KING'S COLLEGE LONDON

15 FEBRUARY 2011

■

Arbitrage Bounds for Prices of Options on Realized Variance

■

MARK DAVIS

Department of Mathematics

Imperial College London

www.ma.ic.ac.uk/∼mdavis

Joint work with VIMAL RAVAL and JAN OBŁÓJ.

AGENDA

- Introduction
  - Model-free bounds for puts and calls
  - Variance swap + one put option
  - Weighted realized variance
- Hedging convex payoffs
  - Lower bound
  - Upper bound
  - Arbitrage conditions
- Examples
- Comparison with market data
- Concluding remarks

Model-free bounds on European option prices

Given a set of traded option prices, is there an arbitrage opportunity? Note: no model is given *a priori*. There are three possibilities

- *There is a model-independent arbitrage.* We can realize a profit by trading at time zero. Example: butterfly spread with negative price.

- *The prices are consistent with absence of arbitrage.* There is a model such that for all options (p_i, H_i) we have $p_i = D\mathbb{E}[H_i]$, where H_i is the possibly path-dependent exercise value. A *model* for an asset price S_t is simply a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{Q})$ carrying an adapted process S_t such that $S_t = F_t M_t$ where F_t is the forward price and M_t is a $\mathbb{Q}$-martingale with $M_0 = 1$. Example: a put option has model price

$$p_K = D\mathbb{E}[(K - S_T)^+].$$

  In normalized units $r = p_K/DF$, $k = K/F$ this is

$$r = \mathbb{E}[(k - M_T)^+].$$

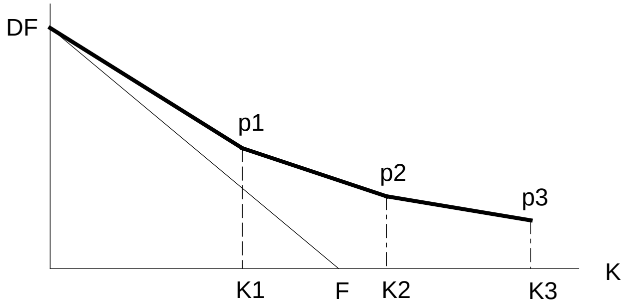
- *There is a model-dependent arbitrage.* Prices are inconsistent with any model but more information is needed to determine the arbitrage strategy. Example: two call options with different strikes but the same price.

We say there is *weak arbitrage* if the 1st or 3rd cases hold. We then have a dichotomy between 'weak arbitrage' and 'consistency with absence of arbitrage'.

Standing assumptions

- Liquid market in underlying asset $S_t,\ t \in [0, T]$.
- No interest rate volatility; time-0 discount factors are $p(0, t) = D_t$ (usually $D \equiv D_T$)
- Uniquely determined forward price F_t (e.g. deterministic dividend yield q) Usually $F = F_T$.
- Options are traded at time 0 at quoted prices. In this talk all options are European with the same exercise time T.

Put and call bounds



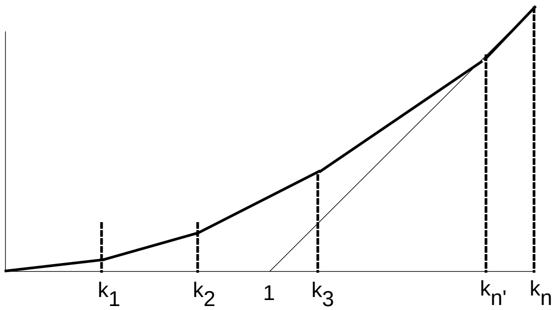
Call options. Normalized prices (k_i, c_i) (with $(k_0, c_0) = (0, 1)$)are consistent with absence of arbitrage if the linear interpolant is strictly decreasing and convex and lies above the line $c = 1 - k$. Model-dependent arbitrage if not *strictly* decreasing. Else model-free arbitrage.

Put options

By put-call parity, the conditions are as shown in the figure. In particular

$$D(K-F)^+ \leq P < DK$$

or in normalized units: $(k-1)^+ \leq p < k$.



The slope is < 1 unless there is some $\bar{n}$ such that $p_{\bar{n}} = k_{\bar{n}} - 1$. In this case, $\mathbb{P}[S_T/F_T > k_{\max}] = 0$ in any model.

Also define $\underline{n} = \max\{i : p_i = 0\}$ and $\mathbf{K} = [\underline{n}, \bar{n}]$.

Adding a Variance swap

From now on we make—at least—the standing assumption

(A) The price S_t is a positive continuous function of $t \in [0, T]$.

Important point: In considering puts and calls, we essentially determine a probability law μ for $M_T = S_T/F_T$. To create a 'model' satisfying (A), we need to specify a continuous martingale whose law at time T is μ. Let B_t be Brownian motion with $B_0 = S_0$. By Skorohod embedding, for any law μ there is a stopping time τ such that $B_\tau \sim \mu$ and $B_{t\wedge\tau}$ is a u.i. martingale. We can now take $M_t = B(\frac{t}{T-t} \wedge \tau)$.

The key point is that imposing (A) *does not change* the arbitrage conditions for plain-vanilla puts and calls.

Variance swap

By standard convention, a variance swap is deemed to be a forward contract in which a cash payment of p_{vs} is exchanged at time T for the realized quadratic variation of returns $< \log S >_T$. By Ito

$$< \log S >_t = -2\log(S_T/S_0) + 2\int_0^T \frac{dS_t}{S_t},$$

and hence in any model

$$p_{\mathrm{vs}} = -2\mathbb{E}[\log(S_T)] + 2\log(S_0).$$

However, without further assumptions we have no *model-free* definition of the variance swap contract (other than the actual market definition!) For this we need *calcul d'Ito sans probabilités*. Strengthen (A) to

(**A**$'$) The price S_t is a positive continuous function on $[0, T]$ having the quadratic variation property.

