

Theory of Quantum Space-Time

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A generalised equivalence principle is put forward according to which *space-time symmetries and internal quantum symmetries are indistinguishable before symmetry breaking*. Based on this principle, a higher-dimensional extension of Minkowski space is proposed and its properties examined. In this scheme the structure of space-time is intrinsically quantum mechanical. It is shown that the causal geometry of such a quantum space-time possesses a rich hierarchical structure. The natural extension of the Poincaré group to quantum space-time is investigated. In particular, we prove that the symmetry group of a quantum space-time is generated in general by a system of irreducible Killing tensors. When the symmetries of a quantum space-time are spontaneously broken, then the points of the quantum space-time can be interpreted as *space-time valued operators*. The generic point of a quantum space-time in the broken symmetry phase thus becomes a Minkowski space-time valued operator. Classical space-time emerges as a map from quantum space-time to Minkowski space. It is shown that the general such map satisfying appropriate causality-preserving conditions ensuring linearity and Poincaré invariance is necessarily a density matrix.

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1. Introduction

The purpose of this paper is to present a novel approach to the unification of space-time physics and quantum theory. We take the view that classical space-time itself is not to be regarded as a primary object which is then subjected to some form of quantisation procedure. In our framework the central object is a mathematical structure that we call a quantum space-time. Intuitively, this structure can be regarded as the space of all space-time valued quantum operators. That is to say, each point in the infinite-dimensional quantum space-time corresponds to a quantum operator with the property that its expectation, in any quantum state, is a space-time point. The space of all such operators has a rich structure that appears to contain all the elements one needs both for a characterisation of the causal structure of relativistic space-time as well as a representation of the phenomena of quantum theory.

Many attempts to unify gravitational physics with other fundamental forces have pursued the idea of extending four-dimensional space-time to higher dimensions. Beginning with the introduction of the Kaluza-Klein theory of gauge potentials, such extensions have typically been carried out by increasing the spatial dimension of the space-time, while retaining the special role played by time. In methodologies of this sort, however, it cannot be said that quantum mechanical characteristics of the fundamental forces are adequately incorporated into the structure of the higher dimensional space-time. Nor can it be said that the space-time itself is being treated in any useful sense as a quantum entity. In what follows, however, we demonstrate that if we take the point of view that the universe is itself intrinsically quantum mechanical in an appropriate sense, then the most natural extension of space-time into higher dimensions has a completely different character from that suggested by the Kaluza-Klein theory and its generalisations.

The basic idea is as follows. The points of Minkowski space are in natural correspondence with two-by-two matrices of the form $x^{\mathbf{AA}'}$ satisfying a Hermitian condition. Lorentz transformations are then given by multiplying $x^{\mathbf{AA}'}$ on the right and on the left by an element of $SL(2, \mathbb{C})$ and its complex conjugate, and the Minkowskian metric for the interval between two points $x^{\mathbf{AA}'}$ and $y^{\mathbf{AA}'}$ is obtained by taking the determinant of their difference. Hermitian matrices are, on the other hand, familiar objects in quantum mechanics in their role as physical observables. In quantum mechanics, the dimensionality of the Hilbert space is directly related to the dimensionality of the corresponding space of Hermitian operators. Thus, if quantum theory is to be unified with space-time physics, it seems natural to extend the space of matrices representing space-time points to higher dimensions, and thus to assume that the dimensionality of space-time is much larger than four, perhaps infinite. A higher-dimensional extension of space-time in this manner is also consistent with the philosophy often put forward that spinors (or twistors) are in some respects just as fundamental, or possibly even more fundamental, than the space-time points themselves.

The framework we introduce here is motivated by the idea that the symmetries of space-time and the symmetries of quantum theory are, at the deeper level, indistinguishable. This generalised ‘equivalence principle’ reflects the notion that the fundamental symmetries or approximate symmetries we observe in nature should have a common origin, and that the breakdown of these symmetries should also have a common cause. In view of the generalised equivalence principle we shall therefore postulate in this investigation that *space-time events*

are themselves infinite-dimensional Hermitian matrices. As in ordinary quantum mechanics, however, it is both legitimate and desirable to consider finite dimensional realisations of the framework in some circumstances. These finite dimensional realisations are given by spaces of r -by- r Hermitian matrices. In this way we obtain for each $r \geq 2$ an r^2 -dimensional space-time \mathcal{H}^{r^2} . The standard four-dimensional Minkowski space \mathcal{M}^4 then emerges as the simplest case.

