Portfolio choices and VaR constraint with a defaultable asset

Emilio Barucci

Department of Mathematics

Politecnico di Milano

Andrea Cosso

Department of Mathematics

Politecnico di Milano

Abstract

Assuming a Constant Elasticity of Variance (CEV) model for the asset price, that is a defaultable asset showing the so called leverage effect (high volatility when the asset price is low), a VaR constraint reevaluated over time induces an agent more risk averse than a logarithmic utility to take more risk than in the unconstrained setting.

Keywords:VaR, Optimal portfolio, regulation

JEL Code:G11, G21, G28

1 Introduction

In this paper we show that assuming a Constant Elasticity of Variance (CEV) model for the asset price, a VaR constraint reevaluated over time induces a bank to take more risk than in the unconstrained setting.

This result contributes to the literature on optimal investment and regulation that has investigated the effect of a VaR constraint on the optimal portfolio. Assuming a defaultable asset showing a leverage effect, as in a CEV model, we show that an agent more risk averse than a logarithmic utility detains a position in the risky asset larger than the one of an uncostrained agent.

The literature on this issue is undeterminate as the effect of a VaR constraint depends on the model for the asset price and on the definition of the risk constraint. According to [Basak and Shapiro (2001)], imposing a static VaR constraint, i.e., the loss refers to the difference between the initial and the terminal wealth, and assuming a constant opportunity set, i.e., a lognormal process for the asset price, we have a portfolio riskier than the portfolio obtained without constraint. Because of the VaR constraint, the agent optimally chooses to insure against intermediate loss states and to incur losses in the worst states of the world. As a matter of fact, under a VaR constraint uninsured states are the worst states. This undesired effect is due to the non coherency of VaR, assuming a coherent risk measure, e.g. the Expected Shortfall, the effect disappears and the agent chooses a less risky portfolio. [Cuoco et al. (2008), Leippold et al. (2006)] point out that the excess risk taking is due to the static nature of the VaR constraint considered in [Basak and Shapiro (2001)]: the VaR constraint is placed in $t = 0$ and concerns the final wealth. This approach has two main drawbacks: the policy is dynamically inconsistent, i.e., the constraint is only placed at the beginning of the optimization horizon and the trader may have the incentive to change the investment policy later on; the probability of the portfolio loss is not updated as time goes, this is different to what happens in practice, as a matter of fact financial institutions reevaluate the VaR on a daily or weekly basis. To address these problems they consider a dynamic VaR constraint: the constraint is posed $\forall t \geq 0$ for a short horizon $\tau > 0$, in the interval $[t, t+\tau]$ the portfolio is kept constant. Assuming that the agent has to satisfy the dynamic VaR constraint, its effect on the optimal investment problem is ambiguous. [Cuoco et al. (2008)] consider a lognormal process for the asset price and prove that the VaR constraint in this case leads the agent to take less risk, i.e., the expected value of losses and the portfolio are lower under a VaR constraint than they would have been without the constraint. [Leippold et al. (2006)] consider a more general incomplete market model with a single risky asset whose dynamics depend on a state variable in such a way that both drift and volatility are stochastic. The complexity of the model forces the authors to consider a utility function nearly logarithmic. The effect of a VaR constraint on the portfolio strategy depends on the opportunity set dynamics. In general they cannot say that the constraint induces the agent to take less risk, however they provide some examples in which it induces banks to increase their exposure in high volatility states.

In this paper we analyze the optimal investment problem with a dynamic VaR constraint as in [Cuoco et al. (2008), Leippold et al. (2006)] assuming a CEV model for the asset price. We consider a CEV model as a good choice looking for a realistic market model. We maintain market completeness and tractability removing the constant opportunity set assumption and allowing for a negative correlation between asset price and volatility. Moreover, differently from the lognormal and stochastic volatility cases, an asset following a CEV model may default (when the asset price touches the zero barrier). We derive a clear cut analysis on the effect of a VaR constraint on the investment policy: for a wide set of parameters, a VaR constraint induces an agent more (less) risk averse than a logarithmic utility to take more (less) risk than a risk unconstrained agent. As in [Leippold et al. (2006)] provide an approximation analysis for a utility function in a neighborhood of a logarithmic utility.

The perverse effect of a VaR constraint is strong when the asset is risky or the risk premium is high. A stronger VaR constraint (low α) induces a strong perverse effect, the only way to limit the effect is to increase the constraint on the VaR in terms of the fraction of wealth. The undesired effect disappears (a VaR constrained agent takes less risk) only when the risk of default is very high, i.e., the price is small enough and the asset return-volatility correlation is strongly negative.

These results contribute to the recent debate on the destabilizing role of VaR and in particular on its role in generating the recent subprime financial crisis, see [Adrian and Shin (2008), Adrian and Shin (2010), Danielsson, et al. (2009), Barucci and Cosso (2010)]. As a matter of fact, there are theoretical and empirical results showing that a VaR constraint leads to a constant leverage ratio and to a portfolio of the risky asset that is positively correlated with the wealth. These features may contribute to destabilize the financial market. Our analysis showing that a VaR constraint exacerbates risky bets when the asset may default contributes to explain why banks before the crisis detained asset backed securities that looked as catastrophe bonds, see [Coval et al. (2009), Coval et al. (2009a)].

The paper is organized as follows. In Section 2 we introduce our setting and the optimization problem. In Section 3 we derive the optimal solution for the VaR constrained problem. In Section 4 we provide a comparative statics of the effect of a VaR constraint on the optimal portfolio. In Appendix A we provide the proofs of the main results.

2 The model

We consider a finite horizon $[0, T]$ market model with two assets: a risk free asset and a risky asset. The peculiarity of our setting is that the risky asset is defaultable, i.e., the asset price can attain the point 0.

The risk-free asset is a bond, its price evolves according to an ordinary differential equation:

$$dB(t) = rB(t), \quad B(0) = 1,$$

where the risk-free interest rate r is a positive constant. The price of the risky asset evolves according to a CEV model, see [Cox (1975)]:

$$dS(t) = (\xi + r)S(t)dt + \sigma S(t)^{1+\beta} dW(t), \quad S(0) = s, \tag{1}$$

where the initial price s, the excess return ξ and σ are all positive constants. We assume $\beta \in (-1, 0)$ which implies that the point 0 is an attainable state for the asset price S. To guarantee uniqueness we assume that after reaching zero, the asset price remains at zero, on this point see [Delbaen and Shirakawa (2002)]. The filtered probability space governing the model is $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration generated by a continuous unidimensional Brownian motion $\{W(t)\}_{t\in[0,T]}$.

We remark two important features of the CEV model. First of all, if the asset price evolves as in (1) then we observe the so called leverage effect, i.e., volatility is negatively correlated with asset returns and is high when the asset price is low. Indeed assuming $\beta < 0$, the diffusion coefficient $\sigma S(t)^\beta$ is inversely proportional to the price $S(t)$. Furthermore, a process like (1) is suitable to describe a default event. As a matter of fact, there is a positive probability that the asset price will reach the zero barrier and therefore that the default event will occur, see [Campi and Sbuelz (2005)].