Definition: A *partition* π of $[0,T]$ is a finite sequence $0 \leq t_0 < t_1 < \cdots < t_k \leq T$. The *mesh size* is $\max_{1 \leq j \leq k}(t_j - t_{j-1})$. A function $S : [0,T] \to \mathbb{R}$ *has the quadratic variation property (QVP)* if there is some sequence π_n of partitions such that the mesh size converges to zero and the sequence of measures

$$\mu_n = \sum_{t_j \in \pi_n} (S(t_{j+1}) - S(t_j))^2 \delta_{t_j}$$

converges weakly to a measure on $[0,T]$ whose distribution function is denoted $< S >_t$.

If S has the QVP and $X_t = F(S_t)$ for $F \in C^1$ then X has the quadratic variation property and

$$< X >_t = \int_0^t (F'(S_u))^2 d < S >_u .$$

In particular if S satisfies $(\mathbf{A}')$ and $F = \log$ we have

$$< \log S >_t = \int_0^t \frac{1}{S_u^2} d < S >_u \tag{1}$$

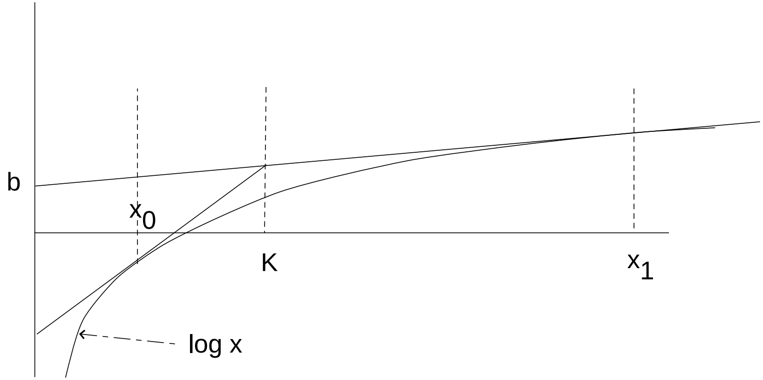
Applying the Föllmer-Ito formula to $\log S_t$ and using (1) we obtain

$$< \log S >_t = -2 \log(S_t/S_0) + 2 \int_0^t \frac{1}{S_u} dS_u.$$

REMARK. If S_t is a continuous semimartingale on some probability space then almost all paths have the QVP. Hence if the sample paths do *not* have the QVP then there is an arbitrage opportunity with or without options—no equivalent martingale measure. So strengthening $(\mathbf{A})$ to $(\mathbf{A}')$ is 'harmless'.

Next question: What's the relation between the prices of the log option and other traded options?

Start with one put option, strike K. We obtain bounds by considering super-replicating strategies.



We can superhedge the log option by a static portfolio containing cash, the underlying asset and a short position in the put option with strike K. For minimum cost the payoff profile of the superhedging portfolio consists of two lines tangent to the log curve as shown in the figure.

The two lines have slopes $1/x_0$ and $1/x_1$, so that

$$b + \frac{1}{x_1} x_1 = 1 + b = \log x_1$$

and hence

$$x_1 = e^{1+b}.$$

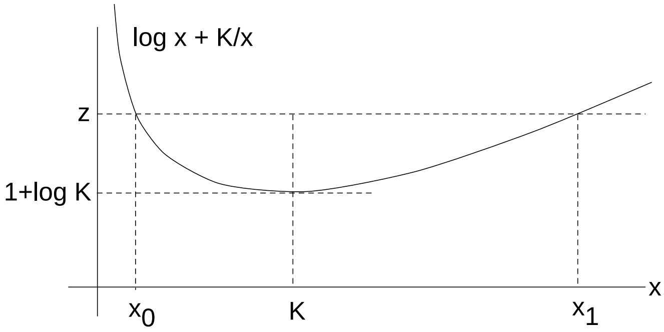
For x_0 we have

$$\log x_0 + \frac{1}{x_0}(K - x_0) = b + \frac{1}{x_1} K = b + K e^{-(1+b)},$$

which implies that

$$\log x_0 + \frac{K}{x_0} = 1 + b + \frac{K}{x_1} = \log x_1 + \frac{K}{x_1} = 1 + b + K e^{-(1+b)}. \tag{2}$$

Thus x_0 and x_1 are the two solutions of $f(x) = z$ where $f(x) = \log x + K/x$ and $z = z(b)$ is the expression on the right of (2). We find that $z(b) \geq 1 + \log K$ for all b.



The value of the superhedging portfolio at time T is

$$v_T = b + \frac{1}{x_1} S_T - \left(\frac{1}{x_0} - \frac{1}{x_1} \right) (K - S_T)^+$$

so its value at time 0 is

$$v_0 = Db + \frac{1}{x_1} DF - \left(\frac{1}{x_0} - \frac{1}{x_1} \right) p_K$$

where p_K is the time-0 put option price. Since the portfolio superhedges, there is an arbitrage opportunity unless $v_{\log} \leq v_0$. We obtain the tightest bound by minimizing v_0 over the one remaining free parameter b, or equivalently z.

Writing $y_0 = 1/x_0$, $y_1 = 1/x_1$ we have, since $y_1 = e^{-(1+b)}$,

$$\begin{aligned} v_0 + D &= D(Fy_1 - \log y_1) - (y_0 - y_1)p_K \\ &= D((F-K)y_1 + z) - (y_0 - y_1)p_K. \end{aligned}$$

Now $dy_j/dz = 1/(K - 1/y_j) = 1/(K - x_j)$, $j = 0, 1$ so

$$\frac{d}{dz}(v_0 + D) = \frac{1}{K - x_1}\left((F - x_1) - \frac{x_0 - x_1}{K - x_0}p_K\right).$$

At the minimum point the derivative is zero, i.e.

$$p_K = D\frac{x_1 - F}{x_1 - x_0}(K - x_0). \tag{3}$$

Letting $\mathbb{Q}$ be the distribution of S_T given by the two-point probability measure $\mathbb{Q} = q\delta_{x_0} + (1-q)\delta_{x_1}$ with $q = (x_1 - F)/(x_1 - x_0)$ we have

- $\mathbb{E}_{\mathbb{Q}}[S_T] = F$
- $p_K = D\mathbb{E}_{\mathbb{Q}}[K - S_T]^+$, from (3)
- $v_0 = D\mathbb{E}_{\mathbb{Q}}[\log S_T]$, since $v_T = \log S_T$ a.s.