As we explain later in the paper, we regard such finite dimensional cases not merely as toy models, but rather as special situations where a finite-dimensional part of the infinite-dimensional space-time is (or effectively can be regarded as) disentangled from the rest of the space-time. In this way the fundamental role played by the Segré embedding in the geometry of quantum theory is carried over to the relativistic domain. Indeed, it is an important feature of our approach that many of the familiar ideas relevant to the geometry of the quantum state space are directly applicable to space-time itself, and as such take on a new physical significance, some aspects of which we explore in what follows.

The structure of the paper is as follows. In §2 we introduce the algebraic formalism appropriate for the manipulation of r -component hyperspinors. The concept of hyperspinors as a natural higher dimensional generalisation of the familiar two-component spinors of relativity theory and as a basis for higher-dimensional space-time theories was introduced by Finkelstein (1986) and Finkelstein *et al.* (1987), and we build on that work here. The r^2 -dimensional quantum space-time \mathcal{H}^{r^2} then arises as the tensor product of the space of r -dimensional hyperspinors with its complex conjugate. In §3 we investigate the causal structure of \mathcal{H}^{r^2} . This structure is shown to arise by virtue of a resolution of the vector separating any two points in \mathcal{H}^{r^2} into a canonical form involving a sum of terms, each of which can be expressed as a product of a hyperspinor with its complex conjugate, together with a plus or minus sign. In particular, we can introduce the concept of future and past pointing time-like and null vectors in \mathcal{H}^{r^2} . This structure exists despite the fact that \mathcal{H}^{r^2} does not possess a pseudo-Riemannian metric. Instead, \mathcal{H}^{r^2} possesses a chronometric form of rank r , which induces a pseudo-Finslerian geometry on \mathcal{H}^{r^2} . In §4 we look at the variational problem for determining the geodesic between two time-like separated points, and show that this reduces to an appropriate linear expression, despite the fact that the chronometric form is itself a polynomial of degree r in the separation vector for the given points.

In §5-6 we study the higher dimensional analogue of the Poincaré

group that acts on \mathcal{H}^{r^2} . We prove that the symmetries of this space are generated by a system of $3r^2 - 2$ irreducible Killing tensors, each of rank $r - 1$. Since \mathcal{H}^{r^2} does not have a pseudo-Riemannian structure for $r > 2$, the link between symmetries and Killing vectors is lost in general, and is replaced by this more subtle manifestation of symmetry. We show that the conserved quantities associated with the hyper-Poincaré group can be obtained in terms of algebraic expressions formed from the Killing tensors. We also derive appropriate hyper-relativistic generalisations of the familiar expressions for the momentum, angular momentum, mass, and spin of a relativistic system.

In §7 we make a detailed study of the algebraic geometry of the complex light-cone at a point of \mathcal{H}^{r^2} , and examine the structures arising for various values of r . We show that the space of complex light-like directions is a complex hypersurface \mathfrak{N}^{r^2-2} of degree r in the complex projective space \mathbb{P}^{r^2-1} . The hypersurface \mathfrak{N}^{r^2-2} can be completely characterised by the fact that it admits a special hyperspinorial subvariety of the form $\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$. In §8 we present a higher dimensional analogue of the Klein representation, and show how \mathcal{H}^{r^2} , when complexified and compactified, can be represented as the Grassmannian of complex $(r - 1)$ -planes in \mathbb{P}^{2r-1} . We show that the causal relations between points of \mathcal{H}^{r^2} can be understood in terms of the intersection properties of the corresponding $(r - 1)$ -planes.

In §9 we show how the conformal symmetry of the geometry of the generalised Klein representation can be reduced to the hyper-Poincaré group by the introduction of elements determining the structure of infinity for this space. The range of possibilities for structure at infinity is considerably larger than it is for four-dimensional space-time. As a consequence, as we show in §10, the choice of structure at infinity can also give rise to interesting classes of cosmological models. We point out that there is a mechanism within our framework whereby the same structures at infinity responsible for the reduction of the symmetry of space-time can also be responsible for the breaking of microscopic symmetries.

In §11 we further explore the notion of symmetry breaking, and introduce the idea of the Segré embedding as the basis of the mechanism according to which space-time degrees of freedom can be disentangled from microscopic or internal degrees of freedom. According to this scheme the dimension of the hyperspinor space is assumed to be even, and each hyperspinor index A with the range $A = 1, 2, \dots, 2n$ is regarded as a clump consisting of a conventional two-component spinor index $\mathbf{A} = 1, 2$ and an “internal” index $i = 1, 2, \dots, n$. It follows as a

consequence of this symmetry breaking scheme that the points of \mathcal{H}^{4n^2} can be interpreted as *space-time valued operators*. This is the sense in which \mathcal{H}^{4n^2} can be regarded as a quantum space-time. Finally, in support of this interpretation, in §12 we consider maps from \mathcal{H}^{4n^2} to Minkowski space \mathcal{H}^4 . We show that if such a map ρ is in a suitably defined sense (i) linear, (ii) Poincaré invariant, and (iii) causal, then ρ is a density matrix, and the map can be interpreted as the expectation. Thus within our scheme an important element of the probabilistic interpretation of quantum theory arises as an emergent property deriving from the causal nature of space-time.