The agent chooses the portfolio of financial assets. The portfolio weight at time t is denoted by $\boldsymbol{\pi}(t) = (1-\pi(t), \pi(t))$, where $\pi(t)$ represents the fraction of wealth invested in the risky asset. Since the portfolio is self-financing, the wealth $V(t)$ evolves according to the following stochastic differential equation

$$dV(t) = (\pi(t)\xi + r)V(t)dt + \pi(t)\sigma S(t)^\beta V(t)dW(t), \quad V(0) = v, \tag{2}$$

where $v > 0$ is the initial wealth. The process $\pi(t)$ is admissible if $\int_0^T |\pi(s)|ds < \infty$ and the resulting wealth process $V(t)$ is such that $V(t) \geq 0$ $\forall t \in [0,T]$. If the portfolio $\pi(t)$ is admissible then we write $\pi \in \mathcal{A}$.

The agent maximizes the expected utility of the final wealth

$$E[u(V(T))].$$

In the sequel we consider a CRRA utility function:

$$u(x) = \begin{cases} \dfrac{x^\gamma - 1}{\gamma}, & \gamma < 0 \text{ and } 0 < \gamma < 1, \\ \ln x, & \gamma = 0. \end{cases}$$

The agent maximizes the expected utility subject to a dynamic VaR constraint as in [Yiu (2004),Cuoco et al. (2008),Leippold et al. (2006)]. Given time horizon $\tau > 0$ and a confidence level $1-\alpha$, the VaR at time t with a constant portfolio over the time interval $[t, t+\tau]$ is defined as

$$\mathrm{VaR}_t^{\alpha,\tau} = \inf\{\ell \in \mathbb{R}^+ : \mathbb{P}(V(t) - \mathcal{V}(t+\tau) > \ell) \leqslant \alpha\}, \tag{3}$$

where $\mathcal{V}(t+\tau)$ is the wealth value at time $t+\tau$ assuming a constant portfolio $\boldsymbol{\pi}(t)$ in the time interval $[t, t+\tau]$. Indeed, we evaluate the VaR considering the frozen portfolio $\boldsymbol{\pi}(t)$ for the interval of time $[t, t+\tau]$.

According to the financial regulation, the VaR should be smaller than a fraction of the assets (wealth). In our analysis we follow [Cuoco et al. (2008),Leippold et al. (2006)] assuming that

$$\mathrm{VaR}_t^{\alpha,\tau} \leqslant \zeta V(t), \qquad \forall t \in [0,T], \tag{4}$$

where $\zeta \in (0,1)$. Hence, the maximum loss with probability $1-\alpha$ is smaller than a fraction of the portfolio value.

We now express the constraint (4) as a constraint on $\pi(t)$. To this end, we follow [Leippold et al. (2006)] performing an Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ centered in $V(t)$. The approximation is provided in Appendix A.1 with a discussion of the order of the approximation error. Thanks to the Itô-Taylor expansion of $\mathcal{V}(t+\tau)$, the following result holds on the VaR constraint with respect to $\pi(t)$.

Proposition 1. *Using the Itô-Taylor expansion (23) of $\mathcal{V}(t+\tau)$, the VaR constraint (4) can be expressed as*

$$\pi^-(S(t)) \leqslant \pi(t) \leqslant \pi^+(S(t)), \qquad t \in [0,T], \tag{5}$$

where

$$\begin{aligned}\pi^{\pm}(S(t)) =& \frac{\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau}}{\sigma^2 S(t)^{2\beta}\tau} \\ &\pm \frac{\sqrt{(\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau})^2 + 2r\sigma^2 S(t)^{2\beta}\tau^2 - 2\ln(1-\zeta)\sigma^2 S(t)^{2\beta}\tau}}{\sigma^2 S(t)^{2\beta}\tau},\end{aligned} \tag{6}$$

and Φ is the cumulative distribution function of the standard normal distribution.

Proof. See Appendix A.1. □

Assuming that the agent has to satisfy the VaR constraint (4) $\forall t \in [0,T]$, the set of viable portfolios in t is provided by $\Pi_{ad}(t)$, i.e., portfolios that are admissible ($\mathcal{A}(t)$) and satisfy the constraints (5) $\forall s \in [t,T]$.

Let J be the value function:

$$J(v,s,t) := \sup_{\pi \in \mathbf{\Pi}_{ad}(t)} \mathbb{E}[u(V(T))|V(t)=v,\, S(t)=s], \tag{7}$$

for every $(v,s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times [0,T]$.

An admissible portfolio $\boldsymbol{\pi}$ which maximizes the expected value (7) in $t=0$ is called the optimal portfolio and is denoted by $\boldsymbol{\pi}^* = (1-\pi^*, \pi^*)$. The value function and the optimal portfolio are fully characterized by an Hamilton-Jacobi-Bellman equation.

Theorem 2. *The Hamilton-Jacobi-Bellman equation for the value function J is*

$$\begin{cases} \dfrac{\partial J}{\partial t} + \sup\limits_{\pi^-(s)\leqslant \pi(t) \leqslant \pi^+(s)} \Big\{ (\pi\xi + r)v\dfrac{\partial J}{\partial v} + (\xi + r)s\dfrac{\partial J}{\partial s} + \tfrac{1}{2}\pi^2\sigma^2 s^{2\beta} v^2 \dfrac{\partial^2 J}{\partial v^2} + \\ \qquad + \pi\sigma^2 s^{1+2\beta} v \dfrac{\partial^2 J}{\partial v \partial s} + \tfrac{1}{2}\sigma^2 s^{2+2\beta}\dfrac{\partial^2 J}{\partial s^2} \Big\} = 0, \qquad \forall\,(v,s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,T), \\ J(v,s,T) = u(v), \qquad\qquad \forall\,(v,s) \in \mathbb{R}^+ \times \mathbb{R}^+. \end{cases}$$

The optimal portfolio in the risky asset has the following expression

$$\pi^*(t) = \begin{cases} \pi^-(S(t)), & \tilde{\pi}(t) \leqslant \pi^-(S(t)), \\ \tilde{\pi}(t), & \pi^-(S(t)) < \tilde{\pi}(t) < \pi^+(S(t)), \\ \pi^+(S(t)), & \tilde{\pi}(t) \geqslant \pi^+(S(t)), \end{cases} \tag{8}$$

where

$$\tilde{\pi}(t) = -\frac{\dfrac{\partial J}{\partial v}}{V(t)\dfrac{\partial^2 J}{\partial v^2}} \frac{\xi}{\sigma^2 S(t)^{2\beta}} - \frac{S(t)\dfrac{\partial^2 J}{\partial v \partial s}}{V(t)\dfrac{\partial^2 J}{\partial v^2}}. \tag{9}$$

Since u is a CRRA utility function, the value function J can be written as follows

$$J(v,s,t) = \begin{cases} \dfrac{e^{\gamma g_\gamma(s,t)} v^\gamma - 1}{\gamma}, & \gamma < 0 \text{ and } 0 < \gamma < 1, \\ g_0(s,t) + \ln v, & \gamma = 0, \end{cases} \tag{10}$$

where g_γ is a function of γ, s and t, but it doesn't depend on v and g_0 is defined as

$$g_0(s,t) := \lim_{\gamma \to 0} g_\gamma(s,t), \qquad \forall\, (s,t) \in \mathbb{R}^+ \times [0,T]. \tag{11}$$

Proof. The derivation of the Hamilton-Jacobi-Bellman equation from the optimization problem follows from classical dynamic programming techniques. As far as the optimal portfolio π^* is concerned, its expression can be obtained considering the second-degree polynomial in π of the Hamilton-Jacobi-Bellman equation. As the second derivative of J with respect to v is negative, the supremum of the polynomial is attained at π^*. Indeed, for a CRRA utility function the homogeneity of the optimization problem (7) J has the expression given in (10). Hence, J is a concave function with respect to v, therefore the second derivative of J with respect to v is negative. □

Substituting the expression of J provided in (10) in (9) we find

$$\widetilde{\pi}(t) = \begin{cases} \dfrac{1}{1-\gamma} \dfrac{\xi}{\sigma^2 S(t)^{2\beta}} + \dfrac{\gamma S(t)}{1-\gamma} \dfrac{\partial g_\gamma}{\partial s}, & \gamma < 0 \text{ and } 0 < \gamma < 1, \\ \dfrac{\xi}{\sigma^2 S(t)^{2\beta}}, & \gamma = 0. \end{cases} \tag{12}$$

Our interest is now to compare this portfolio with that obtained without the VaR constraint and therefore to analyze the consequences of a VaR constraint on the optimal investment problem.

