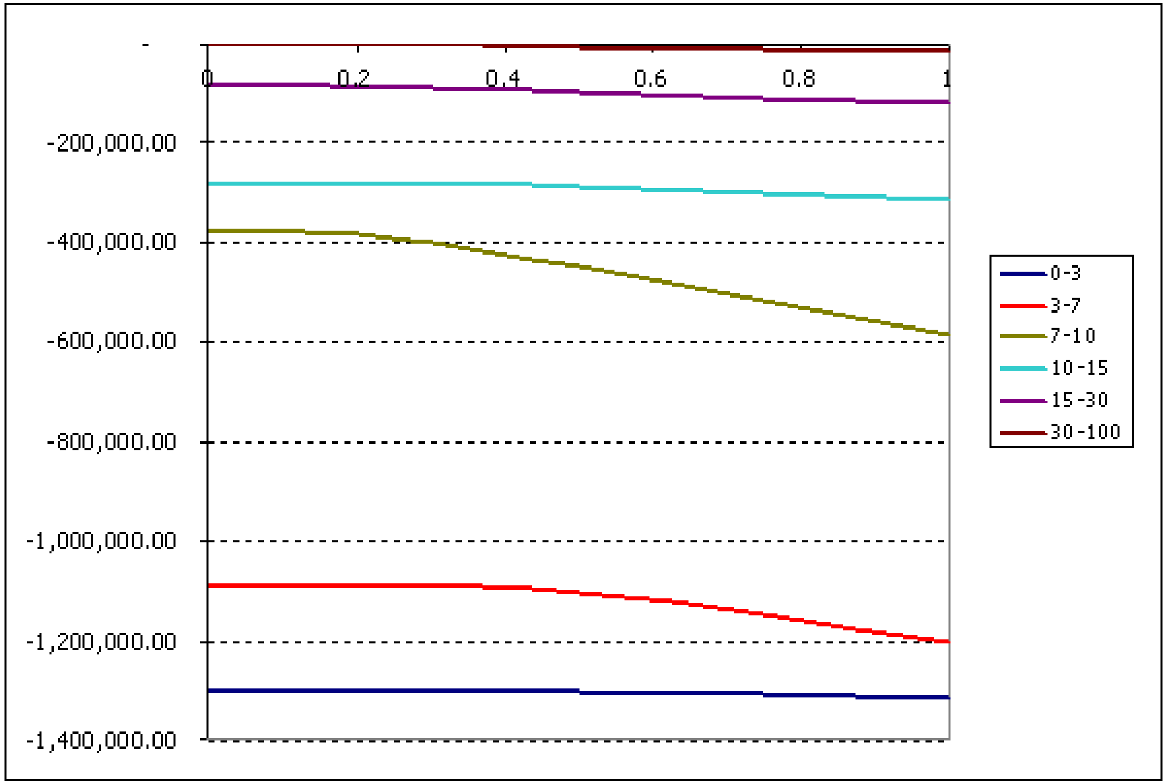
We use the MBIA curve as the counterparty (or collateral) curve: the June-13 upfront is 38.71 and the corresponding SNAC level is 2557 bps.

Unfunded CVA (1)