2. Hyperspinors

What we aim for in this investigation is not a higher dimensional pseudo-Riemannian analogue of four-dimensional space-time, but rather a geometry of a different character that, although richer in structure than Minkowski space, nevertheless retains a definite relation to that space—and as a consequence is in a position to embrace the description of physical phenomena, albeit in a new setting. We find it convenient to proceed in stages. The first step is to introduce a special type of causal geometry for which the space is of dimension $N = r^2$, where $r \geq 2$ is an integer. Later we shall specialise to the case for which r is even, and introduce some additional structure that cements the relationship of the higher dimensional space to ordinary Minkowski space. We shall demonstrate that when r is even there exists a natural symmetry breaking mechanism that embeds Minkowski space in the higher dimensional space. By virtue of the geometry of this embedding we can assign physical properties to elements of the higher-dimensional space.

For general r we refer to this space as \mathcal{H}^N , the quantum space-time of dimension $N = r^2$. If we fix a point of origin in \mathcal{H}^N , then a point of \mathcal{H}^N can be characterised by its position vector with respect to that origin. In Minkowski space \mathfrak{M}^4 such vectors are naturally isomorphic to elements of the vector space obtained by taking the tensor product of a complex vector space $\mathbb{S}^{\mathbf{A}}$ with its complex conjugate $\mathbb{S}^{\mathbf{A}'}$. We recognise $\mathbb{S}^{\mathbf{A}}$ and $\mathbb{S}^{\mathbf{A}'}$ as the spaces of unprimed and primed two-component spinors, respectively. For two-component spinors we use bold upright Roman indices, and we adopt the usual conventions for raising and lowering indices, and for complex conjugation. Thus if $\alpha^{\mathbf{A}} \in \mathbb{S}^{\mathbf{A}}$, then we write $\alpha^{\mathbf{A}}\epsilon_{\mathbf{AB}} = \alpha_{\mathbf{B}}$ and $\alpha^{\mathbf{A}} = \epsilon^{\mathbf{AB}}\alpha_{\mathbf{B}}$, where $\epsilon_{\mathbf{AB}} = -\epsilon_{\mathbf{BA}}$. Likewise, for the complex conjugation map we write $\alpha^{\mathbf{A}} \rightarrow \bar{\alpha}^{\mathbf{A}'}$ where $\alpha^{\mathbf{A}'} \in \mathbb{S}^{\mathbf{A}'}$. A special feature of the two-component spinor algebra is that the epsilon

spinor functions as a nondegenerate symplectic form that can be used to establish a linear map from the spin space \mathbb{S}^A to its dual space \mathbb{S}_A . For further details of the two-component spinor algebra, see, e.g., Pirani (1965), Penrose (1968), Penrose & Rindler (1984,1986).

Our model of quantum space-time generalises the two-component spinor formalism and its relation to Minkowski space by allowing the underlying spin spaces to be higher dimensional complex vector spaces, the elements of which, using the terminology of Finkelstein (1986) we call *hyperspinors*. Let us write \mathbb{S}^A and $\mathbb{S}^{A'}$, respectively, for the complex r -dimensional vector spaces of unprimed and primed hyperspinors. For hyperspinors we use unprimed and primed italic indices. It is assumed that these two spaces admit an anti-linear isomorphism under the operation of complex conjugation. Thus if $\alpha^A \in \mathbb{S}^A$, then under complex conjugation we have $\alpha^A \rightarrow \bar{\alpha}^{A'}$, where $\bar{\alpha}^{A'} \in \mathbb{S}^{A'}$. In a standard basis this map conjugates α^A component by component to give the components of $\bar{\alpha}^{A'}$.

The dual spaces associated with the hyperspin spaces \mathbb{S}^A and $\mathbb{S}^{A'}$ will be denoted \mathbb{S}_A and $\mathbb{S}_{A'}$, respectively. If $\alpha^A \in \mathbb{S}^A$ and $\beta_A \in \mathbb{S}_A$, then their inner product is $\alpha^A \beta_A$. Likewise if $\gamma^{A'} \in \mathbb{S}^{A'}$ and $\delta_{A'} \in \mathbb{S}_{A'}$ then for their inner product we write $\gamma^{A'} \delta_{A'}$.