3 The portfolio strategy

We compare the optimal portfolio derived in the presence of the VaR constraint with the one obtained in its absence. In particular we are interested in analyzing the optimal portfolio in the absence of the VaR constraint and the portfolio $\widetilde{\pi}$, i.e., the optimal portfolio when there is a VaR constraint but is not binding.

The optimal portfolio for the optimal investment problem without VaR constraint (π^f) has been computed in [Battauz and Sbuelz (2010)]:

$$\pi^f(t) = \begin{cases} \dfrac{1}{1-\gamma}\dfrac{\xi}{\sigma^2 S(t)^{2\beta}} + \dfrac{\gamma S(t)}{1-\gamma}\dfrac{\partial g_\gamma^f}{\partial s}, & \gamma < 0 \text{ and } 0 < \gamma < 1, \\ \dfrac{\xi}{\sigma^2 S(t)^{2\beta}}, & \gamma = 0, \end{cases} \tag{13}$$

where the function g_γ^f has the following expression

$$g_\gamma^f(s,t) = \frac{1}{1-\gamma}\frac{\xi^2}{\sigma^2 s^{2\beta}}\frac{1-e^{-q\tau}}{2q - \left(q - 2\beta\frac{\xi + r(1-\gamma)}{1-\gamma}\right)\left(1-e^{-q\tau}\right)}, \qquad \forall\,(s,t) \in \mathbb{R}^+ \times [0,T],$$

and q given by

$$q = \sqrt{4\beta^2\left(r^2 + \frac{1}{1-\gamma}\right)\left((r+\xi)^2 - r^2\right)}.$$

The first component of π^f is the myopic demand of the defaultable asset (when the optimization horizon shrinks to 0). The second component is the intertemporal non myopic demand.

The portfolio π^f is analogous to that of $\widetilde{\pi}$ in (12), the difference is provided by the hedging demand. Note that considering a log-investor, that is when $\gamma = 0$, the two strategies coincide.

As in [Leippold et al. (2006)], we cannot compare $\widetilde{\pi}$ and π^f for a generic CRRA utility function. We restrict our attention to a γ in a neighborhood of 0, i.e., utility in a neighborhood of a logarithmic utility. Taking the difference between $\widetilde{\pi}$ and π^f when $\gamma \neq 0$, we get

$$\widetilde{\pi} - \pi^f = \frac{\gamma S(t)}{1-\gamma}\left(\frac{\partial g_\gamma}{\partial s} - \frac{\partial g_\gamma^f}{\partial s}\right), \tag{14}$$

therefore the difference takes only into account the two hedging demands, namely:

$$\pi_h = \frac{\gamma S(t)}{1-\gamma}\frac{\partial g_\gamma}{\partial s} \qquad \text{and} \qquad \pi_h^f = \frac{\gamma S(t)}{1-\gamma}\frac{\partial g_\gamma^f}{\partial s}.$$

Since γ is in a neighborhood of 0, we expand the functions g_γ and g_γ^f around $\gamma = 0$. When

$\gamma = 0$ the functions g_γ and g_γ^f become g_0 and g_0^f, respectively. Hence we have:

$$g_\gamma = g_0 + O(\gamma) \qquad \text{and} \qquad g_\gamma^f = g_0^f + O(\gamma), \qquad \text{as } \gamma \sim 0.$$

Consequently from (14) we get

$$\pi_h - \pi_h^f \approx \frac{\gamma S(t)}{1-\gamma}\left(\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s}\right), \qquad \gamma \sim 0. \tag{15}$$

Therefore, the sign of the difference between π_h and π_h^f is determined by the difference between $\frac{\partial g_0}{\partial s}$ and $\frac{\partial g_0^f}{\partial s}$: if $0 < \gamma < 1$ then it's equal to the sign of $\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s}$, otherwise, if γ is negative, it is the opposite.

To determine the sign of the difference in (15), we have to compute the difference between g_0 and g_0^f, then we can take the derivative with respect to s. g_0 and g_0^f appear in the value functions of the two problems, exploiting the expression of the the value function J, we can compute the difference between g_0 and g_0^f, as shown in the following Lemma.

Lemma 3. *The difference between the functions g_0 and g_0^f is given by*

$$g_0(s,t) - g_0^f(s,t) = -\frac{1}{2}\int_t^T \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}}(\sigma S(u)^\beta \hat{\pi}(S(u)))^2 \Big| S(t) = s\Big]\,\mathrm{d}u, \tag{16}$$

where

$$\hat{\pi}(S(t)) = \pi^+(S(t)) - \frac{\xi}{\sigma^2 S(t)^{2\beta}}. \tag{17}$$

Proof. See Appendix A.2. □

We can now derive the expression for $g_0(s,t) - g_0^f(s,t)$. The difference between the two portfolios in an approximate way for γ in a neighborhood of 0 is provided in the following Theorem.

Theorem 4. *The difference between the two hedging demands π_h and π_h^f in an approximate way for γ in a neighborhood of* 0 *is given by*

$$\pi_h - \pi_h^f \simeq -\frac{\gamma s}{1-\gamma}\int_t^T C(u)\Big\{\int_{a(u)}^{b(u)} \Big(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^\beta \eta(u) y + \Phi^{-1}(\alpha)\Big)\cdot \tag{18}$$

$$\cdot\left(\frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^{\beta}\eta(u)y+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}}+\right.$$
$$\left.+\sigma\right)e^{-\frac{\eta(u)(1+\beta)^2}{2}y^2}y^{2+\frac{1}{2\beta}}I_{\frac{1}{2\beta}}(y)\,dy\Big\}\,du.$$

where $I_{\frac{1}{2\beta}}$ is the modified Bessel function of the first kind and $C(u)$, $a(u)$, $b(u)$ and $\eta(u)$ are positive functions of u, independent of y, which have the following expressions:

$$C(u)=s^{1+\beta}(1+\beta)^{4-\frac{1}{\beta}}\eta(u)^{3+\frac{1}{4\beta}}\frac{\xi(-\beta)}{\sqrt{\tau}}e^{-\frac{(1+\beta)^{-2}+2s^{-2\beta}}{2\eta(u)}}e^{-\beta(\xi+r)(u-t)} \tag{19}$$

$$a(u)=\frac{-\Phi^{-1}(\alpha)-\sqrt{\Phi^{-1}(\alpha)^2-2r\tau+2\ln(1-\zeta)}}{\xi\sqrt{\tau}\sigma^{-1}\eta(u)}e^{\beta(\xi+r)(u-t)}s^{-\beta} \tag{20}$$

$$b(u)=\frac{-\Phi^{-1}(\alpha)+\sqrt{\Phi^{-1}(\alpha)^2-2r\tau+2\ln(1-\zeta)}}{\xi\sqrt{\tau}\sigma^{-1}\eta(u)}e^{\beta(\xi+r)(u-t)}s^{-\beta} \tag{21}$$

$$\eta(u)=\frac{\beta\sigma^2}{2(\xi+r)}\left(e^{2\beta(\xi+r)(u-t)}-1\right) \tag{22}$$