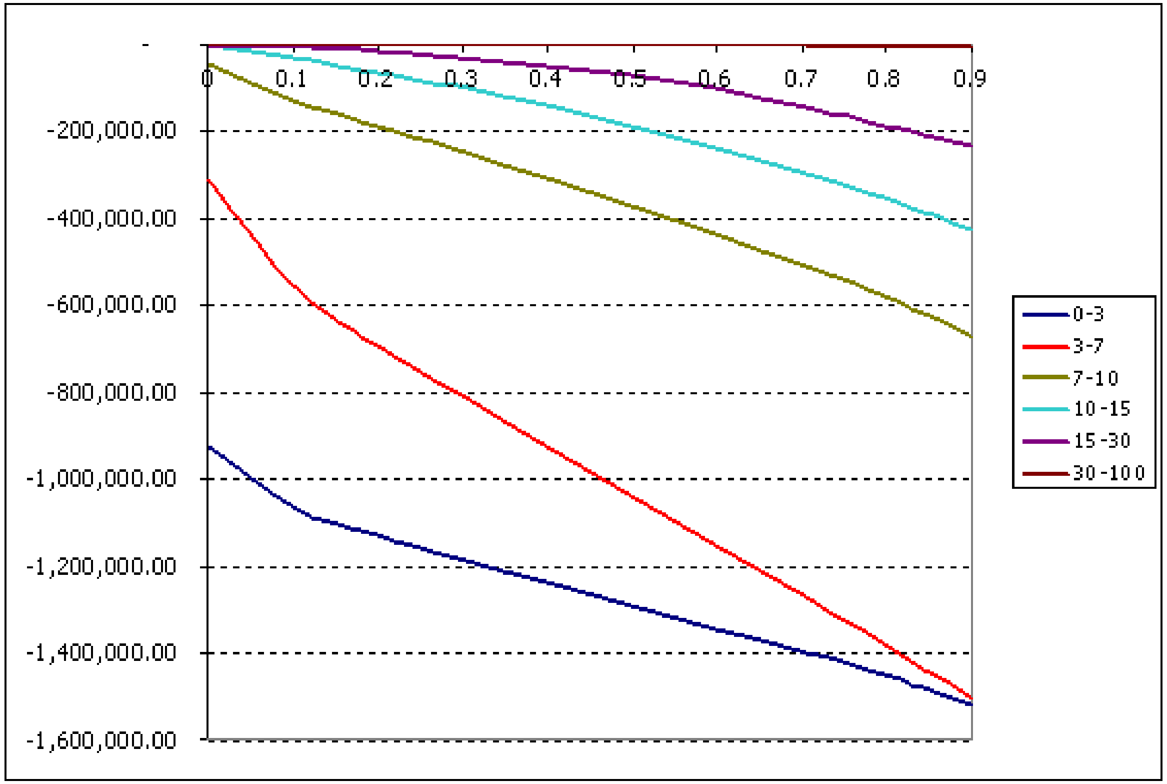
Source: Citi. Unfunded CVA for a non-margined swap with MBIA, as a function of default correlation, for a volatility of 50%

Unfunded CVA (2)



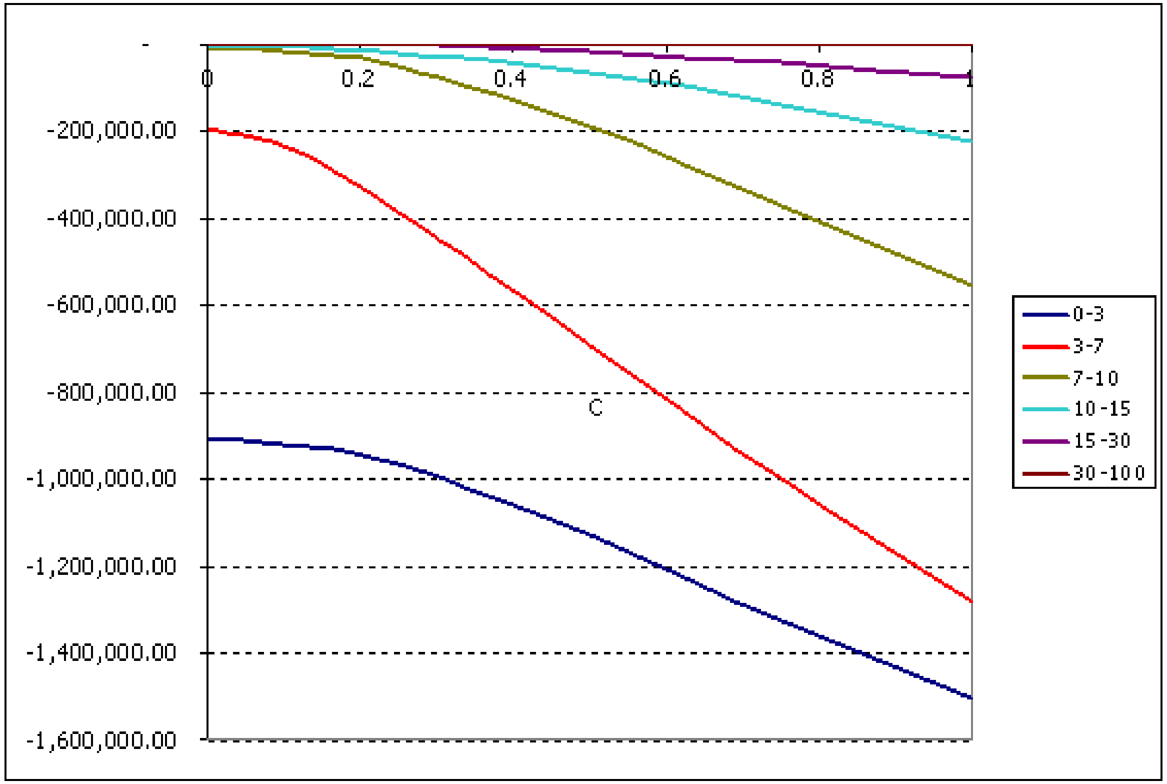
Source: Citi. Unfunded CVA for a non-margined swap with MBIA, as a function of volatility, for a default correlation of 20%