We also introduce the totally antisymmetric hyperspinors of rank r associated with the spaces \mathbb{S}^A , \mathbb{S}_A , $\mathbb{S}^{A'}$, and $\mathbb{S}_{A'}$. These will be denoted $\varepsilon^{AB\dots C}$, $\varepsilon_{AB\dots C}$, $\varepsilon^{A'B'\dots C'}$, and $\varepsilon_{A'B'\dots C'}$. The choice of these antisymmetric hyperspinors is canonical up to an overall scale factor. Once a specific choice has been made for $\varepsilon_{AB\dots C}$, then the other epsilon hyperspinors are determined by the relations $\varepsilon^{AB\dots C} \varepsilon_{AB\dots C} = r!$, $\varepsilon^{A'B'\dots C'} \varepsilon_{A'B'\dots C'} = r!$, and $\varepsilon_{A'B'\dots C'} = \bar{\varepsilon}^{A'B'\dots C'}$, where $\bar{\varepsilon}^{A'B'\dots C'}$ is the complex conjugate of $\varepsilon_{AB\dots C}$. If we introduce a standard basis then it is convenient to set $\varepsilon_{12\dots r} = 1$, which is sufficient to fix the remaining components of the epsilon hyperspinors. The arguments that follow, however, do not depend on a specific choice of scale.

The epsilon hyperspinors play a role similar to that of the two-index epsilon spinors of the two-component spinor algebra; but it should be evident that in the case of r -component hyperspinors the algebra is more elaborate. In particular, for $r \geq 3$ the epsilon spinor no longer has an interpretation as a symplectic structure.

Next we introduce the complex matrix space $\mathbb{C}^{AA'} = \mathbb{S}^A \otimes \mathbb{S}^{A'}$. An element $x^{AA'} \in \mathbb{C}^{AA'}$ is *real* if it satisfies the weak Hermitian property $x^{AA'} = \bar{x}^{A'A}$, where $\bar{x}^{A'A}$ is the complex conjugate of $x^{AA'}$. We denote the linear space of real elements of $\mathbb{C}^{AA'}$ by $\mathbb{R}^{AA'}$. The elements of $\mathbb{R}^{AA'}$

constitute the real quantum space-time \mathcal{H}^N of dimension $N = r^2$. We can then regard $\mathbb{C}\mathcal{H}^N = \mathbb{C}^{AA'}$ as the complexification of \mathcal{H}^N .

3. Chronometric relations on the quantum space-time \mathcal{H}^N

Consider two points $x^{AA'}$ and $y^{AA'}$ in \mathcal{H}^N , and write $r^{AA'} = x^{AA'} - y^{AA'}$ for the corresponding separation vector, which is clearly independent of the choice of origin. In what follows we shall find it useful to introduce an index-clumping convention (see, e.g., Penrose 1968, Penrose & Rindler 1984), and write $a = AA'$, $b = BB'$, and so on, according to which a pair of hyperspinor indices, one primed and the other unprimed, corresponds to a lower case single vector index. Thus we set $x^a = x^{AA'}$, $y^a = y^{AA'}$, $r^a = r^{AA'}$, and so on. Then for the separation vector of the points x^a and y^a in \mathcal{H}^N we write $r^a = x^a - y^a$.

There is a natural causal structure induced on \mathcal{H}^N by the so-called *chronometric tensor* $g_{ab\dots c}$. Making use of the index-clumping convention, we define this tensor as follows:

$$g_{ab\dots c} = \varepsilon_{AB\dots C} \varepsilon_{A'B'\dots C'}. \quad (3.1)$$

The chronometric tensor, which is of rank r , is totally symmetric and is nondegenerate in the sense that for any vector r^a the condition $r^a g_{ab\dots c} = 0$ implies $r^a = 0$. We shall say that x^a and y^a in $\mathbb{R}^{AA'}$ are *null separated* if the *chronometric form* for their separation vanishes:

$$g_{ab\dots c} r^a r^b \dots r^c = 0. \quad (3.2)$$

Null separation is equivalent to the vanishing of the determinant of the matrix $r^{AA'}$:

$$\varepsilon_{AB\dots C} \varepsilon_{A'B'\dots C'} r^{AA'} r^{BB'} \dots r^{CC'} = 0. \quad (3.3)$$

If the hyperspin space has dimension $r = 2$, this reduces to the usual condition for x^a and y^a to be null-separated in Minkowski space. For $r > 2$, however, the situation is more complicated on account of the fact that there are various degrees of nullness that can prevail between two points. More precisely, when two points of quantum space-time are null-separated, we shall define the ‘degree’ of nullness by the rank of the matrix $r^{AA'}$. Null separation of the first degree is the case for which $r^{AA'}$ is of rank one, and thus satisfies a system of quadratic relations of the form

$$\varepsilon_{AB\dots C} \varepsilon_{A'B'\dots C'} r^{AA'} r^{BB'} = 0, \quad (3.4)$$

or equivalently $g_{ab\dots c}r^a r^b = 0$. This implies in the case of a real separation vector that $r^{AA'}$ can be expressed in the form

$$r^{AA'} = \pm \alpha^A \bar{\alpha}^{A'} \quad (3.5)$$

for some hyperspinor α^A . If two points have a separation vector of this form then we say that they are *strongly* null separated. If the sign is positive (*resp.* negative), then x^a lies to the future (*resp.* past) of y^a .