Proof. See Appendix A.2. □

4 Comparative statics

We are now in the position to analyze the effect of the VaR regulation on the optimal portfolio. To this end we determine the sign of the difference $\pi_h-\pi_h^f$ in a neighborhood of $\gamma=0$ evaluating numerically the integral in (18). In our analysis we assume $\xi=0.03,\ r=0.02,\sigma=0.15,T=1,\tau=10/250$ and a confidence level of the Value-at-Risk $1-\alpha$ at 99% (so that $\Phi^{-1}(\alpha)=-2.32635$). The time horizon T corresponds to one year and τ to ten days. The fraction ζ of the portfolio value that appears in the VaR constraint (4) is set equal to 5%.

We can determine numerically the sign of the difference $\pi_h-\pi_h^f$ as a function of the today's price s and the exponent of the CEV model β. Results are reported in Table 1, the magnitude is small as we are considering a utility function nearly logarithmic and for the logarithmic case the difference is 0. From Table 1 we can conclude that in the case of an agent more risk averse than a log-utility investor (table on the left) a VaR constraint leads to increase the holding of the risky asset, i.e., the VaR induces a riskier portfolio strategy. The opposite holds true for

β \ s	0.1	1	10	100
-0.1	+	+	+	+
-0.2	+	+	+	+
-0.3	+	+	+	+
-0.4	+	+	+	+
-0.5	+	+	+	+
-0.6	+	+	+	+
-0.7	−	+	+	+
-0.8	−	+	+	+
-0.9	−	−	+	+

β \ s	0.1	1	10	100
-0.1	−	−	−	−
-0.2	−	−	−	−
-0.3	−	−	−	−
-0.4	−	−	−	−
-0.5	−	−	−	−
-0.6	−	−	−	−
-0.7	+	−	−	−
-0.8	+	−	−	−
-0.9	+	+	−	−

Table 1: Sign of the difference $\pi_h - \pi_h^f$ when $1 - \gamma > 1$ (table in the left) and when $0 < 1 - \gamma < 1$ (table in the right). Parameters: $\xi = 0.03$, $r = 0.02$, $\sigma = 0.15$, $T = 1$, $\tau = 10/250$, $\alpha = 0.01$ and $\zeta = 0.05$.

an agent less risk averse than a log-utility investor (table on the right).

According to these results, the VaR constraint instead of preventing the agent to take risk, encourages him. The effect is stronger as the price decreases and the β goes up in absolute value. This result shows that indeed the VaR has a perverse effect with a strong effect when the default probability is high (low asset price) and when the leverage effect is strong (low β). However, for very low price and low β the effect is reversed. As a consequence, the VaR constraint works against risk when the probability of default is very high.

As far as the other parameters are concerned, we observe that the effect of a VaR constraint goes up with the risk premium and the volatility (ξ and σ go up), as shown in Table 2 and in Table 3. The effect of a VaR constraint instead decreases with the VaR quantile α (Table 4) and with the regulatory parameter ζ (Table 5). These results show two interesting insights:
a) the perverse effect of a VaR constraint is strong when the asset is risky or the premium is high;
b) a stronger VaR constraint (low α) induces a strong perverse effect, the only way to limit the effect is to strengthen the constraint on the VaR in terms of fraction of wealth.

s / β	0.1	1	10	100
-0.1	$+$	$+$	$+$	$+$
-0.2	$+$	$+$	$+$	$+$
-0.3	$-$	$+$	$+$	$+$
-0.4	$-$	$+$	$+$	$+$
-0.5	$+$	$-$	$+$	$+$
-0.6	$-$	$-$	$+$	$+$
-0.7	$-$	$-$	$+$	$+$
-0.8	$+$	$-$	$-$	$+$
-0.9	$-$	$+$	$-$	$-$

s / β	0.1	1	10	100
-0.1	$-$	$-$	$-$	$-$
-0.2	$-$	$-$	$-$	$-$
-0.3	$-$	$-$	$-$	$-$
-0.4	$-$	$-$	$-$	$-$
-0.5	$-$	$-$	$-$	$-$
-0.6	$+$	$-$	$-$	$-$
-0.7	$+$	$-$	$-$	$-$
-0.8	$+$	$-$	$-$	$-$
-0.9	$-$	$+$	$-$	$-$

Table 2: On the Table in the left $\xi = 0.1$, on the Table in the right $\xi = 0.005$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1-\gamma > 1$) with respect to the case with $\xi = 0.03$ (Table 1). The other parameters remain unchanged.

s / β	0.1	1	10	100
-0.1	$+$	$+$	$+$	$+$
-0.2	$+$	$+$	$+$	$+$
-0.3	$+$	$+$	$+$	$+$
-0.4	$+$	$+$	$+$	$+$
-0.5	$+$	$+$	$+$	$+$
-0.6	$+$	$+$	$+$	$+$
-0.7	$+$	$+$	$+$	$+$
-0.8	$-$	$+$	$+$	$+$
-0.9	$-$	$+$	$+$	$+$

s / β	0.1	1	10	100
-0.1	$-$	$-$	$-$	$-$
-0.2	$-$	$-$	$-$	$-$
-0.3	$-$	$-$	$-$	$-$
-0.4	$-$	$-$	$-$	$-$
-0.5	$-$	$-$	$-$	$-$
-0.6	$-$	$-$	$-$	$-$
-0.7	$-$	$-$	$-$	$-$
-0.8	$+$	$-$	$+$	$-$
-0.9	$+$	$+$	$-$	$-$

Table 3: On the Table in the left $\sigma = 0.3$, on the Table in the right $\sigma = 0.01$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1 - \gamma > 1$) with respect to the case with $\sigma = 0.15$ (Table 1). The other parameters remain unchanged.

β \ s	0.1	1	10	100
-0.1	+	+	+	+
-0.2	+	+	+	+
-0.3	+	+	+	+
-0.4	+	+	+	+
-0.5	+	+	+	+
-0.6	+	+	+	+
-0.7	+	+	+	+
-0.8	−	+	+	+
-0.9	−	−	+	+

β \ s	0.1	1	10	100
-0.1	−	−	−	−
-0.2	−	−	−	−
-0.3	−	−	−	−
-0.4	−	−	−	−
-0.5	−	−	−	−
-0.6	−	−	−	−
-0.7	−	−	−	−
-0.8	+	−	+	−
-0.9	+	+	−	−

Table 4: On the Table in the left $\alpha = 0.005$, on the Table in the right $\alpha = 0.05$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1 - \gamma > 1$) with respect to the case with $\alpha = 0.01$ (Table 1). The other parameters remain unchanged.