Weighted realised variance

The weighting function $w : (0, \infty) \mapsto [0, \infty)$ is a measurable map satisfying

$$\int_A \frac{w(a)}{a^2} da < \infty \quad \text{for all compact } A \subset (0, \infty). \tag{4}$$

Lemma For w satisfying Assumption (4) and λ_w a convex function defined by $\lambda''_w(a) = \frac{w(a)}{a^2}$ we have, in any model $\mathcal{M} \in \mathbb{M}$ and stopping time τ,

$$\int_0^\tau w(M_u) \mathrm{d}\langle \ln M \rangle_u = 2\lambda_w(M_t) - 2\lambda_w(1) - 2\int_0^\tau \lambda'_w(M_u) \mathrm{d}M_u \text{ a.s .} \tag{5}$$

(The right hand side of (5) is zero for affine functions so any normalisation can be chosen.)

This lemma is an application of the Meyer-Itô formula, see Rogers & Williams Section IV.45.

For a *pathwise* interpretation of (5) we need a theory of pathwise local time. (Some results available; work in progress.)

For three common specifications of the weight w, all of which satisfy Assumption (4), we specify functions λ_w explicitly as:

1. Realised variance swap - $w \equiv 1$:

$$\lambda_w(x) = -\ln(x).$$

2. Corridor variance swap - $w(x) = \mathbf{1}_{(0,a)}(x)$ or $w(x) = \mathbf{1}_{(a,\infty)}(x)$, where $0 < a < \infty$:

$$\lambda_w(x) = \left(-\ln\left(\frac{x}{a}\right) + \frac{x}{a} - 1\right) w(x).$$

3. Gamma swap - $w(x) = x$:

$$\lambda_w(x) = x\ln(x) - x.$$

Proposition. Let (M_t) be a continuous martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual assumptions, and let σ be a bounded stopping time. For w satisfying (4) it holds

$$\mathbb{E}\left[\int_0^\sigma w(M_t)\mathrm{d}\langle \ln M\rangle_t\right] = 2\mathbb{E}[\lambda_w(M_\sigma)] - 2\lambda_w(1) \tag{6}$$

and in particular

$$\mathbb{E}\left[\int_0^\sigma w(M_t)\mathrm{d}\langle \ln M\rangle_t\right] < \infty \iff \mathbb{E}[|\lambda_w(M_\sigma)|] < \infty.$$

General Results

Hedging of w-weighted realised variance is achieved by dynamically trading the underlying asset and holding a claim paying $\lambda_w(M_T)$ at time T, the key point being that the claim depends only on S_T, the asset price at time T, which is the time at which the traded European options also expire. In view of the previous propositions,

$$\inf_{\mathcal{M}\in\mathbb{M}^E} \mathbb{E}_{\mathcal{M}}\left[\int_0^T w(M_t)\mathrm{d}\langle \ln S\rangle_t\right] = 2\inf_{\mathcal{M}\in\mathbb{M}^E} \mathbb{E}_{\mathcal{M}}\left[\lambda_w(M_T)\right] - 2\lambda_w(1)$$

and

$$\sup_{\mathcal{M}\in\mathbb{M}^E} \mathbb{E}_{\mathcal{M}}\left[\int_0^T w(M_t)\mathrm{d}\langle \ln S\rangle_t\right] = 2\sup_{\mathcal{M}\in\mathbb{M}^E} \mathbb{E}_{\mathcal{M}}\left[\lambda_w(M_T)\right] - 2\lambda_w(1).$$

Consequently, the problem of determining no-arbitrage bounds for the price of the weighted variance swap reduces to proving the bounds for the corresponding European claim.

Hedging convex payoffs

Suppose that, in addition to the n put options, a European option is offered at price p_λ at time 0, with exercise value $\lambda_T(S_T)$ at time T, where λ_T is a convex function.

We work in normalized units throughout. In a model $\mathcal{M}$ we have $S_t = F_t M_t$ where M_t is a $\mathbb{Q}$ martingale. Put options have normalized strikes and prices $k_i = K_i/F_T$, $r_i = p_i/F_T D_T$ so that

$$r_i = \mathbb{E}_{\mathbb{Q}}[k_i - M_T]^+.$$

To achieve a consistent normalization for λ_T we define the convex function λ as

$$\lambda(x) = \frac{1}{F_T}\lambda_T(F_T x).$$

In a model $\mathcal{M}$ we have $p_\lambda = D_T \mathbb{E}[\lambda_T(S_T)] = D_T F_T \mathbb{E}[\lambda(M_T)]$, so the normalized price is

$$r_\lambda = \frac{p_\lambda}{D_T F_T} = \mathbb{E}[\lambda(M_T)].$$

The cost for delivering a payoff $\lambda(M_T)$ is $D_T r_\lambda$.

Let $\mathbb{M}_P$ be the set of models that correctly price the puts. There is a 1-1 correspondence between $\mathbb{M}_P$ and the set of positive measures μ on $\mathbb{R}^+$ (the marginal distributions of M_T) satisfying *the moment conditions*, i.e.,

$$\begin{aligned}\int_{\mathbb{R}^+} \mu(dx) &= 1\\ \int_{\mathbb{R}^+} x\mu(dx) &= 1\\ \int_{\mathbb{R}^+} [k_i - x]^+ \mu(dx) &= r_i\end{aligned}$$

Any μ satisfying the moment conditions has $\mu(\mathbf{K}) = 1$. μ is consistent with absence of arbitrage if it also correctly prices the convex payoff, i.e. if

$$\int_{\mathbb{R}^+} \lambda(x)\mu(dx) = r_\lambda.$$

Static portfolios have time-T values that are linear combinations of cash, underlying M_T and option exercise values $[k_i - M_T]^+$. The prices of units of these components at time 0 are D_T, D_T and $D_T r_i$ respectively, where a unit of cash is \$1.