In the case of nullness of the second degree, $r^{AA'}$ satisfies a set of cubic relations given by $g_{abc\dots d}r^a r^b r^c = 0$. In this case $r^{AA'}$ can be put into one of the following three forms: (a) $r^{AA'} = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'}$, (b) $r^{AA'} = \alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'}$, and (c) $r^{AA'} = -\alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'}$.

In case (a), x^a lies to the future of y^a , and r^a can be thought of as a ‘degenerate’ future-pointing time-like vector. In case (c), x^a lies to the past of y^a , and r^a is a degenerate past-pointing time-like vector. In case (b), r^a can be thought of as a degenerate space-like separation.

A similar analysis can be applied in the case of null separation of other ‘intermediate’ degrees.

If the determinant of $r^{AA'}$ is nonvanishing, and $r^{AA'}$ is thus of maximal rank, then the chronometric form is nonvanishing. In that case $r^{AA'}$ can be represented in the canonical form

$$r^{AA'} = \pm \alpha^A \bar{\alpha}^{A'} \pm \beta^A \bar{\beta}^{A'} \pm \dots \pm \gamma^A \bar{\gamma}^{A'}, \quad (3.6)$$

with the presence of r nonvanishing terms, where the r hyperspinors $\alpha^A, \beta^A, \dots, \gamma^A$ are linearly independent.

Let us write (p, q) for the numbers of plus and minus signs appearing in the canonical form for $r^{AA'}$ given in (3.6). We shall call (p, q) the signature of the vector $r^{AA'}$. The hyperspinors $\alpha^A, \beta^A, \dots, \gamma^A$ are determined by $r^{AA'}$ only up to an overall unitary (or pseudo-unitary) transformation of the form $\alpha_n^A \rightarrow U_n^m \alpha_m^A$, where $n, m = 1, 2, \dots, r$, and $\alpha_n^A = \{\alpha^A, \beta^A, \dots, \gamma^A\}$. The signature (p, q) is, however, an invariant of $r^{AA'}$.

In the cases for which $r^{AA'}$ has signature $(r, 0)$ or $(0, r)$ we say that $r^{AA'}$ is time-like future-pointing or time-like past-pointing, respectively. Then writing

$$\Delta = g_{ab\dots c}r^a r^b \dots r^c \quad (3.7)$$

for the associated chronometric form of degree r , we define the *time interval* between the events x^a and y^a by the formula

$$\|x - y\| = |\Delta|^{\frac{1}{r}}. \quad (3.8)$$

It should be evident that in the case $r = 2$ we recover the standard Minkowskian time-interval between the given events.

In summary, the following classification scheme for the separation between two space-time points can be enunciated. Let $N = r^2$ be the dimension of the quantum space-time and (p, q) the signature of the separation vector r^a . If $p + q = r$ then we say that the separation is nondegenerate; then if $p = r$, the vector r^a is time-like future-pointing, and if $q = r$, then r^a is time-like past-pointing.

If neither p nor q equals r , then we say r^a is space-like of type (p, q) . Note that in the case of ordinary Minkowski space, for which $r = 2$, the fact that a space-like vector is necessarily of type $(1, 1)$ corresponds to the result that any space-like vector in four dimensional space-time can be expressed as the difference between two real null vectors. Any two such representations for the same space-like vector are related by a $U(1, 1)$ transformation.

On the other hand, if $p + q < r$ then we say that r^a is a *degenerate future-pointing vector* if $q = 0$, and a *degenerate past-pointing vector* if $p = 0$, and otherwise a *degenerate space-like vector*. Clearly, all degenerate vectors are null in the sense that the corresponding chronometric form vanishes. If $p = 1$ and $q = 0$ then r^a is future-pointing and strongly null, and if $p = 0$ and $q = 1$ then r^a is past-pointing and strongly null. Strong null separation is the analogue of Minkowskian null separation. The measure of separation, given by (3.8), is nonvanishing if and only if the separation vector is nondegenerate. The causal structure of the quantum space-time, however, also brings into play the various degenerate forms of time-like or null separation. Thus if x^a lies to the future of y^a (i.e. if $x^a - y^a$ is time-like future-pointing, degenerate future-pointing, or strongly null future-pointing), and if y^a lies to the future of z^a , then x^a lies to the future of z^a .