β \ s	0.1	1	10	100
-0.1	−	+	+	+
-0.2	+	+	+	+
-0.3	+	+	+	+
-0.4	+	−	+	+
-0.5	−	+	+	+
-0.6	+	+	+	+
-0.7	−	+	−	+
-0.8	−	+	−	+
-0.9	−	−	+	+

β \ s	0.1	1	10	100
-0.1	−	−	−	−
-0.2	−	−	−	−
-0.3	−	−	−	−
-0.4	−	−	−	−
-0.5	−	−	−	−
-0.6	−	−	−	+
-0.7	−	−	−	−
-0.8	+	−	+	−
-0.9	+	+	−	−

Table 5: On the Table in the left $\zeta = 0.01$, on the Table in the right $\zeta = 0.2$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1 - \gamma > 1$) with respect to the case with $\zeta = 0.05$ (Table 1). The other parameters remain unchanged.

5 Conclusions

The recent subprime financial crisis has shown that VaR limits in banking activity may have a perverse effect generating feedback effects destabilizing the market. We have shown that indeed a VaR constraint in the presence of a defaultable asset may induce the agent to take more risk.

This result contributes to the literature on banking regulation showing a clear cut result on VaR effects: a constraint reevaluated over time as in [Cuoco et al. (2008), Leippold et al. (2006)] may induce a risky strategy when the financial asset may default. In a way we have shown that a VaR constraint induces the agent to take a risky bet. He is conscious that the asset may default, a VaR limit induces him to bet until when the default becomes extremely likely, exactly what happened before the financial crisis. The only way to limit the phenomenon is to strengthen capital requirements.

A Proofs

A.1 Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ and proof of Proposition 1

The Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ is equivalent to the Euler discretization of a stochastic differential equation with time step τ. Since $\ln \mathcal{V}(s)$ solves the following stochastic differential equation

$$\begin{cases} \mathrm{d}\ln \mathcal{V}(s) = (\pi(t)\xi + r)\mathrm{d}s - \frac{1}{2}\pi(t)^2\sigma^2 S(s)^{2\beta}\mathrm{d}s + \pi(t)\sigma S(s)^{\beta}\mathrm{d}W(s), & s > t, \\ \ln \mathcal{V}(t) \;\; = \ln V(t), \end{cases}$$

the Euler discretization with time step τ gives us

$$\ln \mathcal{V}(t+\tau) \approx \ln V(t) + (\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^{\beta}(W(t+\tau) - W(t)).$$

Consequently, $\mathcal{V}(t+\tau)$ is approximated as

$$\mathcal{V}(t+\tau) \approx \widetilde{\mathcal{V}}(t+\tau) = V(t)e^{(\pi(t)\xi+r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^{\beta}(W(t+\tau)-W(t))}. \tag{23}$$

The Euler discretization has an absolute error of order $\sqrt{\tau}$, for a proof see [Kloeden and Platen (1992)], as a consequence, the absolute error of the Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ in (23) is of order $\sqrt{\tau}$:

$$\varepsilon(\tau) := \mathbb{E}[|\mathcal{V}(t+\tau) - \widetilde{\mathcal{V}}(t+\tau)|] = O(\sqrt{\tau}). \tag{24}$$

Proof of Theorem 1. From the definition of VaR in (3), we have to evaluate $\mathbb{P}(V(t) - \mathcal{V}(t+\tau) > \ell)$, where ℓ is a nonnegative real number.

Exploiting the Itô-Taylor expansion (23) of $\mathcal{V}(t+\tau)$, we have that

$$\begin{aligned} \mathbb{P}(V(t) - \widetilde{\mathcal{V}}(t+\tau) > \ell) &= \mathbb{P}\Big(V(t) - V(t)e^{(\pi(t)\xi+r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^{\beta}(W(t+\tau)-W(t))} > \ell\Big) \\ &= \mathbb{P}\Big(e^{(\pi(t)\xi+r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^{\beta}(W(t+\tau)-W(t))} < 1 - \frac{\ell}{V(t)}\Big) \\ &= \mathbb{P}\Big((\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^{\beta}(W(t+\tau) \end{aligned}$$

$$- W(t)) < \ln\left(1 - \frac{\ell}{V(t)}\right)\Big)$$

$$= \mathbb{P}\left(W(t+\tau) - W(t) < \frac{\ln(1 - \frac{\ell}{V(t)}) - (\pi(t)\xi + r)\tau + \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}{\pi(t)\sigma S(t)^\beta}\right).$$

Since $W(t+\tau) - W(t)$ is distributed as $\sqrt{\tau}Z$, where Z is a standard normal random variable, we obtain

$$\mathbb{P}(V(t) - \widetilde{\mathcal{V}}(t+\tau) > \ell) = \mathbb{P}\left(Z < \frac{\ln(1 - \frac{\ell}{V(t)}) - (\pi(t)\xi + r)\tau + \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}{\pi(t)\sigma S(t)^\beta\sqrt{\tau}}\right).$$

Set $\mathbb{P}(V(t) - \widetilde{\mathcal{V}}(t+\tau) > \ell) = \alpha$, we can find the value of ℓ:

$$\frac{\ln(1 - \frac{\ell}{V(t)}) - (\pi(t)\xi + r)\tau + \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}{\pi(t)\sigma S(t)^\beta\sqrt{\tau}} = \Phi^{-1}(\alpha),$$

and therefore

$$\mathrm{VaR}_t^{\alpha,\tau} = V(t)\left(1 - e^{\Phi^{-1}(\alpha)\pi(t)\sigma S(t)^\beta\sqrt{\tau} + (\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}\right).$$

As a consequence, the VaR constraint (4) becomes

$$1 - e^{\Phi^{-1}(\alpha)\pi(t)\sigma S(t)^\beta\sqrt{\tau} + (\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau} \leqslant \zeta$$

which yields

$$\pi(t)^2\sigma^2 S(t)^{2\beta}\tau - 2\pi(t)(\Phi^{-1}(\alpha)\sigma S(t)^\beta\sqrt{\tau} + \xi\tau) + 2\ln(1-\zeta) - 2r\tau \leqslant 0$$

and therefore we obtain

$$\pi^-(S(t)) \leqslant \pi(t) \leqslant \pi^+(S(t)),$$

where

$$\pi^\pm(S(t)) = \frac{\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^\beta\sqrt{\tau}}{\sigma^2 S(t)^{2\beta}\tau}$$

$$\pm \frac{\sqrt{(\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau})^2 + 2r\sigma^2 S(t)^{2\beta}\tau^2 - 2\ln(1-\zeta)\sigma^2 S(t)^{2\beta}\tau}}{\sigma^2 S(t)^{2\beta}\tau}.$$

□

A.2 Proof of Lemma 3 and of Theorem 4

Proof of Lemma 3. As shown in (10), the value function of the log-investor ($\gamma = 0$) is given by

$$J(v, s, t) = g_0(s, t) + \ln v.$$

We can prove that a similar expression holds true in the absence of the VaR constraint:

$$J^f(v, s, t) = g_0^f(s, t) + \ln v,$$

where J^f represents the value function for the optimal investment problem without the VaR constraint.