Funded CVA (1)



Source: Citi. Funded CVA for a swap facing an SPV with MBIA collateral, as a function of default correlation, for a volatility of 50%

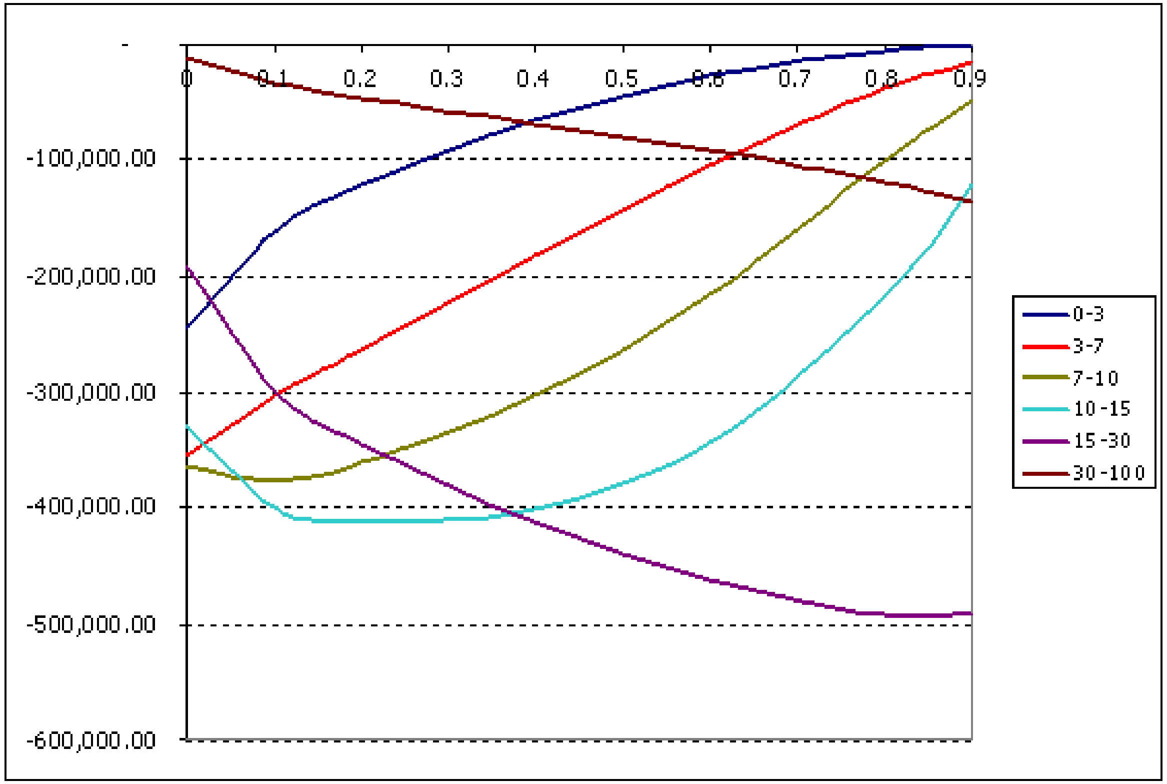
Funded CVA (2)