A striking feature of the causal structure of \mathcal{H}^N is that the essential physical features of the causal structure of Minkowski space are preserved. In particular, the space of future pointing time-like vectors forms a convex cone, and the convex hull of this cone includes the future pointing null vectors of all degrees of degeneracy.

4. Dynamical trajectories

Now suppose that the map $\lambda \mapsto x^{AA'}(\lambda)$ defines a smooth curve Γ in \mathcal{H}^N for $\lambda \in [a, b] \subset \mathbb{R}$. Then Γ will be said to be a time-like curve if

the tangent vector

$$v^{AA'}(\lambda) = \frac{d}{d\lambda}x^{AA'}(\lambda) \quad (4.1)$$

is time-like and future-pointing along Γ . In that case we define the proper time s elapsed along Γ by the integral

$$s = \int_a^b [g_{ab\dots c}v^av^b\dots v^c]^{\frac{1}{r}} d\lambda. \quad (4.2)$$

For convenience, we can also write (4.2) in the infinitesimal form

$$(ds)^r = g_{ab\dots c}dx^adx^b\dots dx^c. \quad (4.3)$$

This expression shows that the geometry under consideration here has a pseudo-Finslerian structure. Finslerian geometries, first considered by Riemann, and studied extensively by Finsler (see, e.g., Bao *et al.* 2000), have from time to time been proposed as the basis for generalisations of the theory of relativity. It is interesting therefore that such a structure arises in a natural way in the present context. One should note, however, that the pseudo-Finslerian structures arising in our framework are of a very particular sort.

For a time-like curve, we can choose the proper time as the parameter along the curve, in which case the resulting affine parameterisation of the curve is determined completely up to a transformation of the form $s \rightarrow s + c$ where c is a constant.

The equation of motion for the situation in which Γ is a time-like geodesic is obtained by varying (4.2) and setting the result to zero. As usual, we assume the variation vanishes at the endpoints; an integration by parts then leads to the desired result. Writing

$$L = (g_{abc\dots d}v^av^bv^c\dots v^d)^{\frac{1}{r}}, \quad (4.4)$$

we find that $x^a(\lambda)$ describes a geodesic if the velocity vector v^a satisfies the Euler-Lagrange equation

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial v^a} \right) = 0. \quad (4.5)$$

A calculation shows that this condition is given more explicitly by

$$g_{abc\dots d} \frac{dv^b}{d\lambda} v^c \dots v^d = \phi g_{abc\dots d} v^b v^c \dots v^d, \quad (4.6)$$

where $\phi = d \ln L / d\lambda$. If λ is chosen to be proper time, then $\phi = 0$ and the geodesic equation takes the form

$$g_{abc\dots d} \dot{v}^b v^c \dots v^d = 0, \quad (4.7)$$

where the dot denotes differentiation with respect to the proper time.

In the case $r = 2$ equation (4.7) reduces immediately to the familiar relation $\dot{v}^a = 0$. To prove that the geodesic equation (4.7) implies $\dot{v}^a = 0$ in the case of a quantum space-time for $r \geq 2$, it suffices to examine the case $r = 3$. Then we have $g_{abc} \dot{v}^b v^c = 0$, which can be expressed in hyperspinor terms as

$$\epsilon_{ABC} \epsilon_{A'B'C'} \dot{v}^{BB'} v^{CC'} = 0. \quad (4.8)$$

This relation in turn can be written

$$\dot{v}^{BB'} v^{CC'} - \dot{v}^{CB'} v^{BC'} - \dot{v}^{BC'} v^{CB'} + \dot{v}^{CC'} v^{BB'} = 0. \quad (4.9)$$

However, because $\det(v^{AA'}) \neq 0$ we know that $v^{AA'}$ has an inverse $u_{AA'}$ satisfying $v^{AA'} u_{BA'} = \delta_B^A$ and $v^{AA'} u_{AB'} = \delta_{B'}^{A'}$. Therefore, contracting (4.9) with $u_{BB'}$ we obtain

$$(u_{BB'} \dot{v}^{BB'}) v^{CC'} + (r - 2) \dot{v}^{CC'} = 0. \quad (4.10)$$

This equation shows that if $\dot{v}^{CC'}$ were not zero, then it would have to be proportional to $v^{CC'}$. However, if that were the case, then (4.8) would imply that $\det(v^{CC'}) = 0$, which is contrary to the assumption that $v^{CC'}$ is time-like. It follows that $\dot{v}^a = 0$. That concludes the proof for $r = 3$. A similar argument then establishes for all $r \geq 2$ that (4.7) implies $\dot{v}^a = 0$. Hence we have deduced the following result:

Theorem 4.1. *Let A^a and B^a be quantum space-time points with the property that $A^a - B^a$ is time-like and future-pointing. Then the affinely parametrised geodesic connecting these points in \mathcal{H}^N is the curve*

$$X^a(s) = B^a + \frac{A^a - B^a}{[\Delta(A, B)]^{1/r}} s \quad (4.11)$$

for $-\infty < s < \infty$, where $\Delta = g_{ab\dots c} (A^a - B^a)(A^b - B^b) \dots (A^c - B^c)$.