Lower bound

A *sub-replicating portfolio* is a static portfolio formed at time 0 such that its value at time T is majorized by $\lambda(M_T)$ for all values of M_T. It turns out that the options k_i with $i \leq \underline{n}$ or $i \geq \bar{n}$ are redundant, so the assets in the portfolio are indexed by $k = 1, \ldots, m$ where

$$m = (n+1) \wedge \bar{n} - \underline{n} + 1$$

and the time-T values of these assets, as functions of $x = M_T$ are

$$\begin{aligned} a_1(x) &= 1 \quad \text{(Cash)} \\ a_2(x) &= x \quad \text{(Underlying)} \\ a_{i+2}(x) &= [k_{\underline{n}+i} - x]^+, \quad i = 1, \ldots, m-2 \quad \text{(Options)}. \end{aligned} \tag{7}$$

The set-up costs for each of these components is

$$\begin{aligned} c_1 &= D_T \\ c_2 &= D_T \\ c_{i+2} &= D_T\, r_{\underline{n}+i}, \quad i = 1, \ldots, m-2. \end{aligned} \tag{8}$$

The corresponding *forward* prices are $1, 1, r_i$, and we define $\mathbf{c}^0$ as the m-vector with components $\mathbf{c}^0_1 = \mathbf{c}^0_2 = 1$, $\mathbf{c}^0_{i+2} = r_i$, $i = 1, \ldots, m-2$, and $\mathbf{a}(x)$ as the m-vector with components $a_k(x)$. A static portfolio is defined by a vector $\mathbf{y}$ whose kth component is the number of units of the kth asset in the portfolio. The forward set-up cost is $\mathbf{y}^\top \mathbf{c}^0$ and the value at T is $f_{\mathbf{y}}(M_T)$ where $f_{\mathbf{y}}(x) = \mathbf{y}^\top \mathbf{a}(x)$.

With this notation, the problem of determining the most expensive sub-replicating portfolio is equivalent to solving the (primal) *semi-infinite linear program*

$$P_{\mathrm{LB}}: \quad \sup_{\mathbf{y} \in \mathbb{R}^m} \mathbf{y}^\top \mathbf{c}^0 \quad \text{subject to} \quad \mathbf{y}^\top \mathbf{a}(x) \le \lambda(x) \; \forall x \in \mathbf{K}.$$

The constraints are enforced only for $x \in \mathbf{K}$. If $\underline{n} > 0$ [$\bar{n} \le n$] we have a free put with strike $k_{\underline{n}}$ [call with strike $k_{\bar{n}}$], so we can extend the sub-replicating portfolio to all of $\mathbb{R}^+$ at no cost.

Key result: Karlin-Isii duality theorem of semi-infinite linear programming.

The dual program corresponding to P_{LB} is

$$D_{\mathrm{LB}}: \quad \inf_{\mu \in \mathbb{M}} \int_{\mathbf{K}} \lambda(x)\mu(dx) \quad \text{subject to} \quad \int_{\mathbf{K}} \mathbf{a}(x)\mu(dx) = \mathbf{c}^0,$$

where $\mathbb{M}$ is the set of Borel measures such that each a_i is integrable. Any $\mu \in \mathbb{M}_P$ has support in $\mathbf{K} = [\underline{n}, \bar{n}]$. The constraints in D_{LB} can be written $\mu \in \mathbb{M}'_P$, where $\mathbb{M}'_P$ is the set of measures satisfying the moment conditions for $\underline{n} < i < \bar{n}$.

Let V_P^L and V_D^L be the values of the primal and dual problems respectively. It is a general and easily proved fact that $V_P^L \leq V_D^L$. The 'duality gap' is $V_D^L - V_P^L$.

Karlin-Isii Theorem: Suppose that

(i) $a_1, \ldots, a_m$ are linearly independent over $\mathbf{K}$,

(ii) $\mathbf{c}^0$ is an interior point of M_m and

(iii) V_D^L is finite.

Then $V_P^L = V_D^L$ and P_{LB} has a solution.

Proposition We suppose as above that $\lambda(x)$ is a convex function on $\mathbb{R}^+$, finite for all $x > 0$, and that (k_i, r_i) is a set of normalized put option strike and price pairs that is consistent with absence of arbitrage. If $\lambda(x)$ is unbounded as $x \to 0$ and $\underline{n} = 0$ then we further assume that $r_1/k_1 < r_2/k_2$. Then $V_D^L = V_P^L$ and there exists a maximizing vector $\hat{\mathbf{y}}$. The most expensive sub-replicating portfolio has weights in, respectively, cash, forward and options $\psi^\dagger = F_T \hat{\mathbf{y}}_1, \phi^\dagger = \hat{\mathbf{y}}_2, \pi^\dagger_{\underline{n}+i} = \hat{\mathbf{y}}_{2+i}$ for $i = 1, \ldots, m-2$ and $\pi_i^\dagger = 0$ otherwise. For this portfolio, $X_0^\dagger = D_T F_T V_D^L$.

The value V_D^L can be represented as

$$V_D^L = \int_{\mathbf{K}} \lambda(x) \mu^\dagger(dx)$$

where $\mu^\dagger$ is an atomic measure such that

$$\mu^\dagger(\{x\}) > 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \pi_i^\dagger [K_i - F_T x]^+ + \phi^\dagger F_T x + \psi^\dagger = \lambda_T(F_T x). \tag{9}$$

Either $\mu^\dagger \in \mathbb{M}_P$ and we have existence in the dual problem, or $\mu^\dagger$ is a probability measure that prices the puts, but not the forward, correctly and $\phi^\dagger = 0$.

Proof. The first part of the proposition is an application of the Karlin-Isii theorem. The primal problem P_{LB} is feasible because any support line corresponds to a portfolio (containing no options). The functions $a_1, \ldots, a_m$ are linearly independent. If $\{(k_i, r_i)\}$ are consistent with absence of arbitrage then there is a measure μ satisfying the moment conditions and such that μ is a finite weighted sum of Dirac measures. There need be no mass at 0 except in the case excluded in the theorem. In every other case there is a realizing measure μ such that $\mu(\{0\}) = 0$. Thus V_D^L is finite under the conditions we have stated.