By definition, we have

$$J(v, s, t) = \mathbb{E}[\ln(V^*(T))|V^*(t) = v,\ S(t) = s]$$

and

$$J^f(v, s, t) = \mathbb{E}[\ln(V^f(T))|V^f(t) = v,\ S(t) = s]$$

where $V^*(T)$ is the wealth at time T in the presence of the VaR constraint, and $V^f(T)$ is the wealth at time T in the absence of the VaR constraint. From (2) we have that $V^*(T)$ and $V^f(T)$ are given by:

$$V^*(T) = ve^{\int_t^T \left(\pi^*(S(u))\xi + r - \frac{1}{2}\pi^*(S(u))^2\sigma^2 S^{2\beta}(u)\right) \mathrm{d}u + \int_t^T \pi^*(S(u))\sigma S^{\beta}(u)\,\mathrm{d}W(u)}$$

and

$$V^f(T) = ve^{\int_t^T \left(\pi^f(S(u))\xi + r - \frac{1}{2}\pi^f(S(u))^2\sigma^2 S^{2\beta}(u)\right) \mathrm{d}u + \int_t^T \pi^f(S(u))\sigma S^{\beta}(u)\,\mathrm{d}W(u)}$$

where, from (8) and (12), we have

$$\pi^*(S(t)) = \begin{cases} \dfrac{\xi}{\sigma^2 S(t)^{2\beta}}, & \dfrac{\xi}{\sigma^2 S(t)^{2\beta}} < \pi^+(S(t)), \\ \pi^+(S(t)), & \dfrac{\xi}{\sigma^2 S(t)^{2\beta}} \geqslant \pi^+(S(t)), \end{cases}$$

as $\dfrac{\xi}{\sigma^2 S(t)^{2\beta}}$ is always greater than $\pi^-(S(t))$. From (13), we get

$$\pi^f(S(t)) = \frac{\xi}{\sigma^2 S(t)^{2\beta}}.$$

By inserting the expression of $V^*(T)$ in J, we obtain

$$J(v,s,t) = \mathbb{E}[\ln(V^*(T))|V^*(t) = v,\, S(t) = s] = \ln v + \\ + \int_t^T \mathbb{E}[\pi^*(S(u))\xi + r - \tfrac{1}{2}\pi^*(S(u))^2\sigma^2 S^{2\beta}(u)|S(t) = s]\,\mathrm{d}u.$$

From the expression of J given at the beginning of the proof, we deduce that

$$g_0(s,t) = \int_t^T \mathbb{E}[\pi^*(S(u))\xi + r - \tfrac{1}{2}\pi^*(S(u))^2\sigma^2 S^{2\beta}(u)|S(t) = s]\,\mathrm{d}u. \tag{25}$$

Analogously it can be proved that g_0^f is given by

$$g_0^f(s,t) = \int_t^T \mathbb{E}[\pi^f(S(u))\xi + r - \tfrac{1}{2}\pi^f(S(u))^2\sigma^2 S^{2\beta}(u)|S(t) = s]\,\mathrm{d}u. \tag{26}$$

Taking the difference between g_0 and g_0^f, we get

$$g_0(s,t) - g_0^f(s,t) =$$

$$\int_t^T \mathbb{E}\Big[\Big(\pi^*(S(u)) - \pi^f(S(u))\Big)\xi - \tfrac{1}{2}\Big(\pi^*(S(u))^2 - \pi^f(S(u))^2\Big)\sigma^2 S^{2\beta}(u)\Big|S(t) = s\Big]\,\mathrm{d}u =$$

$$= \int_t^T \mathbb{E}\Big[\Big(\pi^*(S(u)) - \pi^f(S(u))\Big)\Big(\xi - \tfrac{1}{2}\Big(\pi^*(S(u)) + \pi^f(S(u))\Big)\sigma^2 S^{2\beta}(u)\Big)\Big|S(t) = s\Big]\,\mathrm{d}u.$$

We define $\hat{\pi}(S(t)) := \pi^+(S(t)) - \pi^f(S(t))$. Consequently, we have

$$\pi^*(S(t)) - \pi^f(S(t)) = 1_{\{\hat{\pi}(S(t))<0\}}\hat{\pi}(S(t))$$

and

$$\pi^*(S(t)) + \pi^f(S(t)) = 1_{\{\hat{\pi}(S(t))<0\}}\hat{\pi}(S(t)) + 2\pi^f(S(t)) = 1_{\{\hat{\pi}(S(t))<0\}}\hat{\pi}(S(t)) + 2\frac{\xi}{\sigma^2 S(t)^{2\beta}}.$$

Hence we have

$$g_0(s,t) - g_0^f(s,t) =$$
$$\int_t^T \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}}\hat{\pi}(S(u))\Big(\xi - \tfrac{1}{2}\Big(1_{\{\hat{\pi}(S(u))<0\}}\hat{\pi}(S(u)) + 2\tfrac{\xi}{\sigma^2 S^{2\beta}(u)}\Big)\sigma^2 S^{2\beta}(u)\Big)\Big|S(t)=s\Big]\,\mathrm{d}u$$
$$= -\frac{1}{2}\int_t^T \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}}(\sigma S^\beta(u)\hat{\pi}(S(u)))^2\Big|S(t)=s\Big]\,\mathrm{d}u,$$

as stated in the Lemma. □

Proof of Theorem 4. From Lemma 3, we know the expression of the difference $g_0 - g_0^f$. We begin by taking the derivative of $S(u)^\beta\hat{\pi}(S(u))$ with respect to s. By (17) and (6) we get

$$\begin{aligned}
\frac{\partial(S(u)^\beta\hat{\pi}(S(u)))}{\partial s} &= \frac{\partial\Big(S(u)^\beta\Big(\pi^+(S(u)) - \frac{\xi}{\sigma^2 S(u)^{2\beta}}\Big)\Big)}{\partial s} \\
&= \frac{\partial\Big(S(u)^\beta\Big(\frac{\Phi^{-1}(\alpha)}{\sigma S(u)^\beta\sqrt{\tau}}\Big)\Big)}{\partial s} + \\
&\quad + \frac{\partial\Big(S(u)^\beta\Big(\frac{\sqrt{(\xi\tau+\Phi^{-1}(\alpha)\sigma S(u)^\beta\sqrt{\tau})^2+2r\sigma^2 S(u)^{2\beta}\tau^2-2\ln(1-\zeta)\sigma^2 S(u)^{2\beta}\tau}}{\sigma^2 S(u)^{2\beta}\tau}\Big)\Big)}{\partial s} \\
&= \frac{\partial\Big(\frac{\Phi^{-1}(\alpha)}{\sigma\sqrt{\tau}}\Big)}{\partial s} + \\
&\quad + \frac{\partial\Big(\frac{\sqrt{(\xi\tau+\Phi^{-1}(\alpha)\sigma S(u)^\beta\sqrt{\tau})^2+2r\sigma^2 S(u)^{2\beta}\tau^2-2\ln(1-\zeta)\sigma^2 S(u)^{2\beta}\tau}}{\sigma S(u)^\beta\tau}\Big)}{\partial s} \\
&= \frac{\partial\Big(\frac{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}}{\sqrt{\tau}}\Big)}{\partial s} \\
&= \frac{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))\xi(-\beta)\sigma^{-1}S(u)^{-\beta-1}}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}}\frac{\partial S(u)}{\partial s}.
\end{aligned}$$

As a consequence, the derivative of (16) with respect to s becomes

$$\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = -\int_t^T \mathbb{E}\Bigg[1_{\{\hat{\pi}(S(u))<0\}}\sigma^2 S(u)^\beta \hat{\pi}(S(u))\cdot \tag{27}$$

$$\cdot \frac{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))\xi(-\beta)\sigma^{-1}S(u)^{-\beta-1}}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}} \frac{\partial S(u)}{\partial s}\Bigg|S(t) = s\Bigg] \,\mathrm{d}u.$$