5. The Hyper-Poincaré group

The chronometric form Δ for the separation between two points in \mathcal{H}^N is invariant when the points of \mathcal{H}^N are subjected to transformations of the following type:

$$x^{AA'} \rightarrow \lambda_B^A \bar{\lambda}_{B'}^{A'} x^{BB'} + b^{AA'}. \quad (5.1)$$

Here $b^{AA'}$ represents an arbitrary translation in quantum space-time, λ_B^A is an element of $SL(r, \mathbb{C})$, and $\bar{\lambda}_{B'}^{A'}$ is the complex conjugate of λ_B^A . The relation of this group of transformations to the Poincaré group in the case $r = 2$ is evident. Indeed, one of the attractions of the extension of space-time geometry under consideration here is that the resulting hyper-Poincaré group admits such a description.

More generally, we observe that the (proper) hyper-Poincaré group preserves the signature of any space-time interval $r^{AA'}$, whether or not the interval is degenerate, and hence leaves the causal relations between events unchanged.

We refer to a transformation of the form $r^a \rightarrow L_b^a r^b$ as a hyper-Lorentz transformation if $L_b^a = \lambda_B^A \bar{\lambda}_{B'}^{A'}$ for some element $\lambda_B^A \in SL(r, \mathbb{C})$. The (real) dimension of the hyper-Lorentz group is $2r^2 - 2$, and the dimension of the hyper-Poincaré group is thus $3r^2 - 2$. It is interesting to observe that the dimension of the hyper-Poincaré group grows linearly with the dimension of the quantum space-time itself, which is given by r^2 . This can be contrasted with the dimension of the group arising if we endow \mathbb{R}^N with a standard Lorentzian metric with signature $(1, r^2 - 1)$. In that case the pseudo-orthogonal group has real dimension $\frac{1}{2}r^2(r^2 - 1)$, which together with the translation group gives a total dimension of $\frac{1}{2}r^2(r^2 + 1)$. The parsimonious dimensionality of the hyper-Poincaré group is due to the fact that it preserves a rather delicate system of causal relations holding between pairs of points in quantum space-time.

6. Symmetries and conservation laws

In Minkowski space the symmetries of the Poincaré group are associated with a ten-parameter family of Killing vectors. That is to say, for $r = 2$ we have the Minkowski metric $g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$, and the Poincaré group is generated by the ten-parameter family of vector fields $\xi^a(x)$ on \mathfrak{M}^4 satisfying $\mathcal{L}_\xi g_{ab} = 0$, where \mathcal{L}_ξ denotes the Lie derivative with respect to $\xi^a(x)$. Now for any vector field ξ^a and any symmetric tensor field g_{ab} we have

$$\mathcal{L}_\xi g_{ab} = \xi^c \nabla_c g_{ab} + 2g_{c(a} \nabla_{b)} \xi^c. \quad (6.1)$$

If g_{ab} is the metric and ∇_a denotes the associated Christoffel derivative satisfying $\nabla_a g_{bc} = 0$, we obtain $\nabla_{(a} \xi_{b)} = 0$, where $\xi_a = g_{ab} \xi^b$. The condition $\mathcal{L}_\xi g_{ab} = 0$ therefore implies that ξ^a is a Killing vector.

We have taken the trouble to spell out the case for $r = 2$ in order to highlight the contrast with the situation for general r . Clearly for $r > 2$ we have no Riemannian metric, and the usual relation between

symmetries and Killing vectors is lost. What survives, however, is of considerable interest. More specifically, to generate a symmetry of the quantum space-time \mathcal{H}^N the vector field ξ^a has to satisfy

$$\mathcal{L}_\xi g_{ab\dots c} = 0, \quad (6.2)$$

where $g_{ab\dots c}$ is the chronometric tensor. Now for a general vector field ξ^a and a general symmetric tensor $g_{ab\dots c}(x)$ we have

$$\mathcal{L}_\xi g_{ab\dots c} = \xi^d \nabla_d g_{ab\dots c} + r g_{d(a\dots b} \nabla_c) \xi^d. \quad (6.3)$$

In the case of a quantum space-time we let ∇_a be the flat connection for which $\nabla_a g_{bc\dots d} = 0$. Then to generate a symmetry of the chronometric structure of \mathcal{H}^N the vector field ξ^a must satisfy

$$g_{d(a\dots b} \nabla_c) \xi^d = 0, \quad (6.4)$$

which serves as the analogue of Killing's equation. Equation (6.4) can be written in a more suggestive form if we define a symmetric tensor $\xi_{ab\dots c}$ of rank $r - 1$ by

$$\xi_{ab\dots c} = g_{ab\dots cd} \xi^d. \quad (6.5)$$

Then (6.4) says that $\xi_{ab\dots c}$ satisfies the conditions for a symmetric *Killing tensor*:

$$\nabla_{(a} \xi_{bc\dots d)} = 0. \quad (6.6)$$

Thus we see that \mathcal{H}^N provides an example of a symmetry group generated by a family of Killing tensors.