It remains to verify that the vector $\mathbf{c}^0$ belongs to the interior of the moment cone M_m. For this, it suffices to note that for all i such that $k_i \in (k_{\underline{n}}, k_{\bar{n}})$ it holds that $[k_i - 1]^+ < r_i < k_i$, and so the condition is satisfied. We can now apply the Karlin-Isii theorem to conclude that $V_P^L = V_D^L$ and that we have existence in the primal problem.

For the last part of the proposition, note first that the measure $\mu^\dagger$ to be determined satisfies

$$\int_{\mathbf{K}} \lambda(x)\mu^\dagger(\mathrm{d}x) = \inf_{\mu\in\mathbb{M}_P} \left\{ \int_{\mathbf{K}} \lambda(x)\mu(\mathrm{d}x) \right\}.$$

Recall $\mathbf{K} = [k_{\underline{n}}, k_{\bar{n}}]$ and partition $\mathbf{K}$ into intervals $I_{\underline{n}+1}, \ldots, I_{\bar{n}\wedge n+1}$ defined by

$$I_i = [k_{i-1}, k_i) \;\text{ for } i = \underline{n}+1, \ldots \bar{n}\wedge n \text{ and } I_{\bar{n}\wedge n+1} = [k_{\bar{n}\wedge n}, k_{\bar{n}}),$$

so $I_{\bar{n}\wedge n+1} = \emptyset$ if $\bar{n} \leq n$.

The main ingredient in the proof is the following lemma.

Lemma

Let $\mu \in \mathbb{M}_P$ and suppose $\int_{\mathbf{K}} |\lambda(x)|\mu(\mathrm{d}x) < \infty$. Define

$$\mathcal{I}_\mu = \{i \leq \bar{n} \wedge n + 1 | \mu(I_i) > 0\}. \tag{10}$$

Now let μ' be the measure

$$\mu' = \sum_{i \in \mathcal{I}_\mu} \mu(I_i)\delta_{x_i},$$

in which δ_x denotes the Dirac measure at x, and for an index $i \in \mathcal{I}$, $x_i = \frac{\int_{I_i} x d\mu(x)}{\mu(I_i)}$.
Then $\mu' \in \mathbb{M}_P$ and

$$\int_{\mathbf{K}} \lambda(x)\mu'(dx) \leq \int_{\mathbf{K}} \lambda(x)\mu(dx). \tag{11}$$

This result implies that, in $\mathbb{M}_P$, it suffices to search over atomic measures with at most one atom per interval I_i. We take a minimizing sequence of measures $\mu_n \in \mathbb{M}_P$, use compactness to extract a convergent subsequence, and see what happens. We find:

- If $\bar{n} \leq n$ then there is a limit $\mu^\dagger \in \mathbb{M}_P$, so we have existence in the dual problem.
- If $\bar{n} = \infty$ then we find a limiting measure $\mu^\dagger$, but this may not satisfy

$$\int_{\mathbf{K}} x\mu^\dagger(dx) = 1. \tag{12}$$

  If (12) fails, then $\phi = 0$ in the (primal) optimal portfolio.

In general, we have to actually solve the linear program in order to determine whether, in the notation of the above proof, $\iota_1 < 1$ or $\iota_1 = 1$. However, there is an important special case (because it covers the case $\lambda_T(s) = -\log s$) in which the answer is immediate.

Corollary. If $\lim_{s\to\infty} \lambda_T(s) = -\infty$ then $\mu^\dagger \in \mathbb{M}_P$.

Example

This example shows how existence in the dual problem can fail. We take normalized prices, i.e. $D_T = F_T = 1$ and a single put option with strike $k = 1.2$. The convex function is $\lambda(x) = 1/x$.

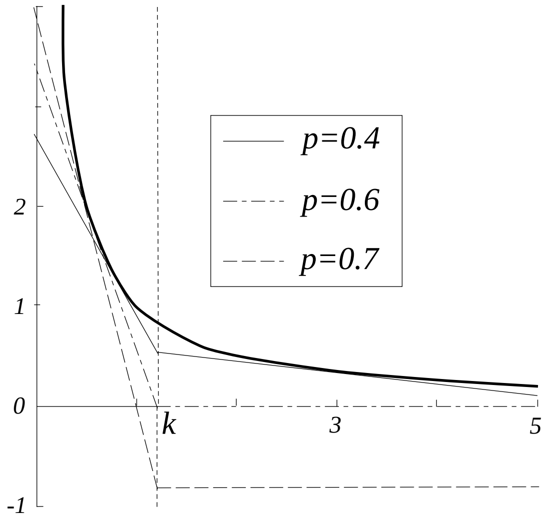
$p = 0.4$ is a 'regular' case: we have tangent lines at x_0, x_1 and the solution to the dual problem puts weights 8/9, 1/9 respectively on these points. As p increases it is advantageous to include more puts in the portfolio, so x_1 increases.

At $p = 0.6$ we reach a boundary case where $\psi = \phi = 0$, and the put is correctly priced by the Dirac measure with weight 1 at $x_0 = 0.6 = k/2$. Obviously, this measure does not correctly price the forward.

When $p > 0.6$, the only way to increase the put component further is to take $\psi < 0$ (and then clearly the optimal value of ϕ is 0.) When $p = 0.7$ the optimal value is $\psi = -0.8$ and we find that in this and every other such case the implied weight is $w_0 = 1$, as the general theory predicts.

p	x_0	x_1	value	ψ	ϕ	w_0	w_1
0.4	0.75	3	1.2222	0.6667	-0.1111	0.8889	0.1111
0.6	0.6	-	1.6667	0	0	1.00	-
0.7	0.5	-	2.00	-0.8	0	1.00	-

Table 1: Data for Example



Upper bound

To compute the cheapest super-replicating portfolio we have to solve the linear program

$$P_{\mathrm{UB}}: \inf_{\mathbf{y}\in\mathbb{R}^m} \mathbf{y}^{\scriptscriptstyle T}\mathbf{c}^0 \quad \text{subject to} \quad \mathbf{y}^{\scriptscriptstyle T}\mathbf{a}(x) \geq \lambda(x)\ \forall x \in \mathbf{K}.$$