Since

$$S(u)^\beta \hat{\pi}(S(u)) = \frac{\Phi^{-1}(\alpha)}{\sigma\sqrt{\tau}} + \frac{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}{\sqrt{\tau}},$$

then we get

$$\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = -\int_t^T \mathbb{E}\Bigg[1_{\{\hat{\pi}(S(u))<0\}}\frac{\xi(-\beta)}{\sqrt{\tau}}S(u)^{-\beta-1}\frac{\partial S(u)}{\partial s}(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))\Bigg(\sigma+ \tag{28}$$

$$+ \frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}\Bigg)\Bigg|S(t) = s\Bigg] \,\mathrm{d}u.$$

Now we have to find the expression of the derivative of $S(u)$ with respect to s and its density function. We start determining the derivative of $S(u)$ with respect to s. By (1) we know that $S(u)$ solves the following stochastic integral equation

$$S(u) = s + \int_t^u (\xi + r)S(u)\,\mathrm{d}u + \int_t^u \sigma S(u)^{1+\beta}\,\mathrm{d}W(u).$$

Taking the derivative with respect to s, we get

$$\frac{\partial S(u)}{\partial s} = 1 + \int_t^u (\xi + r)\frac{\partial S(u)}{\partial s}\,\mathrm{d}u + \int_t^u \sigma(1+\beta)S(u)^\beta \frac{\partial S(u)}{\partial s}\,\mathrm{d}W(u),$$

which in differential form becomes

$$\begin{cases} \mathrm{d}\Big(\dfrac{\partial S(u)}{\partial s}\Big) = (\xi + r)\dfrac{\partial S(u)}{\partial s}\mathrm{d}u + \sigma(1+\beta)S(u)^\beta \dfrac{\partial S(u)}{\partial s}\mathrm{d}W(u), & u > t, \\ \qquad \dfrac{\partial S(t)}{\partial s} = 1. \end{cases}$$

Considering the logarithm of the derivative, we get the following stochastic differential equation

$$\begin{cases} \mathrm{d}\ln\left(\dfrac{\partial S(u)}{\partial s}\right) = \left(\xi + r - \dfrac{1}{2}\sigma^2(1+\beta)^2 S(u)^{2\beta}\right)\mathrm{d}u + \sigma(1+\beta)S(u)^{\beta}\mathrm{d}W(u), \qquad u > t, \\ \ln\left(\dfrac{\partial S(t)}{\partial s}\right) = 0. \end{cases}$$

As a consequence, the derivative of $S(u)$ with respect to s is given by

$$\frac{\partial S(u)}{\partial s} = e^{(\xi+r)(u-t) - \frac{1}{2}\sigma^2(1+\beta)^2 \int_t^u S(u)^{2\beta}\,\mathrm{d}u + \sigma(1+\beta)\int_t^u S(u)^{\beta}\,\mathrm{d}W(u)}. \tag{29}$$

This expression is very similar to that of $S(u)$. We want to prove that $\frac{\partial S(u)}{\partial s}$ is a random variable with a distribution law related to that of $S(u)$. In particular, from [Delbaen and Shirakawa (2002)] we know that $S(u)$ can be expressed as

$$S(u) = e^{(\xi+r)(u-t)}\left(X^{(\frac{1}{\beta}+2)}(\eta(u))\right)^{-\frac{1}{2\beta}}, \tag{30}$$

where $X^{(\frac{1}{\beta}+2)}$ is a $\left(\frac{1}{\beta}+2\right)$-dimensional squared Bessel process and $\eta(u)$ is given in (22). The random variable $X^{(\frac{1}{\beta}+2)}(\eta(u))$ has the following density function

$$f_X(x) = \frac{1}{2}\sqrt{s}e^{-\frac{s^{-2\beta}+x}{2\eta(u)}} x^{\frac{1}{4\beta}} I_{\frac{1}{2\beta}}\left(\frac{s^{-\beta}\sqrt{x}}{\eta(u)}\right), \tag{31}$$

where $I_{\frac{1}{2\beta}}$ is the modified Bessel function of the first kind. As mentioned above, $\frac{\partial S(u)}{\partial s}$ has an expression very similar to that of $S(u)$, indeed it can be seen as the price of an asset, whose price today is equal to 1 instead of s and whose volatility is $\sigma(1+\beta)S(u)^{\beta}$ instead of $\sigma S(u)^{\beta}$. Therefore, we have

$$\frac{\partial S(u)}{\partial s} = e^{(\xi+r)(u-t)}\left(Z^{(\frac{1}{\beta}+2)}(\eta(u)(1+\beta)^2)\right)^{-\frac{1}{2\beta}},$$

where $Z^{(\frac{1}{\beta}+2)}$ is a $\left(\frac{1}{\beta}+2\right)$-dimensional squared Bessel process as well. To get the density function of $Z^{(\frac{1}{\beta}+2)}(\eta(u)(1+\beta)^2)$ from the density function of $X^{(\frac{1}{\beta}+2)}(\eta(u))$ we need only to swap s for 1 and $\eta(u)$ for $\eta(u)(1+\beta)^2$ in (31). Hence, the density function of $Z^{(\frac{1}{\beta}+2)}(\eta(u)(1+\beta)^2)$

is given by

$$f_Z(z) = \frac{1}{2} e^{-\frac{1+z}{2\eta(u)(1+\beta)^2}} z^{\frac{1}{4\beta}} I_{\frac{1}{2\beta}}\Big(\frac{\sqrt{z}}{2\eta(u)(1+\beta)^2}\Big).$$

We can express f_Z in terms of f_X; indeed, set $x = z(1+\beta)^{-4}s^{2\beta}$, we get

$$\begin{aligned} f_Z(z) &= \frac{1}{2} e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}(1+\beta)^2 x}{2\eta(u)}} \Big(s^{-2\beta}(1+\beta)^4\Big)^{\frac{1}{4\beta}} x^{\frac{1}{4\beta}} I_{\frac{1}{2\beta}}\Big(\frac{s^{-\beta}\sqrt{x}}{\eta(u)}\Big) \\ &= \frac{1}{s}(1+\beta)^{\frac{1}{\beta}} e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)}x} f_X(x) \end{aligned}$$

As a consequence, to calculate the expected value of $\frac{\partial S(u)}{\partial s}$, we can express it in terms of $S(u)$:

$$\begin{aligned} \mathbb{E}\Big[\frac{\partial S(u)}{\partial s}\Big| S(t)=s\Big] &= \int_0^\infty e^{(\xi+r)(u-t)} z^{-\frac{1}{2\beta}} f_Z(z)\,\mathrm{d}z = \int_0^\infty e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} \cdot \\ &\quad \cdot e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)}x} e^{(\xi+r)(u-t)} x^{-\frac{1}{2\beta}} f_X(x) s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}}\,\mathrm{d}x \\ &= \mathbb{E}\Big[e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} S(u) e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)} S(u)^{-2\beta}} e^{2\beta(\xi+r)(u-t)} \cdot \\ &\quad \cdot s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}} \Big| S(t)=s\Big]. \end{aligned}$$

Hence, the derivative of $g_0 - g_0^f$ with respect to s becomes

$$\begin{aligned} \frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} &= -\int_t^T s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}} \frac{\xi(-\beta)}{\sqrt{\tau}} e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}} S(u)^{-\beta} \cdot \\ &\quad \cdot e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)} S(u)^{-2\beta}} e^{2\beta(\xi+r)(u-t)} (\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))\Big(\sigma + \\ &\quad + \frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}\Big)\Big| S(t)=s\Big]\,\mathrm{d}u. \end{aligned} \tag{32}$$