Source: Citi. Funded CVA for a swap facing an SPV with MBIA collateral, as a function of volatility, for a default correlation of 20%

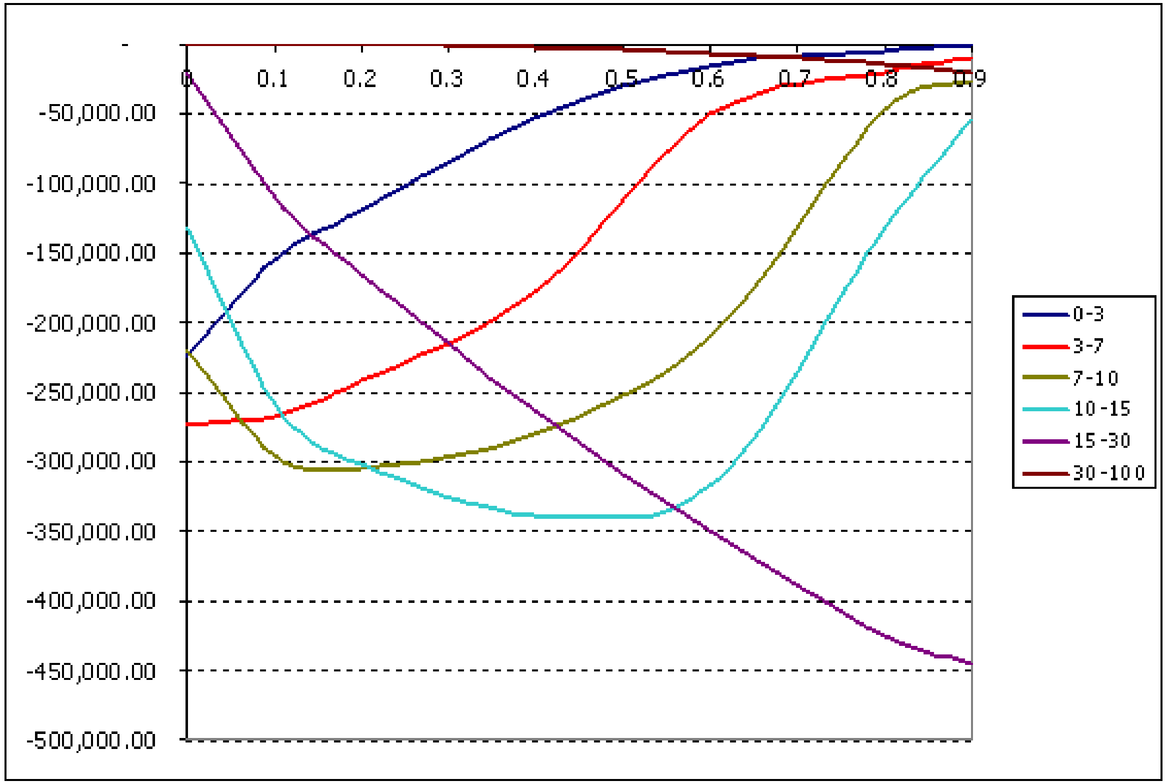
HY Portfolio (1)

We run the CVA on the same set of trades with the HY10 portfolio.



Source: Citi. Unfunded CVA for a non-margined swap with MBIA, as a function of default correlation, for a volatility of 50%

HY Portfolio (2)



Source: Citi. Funded CVA for a swap facing an SPV with MBIA collateral, as a function of default correlation, for a volatility of 50%

Conclusion

- We have reviewed the most commonly used CVA concepts and definitions, and we have developed a general methodology for pricing CVA on CDO Tranches

- We have used a variety of modelling techniques that were developed over the last few years, which include: dynamic loss models, CDO-Squared methodologies, and Option pricing

- We have derived closed-form solutions for funded and unfunded CVA on CDS using standard change of measure techniques, and introducing the Conditional Forward Annuity Measure

- We have addressed the issues involved in CDO CVA modelling and derived pricing formulas based on a Black model and a Markovian Dynamic model.

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