The corresponding dual program is

$$D_{UB}: \sup_{\mu\in\mathbb{M}} \int_{\mathbf{K}} \lambda(x)\mu(\mathrm{d}x) \quad \text{subject to} \quad \int_{\mathbf{K}} \mathbf{a}(x)\mu(dx) = \mathbf{c}^0.$$

It turns out that in this case the minimizing vector $\check{\mathbf{y}}$ is explicitly computable, and the only fact about duality that is needed is the inequality $V_D^U \leq V_P^U$ between the primal and dual values. This implies that if for some $\mathbf{y}$ it holds that for any $\epsilon > 0$ there is a feasible μ in the dual problem such that $\int_{\mathbf{K}} \lambda\, d\mu > \mathbf{y}^{\scriptscriptstyle T}\mathbf{c}^0 - \epsilon$, then $\mathbf{y}$ is optimal for P_{UB}.

Consider the conditions

$$\begin{aligned} &(a) \quad \underline{n} > 0, \text{ or } \underline{n} = 0 \text{ and } \lambda(x) = O(1) \ \text{ as } x \to 0, \ \textit{and} \\ &(b) \quad \bar{n} \leq n, \text{ or } \bar{n} = \infty \text{ and } \lambda(x) = O(x) \ \text{ as } x \to \infty \end{aligned} \tag{13}$$

The function $x \mapsto \mathbf{y}^T\mathbf{a}(x)$ is piecewise linear with a finite number of pieces, and it is evident that no such function can majorize the convex function λ over $\mathbb{R}^+$ unless conditions (13) hold. Recall also that super-replication is required only over $x \in \mathbf{K} = [\underline{n}, \bar{n}]$; generally, $\check{\mathbf{y}}^T\mathbf{a}(x)$ will not majorize $\lambda(x)$ over $\mathbb{R}^+$ if $\mathbf{K} \neq \mathbb{R}^+$, but then $\mu(\mathbb{R}^+ \backslash \mathbf{K}) = 0$ for $\mu \in \mathbb{M}_P$, so we get *almost sure* super-replication in any compatible model.

Proposition. If conditions (13) hold then there exists a solution $\check{\mathbf{y}}$ to the linear program P_{UB}. The function $\check{\mathbf{y}}^T\mathbf{a}(x)$ is the linear interpolation of the points $(\underline{n}, \lambda(k_{\underline{n}})), \ldots, (\bar{n}\wedge n, \lambda(k_{\bar{n}\wedge n}))$ together with, if $\bar{n} = \infty$, the line $l(x) = \check{\mathbf{y}}^T\mathbf{a}(k_n) + (x - k_n)\gamma$ for $x \geq k_n$, where

$$\gamma = \lim_{x\to\infty} \frac{\lambda(x)}{x}.$$

If the conditions (13) are not satisfied, there is no feasible solution.

An explanation: Suppose for simplicity that λ is C^2 and that $\bar{n} = n$. We have for $x, x_0 \in \mathbb{R}^+$

$$\lambda(x) = \lambda(x_0) + \lambda'(x_0)(x - x_0) + \int_0^{x_0} [k - x]^+ \lambda''(k) dk + \int_{x_0}^{\infty} [x - k]^+ \lambda''(k) dk.$$

Evaluating this at $x = M_T, x_0 = k_n$ and taking expectations with respect to a probability measure μ on $[0, k_n]$ with expectation 1, we obtain, since call options with strike $k \geq k_n$ have value 0,

$$\mathbb{E}_{\mu}[\lambda(M_T)] = \lambda(k_n) + (1 - k_n)\lambda'(k_n) + \int_0^{k_n} r(k)\lambda''(k) dk,$$

where $r(k)$ is the normalized put option price for strike k in model μ. Hence the dual problem D_{UB} amounts to finding measures that maximize the put option values $r(k)$. The measure in Davis-Hobson gives $r(k)$ as the linear interpolation of r_{j-1} and r_j when $k \in [k_{j-1}, k_j]$ and it is well known that this is the maximum arbitrage-free price consistent with r_{j-1}, r_j.

We can now summarize the results for the cheapest super-replicating portfolio.

Proposition. Under conditions (13) there is a cheapest super-replicating portfolio $(\psi^*, \phi^*, \pi_i^*)$ whose value is

$$X_0^* = \sup_z D_T \int_{\mathbb{R}^+} \lambda_T(F_T x)\mu_z(dx).$$

The underlying component is $\phi^* = \gamma$ if $\bar{n} = \infty$, $\phi^* = 0$ otherwise. The cash component is $\psi^* = F_T(\lambda(k_{\bar{n}\wedge n}) - \gamma k_n \mathbf{1}_{\bar{n}=\infty})$. The option components are $\pi_i^* = \check{\mathbf{y}}_{2+i}$ for $i = \underline{n}, \ldots, \bar{n} \wedge n$ and $\pi_i^* = 0$ otherwise. If conditions (13) are not satisfied, there is no super-replicating portfolio and $X_0^* = +\infty$.

Arbitrage conditions

We can now state the arbitrage relationships when a European option whose exercise value at T is a convex function $\lambda_T(S_T)$ can be traded at time 0 at price p_λ in a market containing traded put options, whose prices p_i are consistent with absence of arbitrage. Recall that $X_0^\dagger$ and X_0^* are respectively the setup costs of the most expensive sub-replicating and cheapest super-replicating portfolios, with $X_0^* = +\infty$ when no super-replicating portfolio exists.

Theorem. The prices $p_\lambda, p_1, \ldots, p_n$ are consistent with absence of arbitrage if and only if

$$p_2 > \frac{K_2}{K_1} p_1 \text{ if } \underline{n} = 0 \text{ and } \lambda \text{ is unbounded at the origin,} \tag{14}$$

and either $p_\lambda \in (X_0^\dagger, X_0^*)$, or $p_\lambda = X_0^\dagger$ and existence holds in D_{LB}, or $p_\lambda = X_0^* < \infty$ and existence holds in D_{UB}.