Finally we find the values of $S(u)$ for which $\hat{\pi}(S(u)) < 0$:

$$\begin{aligned} \hat{\pi}(S(u)) &= \pi^+(S(u)) - \frac{\xi}{\sigma^2 S(u)^{2\beta}} = \frac{\Phi^{-1}(\alpha)}{\sigma S(u)^\beta \sqrt{\tau}} + \\ &\quad + \frac{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}{\sigma S(u)^\beta\sqrt{\tau}} < 0, \end{aligned}$$

which yields

$$\Phi^{-1}(\alpha) + \sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)} < 0,$$

therefore we get

$$(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta) < \Phi^{-1}(\alpha)^2,$$

and

$$\xi^2\tau\sigma^{-2}S(u)^{-2\beta} + 2\xi\sqrt{\tau}\Phi^{-1}(\alpha)\sigma^{-1}S(u)^{-\beta} + 2r\tau - 2\ln(1-\zeta) < 0.$$

Hence, solving this inequality for the unknown $S(u)^{-\beta}$, we find the following pair of inequalities:

$$\begin{aligned} A := \frac{-\Phi^{-1}(\alpha) - \sqrt{\Phi^{-1}(\alpha)^2 - (2r\tau - 2\ln(1-\zeta))}}{\xi\sqrt{\tau}\sigma^{-1}} < S(u)^{-\beta} < \\ < \frac{-\Phi^{-1}(\alpha) + \sqrt{\Phi^{-1}(\alpha)^2 - (2r\tau - 2\ln(1-\zeta))}}{\xi\sqrt{\tau}\sigma^{-1}} =: B. \end{aligned} \tag{33}$$

Therefore, exploiting (30) and (31), (32) can be rewritten in the following way

$$\begin{aligned} \frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = &- \int_t^T s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}} \frac{\xi(-\beta)}{\sqrt{\tau}} e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} \Bigg\{ \int_{A^2 e^{2\beta(\xi+r)(u-t)}}^{B^2 e^{2\beta(\xi+r)(u-t)}} \sqrt{x} \cdot \\ &\cdot \left(\sigma + \frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}\sqrt{x} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}} \right) \cdot \\ &\cdot e^{-\beta(\xi+r)(u-t)} e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)}x} (\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}\sqrt{x} + \Phi^{-1}(\alpha)) \cdot \\ &\cdot \frac{1}{2}\sqrt{s} e^{-\frac{s^{-2\beta}+x}{2\eta(u)}} x^{\frac{1}{4\beta}} I_{\frac{1}{2\beta}}\left(\frac{s^{-\beta}\sqrt{x}}{\eta(u)} \right) \mathrm{d}x \Bigg\} \mathrm{d}u. \end{aligned} \tag{34}$$

Let y be defined as

$$y := \frac{s^{-\beta}\sqrt{x}}{\eta(u)}, \tag{35}$$

then we obtain

$$a(u) := Ae^{\beta(\xi+r)(u-t)}\frac{s^{-\beta}}{\eta(u)} < y < Be^{\beta(\xi+r)(u-t)}\frac{s^{-\beta}}{\eta(u)} =: b(u),$$

where $a(u)$ and $b(u)$ are also reported in (20) and (21), respectively.

Now we change variable inside the integral in (34), from x to y, obtaining the following expression:

$$\begin{aligned}\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = &-\int_t^T s^{1+\beta}(1+\beta)^{4-\frac{1}{\beta}}\eta(u)^{3+\frac{1}{4\beta}}\frac{\xi(-\beta)}{\sqrt{\tau}}e^{-\frac{(1+\beta)^{-2}+2s^{-2\beta}}{2\eta(u)}}e^{-\beta(\xi+r)(u-t)}\cdot\\ &\cdot\left\{\int_{a(u)}^{b(u)}\left(\sigma+\frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^{\beta}\eta(u)y+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}}\right)\cdot\right.\\ &\left.\cdot\left(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^{\beta}\eta(u)y+\Phi^{-1}(\alpha)\right)e^{-\frac{\eta(u)(1+\beta)^2}{2}y^2}y^{2+\frac{1}{2\beta}}I_{\frac{1}{2\beta}}(y)\,\mathrm{d}y\right\}\mathrm{d}u.\end{aligned} \tag{36}$$

Introducing $C(u)$, as in (19), we get the thesis. □

References

[Adrian and Shin (2008)] Adrian, T. and Shin, H. S. (2008) Financial Intermediary Leverage and Value-at-Risk, Federal Reserve Bank of New York Staff Report no. 338.

[Adrian and Shin (2010)] Adrian, T. and Shin, H. S. (2010) Liquidity and leverage, *Journal of Financial Intermediation*, 19: 418-437.

[Barucci and Cosso (2010)] Barucci, E. and Cosso, A. (2010) Does a equity holding tax help to stabilize a VaR regulated financial market?, mimeo.

[Basak and Shapiro (2001)] Basak, S. and Shapiro, A. (2001) Value-at-Risk Based Risk Management: Optimal Policies and Asset Prices. *Review of Financial Studies*, 14:371-405.

[Battauz and Sbuelz (2010)] Battauz, S. and Sbuelz, A. (2010) Reaching Nirvana with a Defaultable Asset?, Working Paper, Bocconi University.

[Campi and Sbuelz (2005)] Campi, L. and Sbuelz, A. (2005) Closed-form Pricing of Benchmark Equity Default Swaps under the CEV Assumption. *Risk Letters*, 1.

[Coval et al. (2009)] Coval, J. D., Jurek, J.W. and Stafford, E. (2009) The Economics of Structured Finance, *Journal of Economic Perspectives* 23, 1: 3-25.

[Coval et al. (2009a)] Coval, J. D., Jurek, J.W. and Stafford, E. (2009) Economic Catastrophe Bonds, *The American Economic Review* 99.

[Cox (1975)] Cox, J.C. (1975) Notes on option pricing I: constant elasticity of variance diffusions. Mimeo.

[Cuoco et al. (2008)] Cuoco, D., He, H., Isaenko, S. (2008) Optimal Dynamic Trading Strategies with Risk Limits. *Operations Research*, 56:358-368.

[Danielsson and Shin (2003)] Danielsson, J. and Shin, H. S. (2003) Endogenous Risk, in Modern risk management: A history, Risk Books, 2003.

[Danielsson, et al. (2004)] Danielsson, J., Shin, H. S., Zigrand, J.-P. (2004) The impact of risk regulation on price dynamics, *Journal of Banking and Finance*, 28: 1069-1087.

[Danielsson, et al. (2009)] Danielsson, J., Shin, H. S., Zigrand, J.-P. (2009) Risk Appetite and Endogenous Risk, Working Paper.

[Delbaen and Shirakawa (2002)] Delbaen, F. and Shirakawa, H. (2002) A Note on Option Pricing for the Constant Elasticity of Variance Model. *Asia-Pacific Financial Markets*, 9:85-99.

[Kloeden and Platen (1992)] Kloeden, P. and Platen, E. (1992) *Numerical Solution of Stochastic Differential Equations*. Springer, Berlin.

[Leippold et al. (2006)] Leippold, M., Trojani, F., Vanini, P. (2006) Equilibrium Impact of Value-at-Risk Regulation. *Journal of Economic Dynamics and Control*, 30:1277-1313.

[Yiu (2004)] Yiu, K. (2004) Optimal portfolios under a value at risk constarint. *Journal of Economic Dynamics and Control*, 28:1317-1334.