If (14) holds and either $p_\lambda \notin [X_0^\dagger, X_0^*]$ or $p_\lambda = X_0^\dagger, X_0^*$ and existence fails in $D_{\mathrm{LB}}, D_{\mathrm{UB}}$ respectively, then there is a model-independent arbitrage. If (14) fails there is weak arbitrage.

Proof. Suppose first that condition (14) holds. This condition (under its equivalent form $r_2 > (k_2/k_1)r_1$) guarantees the existence of a sub-replicating portfolio with value $X_0^\dagger$. If $p_\lambda \in (X_0^\dagger, X_0^*)$ then there exists $\epsilon > 0$ such that $p_\lambda \in (X_0^\dagger + \epsilon, X_0^* - \epsilon)$ and, since there is no duality gap, there are measures $\mu_1, \mu_2 \in \mathbb{M}_P$ such that $D_T \mathbb{E}_{\mu_1}[\lambda_T(S_T)] < X_0^\dagger + \epsilon$ and $D_T \mathbb{E}_{\mu_2}[\lambda_T(S_T)] > X_0^* - \epsilon$. A convex combination μ of μ_1 and μ_2 then satisfies $D_T \mathbb{E}_\mu[\lambda_T(S_T)] = p_\lambda$. If existence holds in D_{LB} then the minimizing measure $\mu^\dagger$ satisfies

$$X_0^\dagger = D_T \int_{\mathbf{K}} \lambda_T(F_T x) \mu^\dagger(dx),$$

so that if $p_\lambda = X_0^\dagger$ then $\mu^\dagger$ is a martingale measure that consistently prices the convex payoff λ_T and the given set of put options. The same argument applies on the upper bound side.

It is clear that if $p_\lambda \notin [X_0^\dagger, X_0^*]$ then there is a model-independent arbitrage.

Next, suppose that condition (14) holds and $p_\lambda = X_0^\dagger$ but no minimizing measure μ exists in D_{LB}. Then $\lambda_T(s) \geq X_T^\dagger(s)$ for all $s \in \mathbb{R}^+$ but for any measure $\mu \in \mathbb{M}_P$

$$D_T\mathbb{E}_\mu[\lambda_T(S_T)] > D_T\mathbb{E}_\mu[X_T^\dagger(S_T)] = X_0^\dagger,$$

and hence $\mu\{\lambda_T(S_T) > X_T^\dagger(S_T)\} > 0$. If $\hat{\mu}$ is any measure on $\mathbb{R}^+$ equivalent to μ, there exists $\epsilon > 0$ and $A \in \mathcal{B}(\mathbb{R}^+)$ such that $\hat{\mu}(A) > 0$ and $\lambda_T(s) - X_T(s) > \epsilon$ for $s \in A$. If $p_\lambda = X_0^\dagger$ we can, at zero initial cost, buy $\lambda_T(S_T)$ and sell the portfolio $X_T^\dagger(S_T)$ and this strategy realizes an arbitrage in any model $(\Omega, \mathbb{F}, \mathbb{P}, (S_t = F_t M_t))$ such that $\mathbb{P} \circ M_T^{-1} = \hat{\mu}$. The same argument shows that there is an arbitrage if $X_0^* < \infty$ and $p_\lambda = X_0^*$, in the absence of any maximizing measure in the dual problem P_{UB}.

Now suppose (14) does not hold, so that $p_2/K_2 = p_1/K_1$ (this is the only case other than (14) consistent with absence of arbitrage among the put options). Consider portfolios with exercise values

$$\begin{aligned} H_1(S_T) &= [K_2 - S_T]^+ - \frac{K_2}{K_1}[K_1 - S_T]^+ \\ H_2(S_T) &= \lambda_T(S_T) - \lambda_T(K_2) - \lambda_0'(K_2)(S_T - K_2) - \frac{1}{p_1}\left(p_\lambda - \lambda(K_2) - \lambda_0'(F_T - K_2)\right)[K_1 - \end{aligned}$$

where λ_0' denotes the left derivative. The setup cost for each of these is zero, and $H_1(s) > 0$ for $s \in (0, K_2)$ while $H_2(s) \to \infty$ as $s \to 0$. There is a number $\theta \geq 0$ such that $H(s) = \theta H_1(s) + H_2(s) > 0$ for $s \in (0, K_2)$. Weak arbitrage is realized in a model as above by a portfolio whose exercise value is

$$X_T(S_T) = \begin{cases} -[K_2 - S_T]^+ & \text{if } \mathbb{P}[S_T \in [0, K_2)] = 0 \\ H(S_T) & \text{if } \mathbb{P}[S_T \in [0, K_2)] > 0. \end{cases}$$

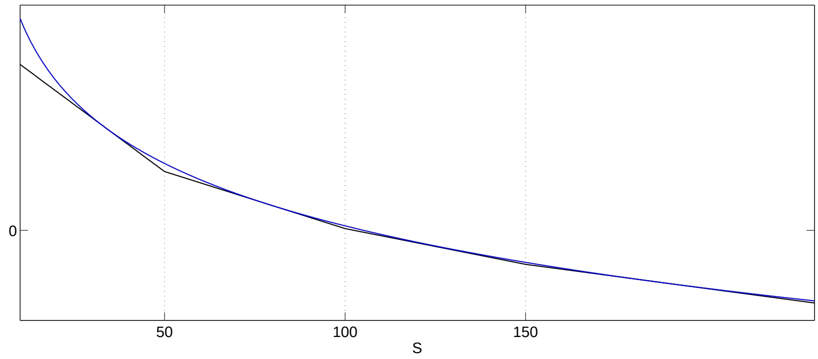
This completes the proof. □

Example

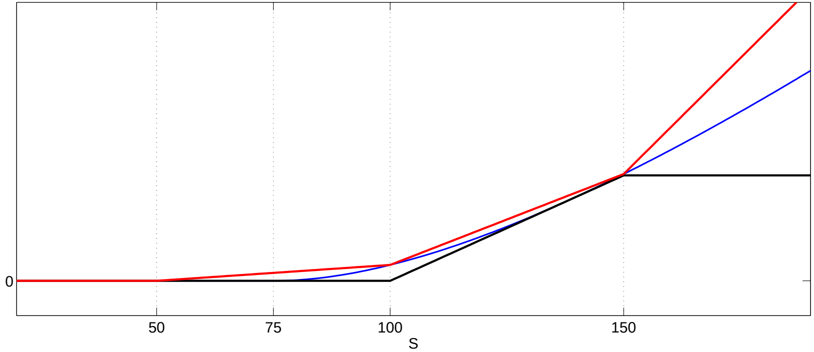
To illustrate an application of our results, consider a fictional market with an index S that trades three European put options maturing in 1 year. The data are $S_0 = 100$, $F_T = 105$, $D_T = \exp(-0.03)$, $K_i = 50, 100$ and 150, $p_1 = 1.127$, $p_2 = 18.006$ and $p_3 = 53.326$. The range of (weak) arbitrage-free prices for a vanilla variance swap, corridor variance swap and gamma swap have been determined and summarised as follows.

Variance Swap Type	$w(x)$	$\lambda_w(x)$	Arbitrage bounds for k_w (£)
Vanilla Swap	1	$-\ln(x)$	$[0.224, \infty)$
Corridor Variance Swap	$\mathbf{1}_{\left[\frac{75}{F_T},\infty\right)}(x)$	$[-\ln(\frac{xF_T}{75}) + \frac{F_T x}{75} - 1]w(x)$	$(0.038, 0.340)$
Gamma Swap	x	$x\ln(x) - x$	$(0.125, \infty)$

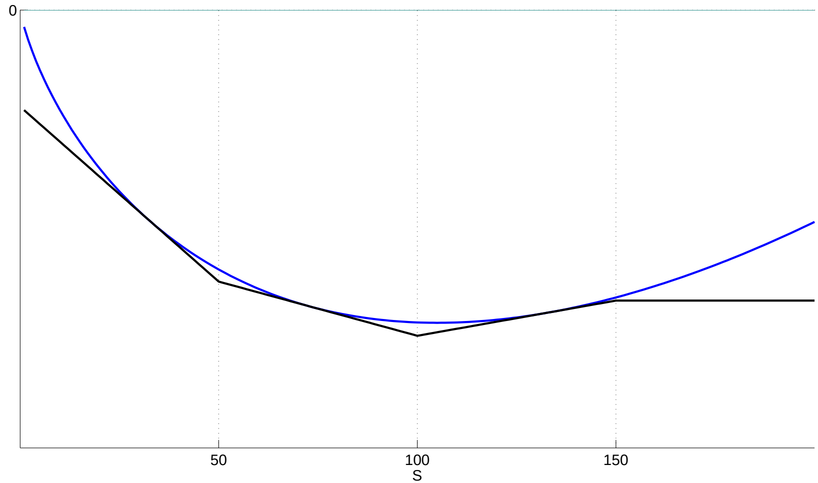
Log contract: The figure shows the log contract payoff $-\ln(S_T/F_T)$ (blue line) and the consequent sub-hedging portfolio (black line). The portfolio is given by $\pi_1^* = 0.01706$, $\pi_2^* = 0.00472$, $\pi_3^* = 0.00259$, $\phi^\dagger = -0.00536$ and $\psi^\dagger = 0.42517$.



Corridor variance swap: The figure shows the payoff $[-\ln(\frac{S_T}{75}) + \frac{S_T}{75} - 1]\mathbf{1}_{\left[\frac{75}{F_T},\infty\right)}(S_T/F_T)$ (blue line), the consequent sub-hedging portfolio (black line) and super-hedging portfolio (red line). The portfolios are given by $\pi_1^\dagger = 0$, $\pi_2^\dagger = 0.00606$, $\pi_3^\dagger = -0.00606$, $\phi^\dagger = 0$ and $\psi^\dagger = 0.30293$ for the sub-hedge, and for the super-hedge: $\pi_1^* = 0.00091$, $\pi_2^* = 0.00431$, $\pi_3^* = 0.00811$, $\phi^* = 0.01333$ and $\psi^* = -1.69315$.



Gamma swap: Here we see the payoff $\frac{S_T}{F_T} \ln\left(\frac{S_T}{F_T}\right) - \frac{S_T}{F_T}$ (blue line) and the consequent sub-hedging portfolio (black line). The portfolio is given by $\pi_1^\dagger = 0.00772$, $\pi_2^\dagger = 0.00571$, $\pi_3^\dagger = -0.00225$, $\phi^\dagger = 0$ and $\psi^\dagger = -0.92899$.



Comment

- We have $p_3 = 53.326 > 43.670 = D_T(K_3 - F_T)$.
- The interval for the vanilla variance swap is half closed and unbounded.
- For the corridor variance swap, the sub-hedging strategy does not hold the underlying index and so indicating the lower bound is not attained. Note that for the choice of corridor in the weight of the corridor variance swap, the swap price has a finite upper bound. In particular the upper bound is not attained due to the inequality $p_3 > D_T(K_3 - F_T)$.
- The Gamma swap has no finite upper bound and the lower bound is not an admissible price, which again is indicated by the corresponding sub-hedge for the convex payoff not holding the index.

Empirical results

Variance swaps on S&P500 Index

Term	Quote date	VS quote	LB	No. of puts
2M	20/04/2008	21.78	18.73	58
2M	19/07/2008	23.6	21.18	51
2M	19/10/2008	57.97	57.07	101
2M	20/01/2008	52.88	47.68	82
3M	20/03/2008	27.22	26.33	48
3M	19/06/2008	22.33	19.24	40
3M	19/09/2008	26.78	26.02	58
3M	20/12/2008	45.93	65.81	137
6M	19/03/2008	25.63	22.97	25
6M	19/06/2008	22.88	21.76	28

Concluding remarks

To get a finite upper bound for the plain-vanilla variance swap requires more 'left-wing' information. See paper.

There are several issues that remain unaddressed, for example:

- The situation with multiple exercise times.
- The effect of jumps.
- Quantifying the error due to discrete sampling (there are some calculations in Gatheral's book).