Theorem 6.1. *The symmetries of the quantum space-time \mathcal{H}^N are generated by a system of $3r^2 - 2$ irreducible symmetric Killing tensors of rank $r - 1$.*

The significance of Killing tensors is that they are associated with the existence of conserved quantities. A well-known example of a conserved quantity associated with an irreducible Killing tensor (that is to say, a Killing tensor that cannot be expressed as a sum of products of Killing vectors) is Carter's fourth integral of the equations of motion for geodesics and charged-particle orbits in the Kerr and Kerr-Newman solutions of Einstein's equations (Carter 1968, Walker & Penrose 1970, Hughston, *et al.* 1971, Hughston & Sommers 1973, Penrose & Rindler 1986).

In the present context it follows that if the vector field $v^a(x)$ satisfies the geodesic equation, which on a chronometric space of dimension r^2 is given by

$$g_{abc\dots d} (v^e \nabla_e v^b) v^c \dots v^d = 0, \quad (6.7)$$

and if $\xi_{ab\dots c}$ is the Killing tensor of rank $r - 1$ given by (6.5), then we have the conservation law

$$v^e \nabla_e (\xi_{ab\dots c} v^a v^b \dots v^c) = 0. \quad (6.8)$$

In other words, $g_{ab\dots cd} v^a v^b \dots v^c \xi^d$ is a constant of the motion.

Thus in higher-dimensional quantum space-times the apparatus of conservation laws and symmetry principles remains intact in the absence of a pseudo-Riemannian metric. In particular, the conservation of hyper-relativistic momentum and angular momentum for a system of interacting particles can be given a well-defined formulation, the basic principles of which are similar to those applicable in the Minkowskian case. For this purpose it is useful to introduce the notion of an ‘elementary system’ or particle in hyper-relativistic mechanics. Such a system is defined by its hyper-relativistic momentum and angular momentum.

The hyper-relativistic momentum of an elementary system is given by a momentum covector P_a . The associated mass m is then defined by the invariant

$$m = (g^{ab\dots c} P_a P_b \dots P_c)^{\frac{1}{r}}. \quad (6.9)$$

The hyper-relativistic angular momentum of an elementary system is given by a tensor L_a^b of the form

$$L_a^b = l_A^B \delta_{A'}^{B'} + \bar{l}_{A'}^{B'} \delta_A^B, \quad (6.10)$$

where the hyperspinor l_A^B is required to be trace-free: $l_A^A = 0$. The angular momentum is defined with respect to a choice of origin, in such a manner that under a change of origin defined by a shift vector b^a we have $l_A^B \rightarrow l_A^B + P_{AC'} b^{BC'}$. In the case $r = 2$ these formulae reduce to the usual formulae for relativistic momentum and angular momentum in a Minkowskian setting. The real covector

$$S_{AA'} = im^{-1} \left(l_A^B P_{A'B} - \bar{l}_{A'}^{B'} P_{AB'} \right) \quad (6.11)$$

is invariant under a change of origin, and carries the interpretation of the intrinsic spin of the elementary system. The magnitude S of the spin is then defined by $S = |g^{ab\dots c} S_a S_b \dots S_c|^{\frac{1}{r}}$.

In the case of a set of interacting hyper-relativistic systems we require that the total momentum and angular momentum should both be conserved. This then implies conservation of the total mass and spin.

7. Geometry of complex null-separation

In Minkowski space it is useful to examine the geometry of the space of complex null vectors at a point in the space-time. Thus we take the case $r = 2$, and consider complex vectors z^a satisfying $g_{ab}z^az^b = 0$. In spinor terms this implies that z^a can be written in the form

$$z^{AA'} = \alpha^A \beta^{A'}. \quad (7.1)$$

In we take a projective point of view, then up to overall scale the space of complex vectors at a point in Minkowski space can be regarded as a complex projective space \mathbb{P}^3 . The null directions constitute a quadric \mathbb{Q}^2 in that space, which owing to the decomposition (7.1) has the structure of a doubly ruled surface $\mathbb{Q}^2 = \mathbb{P}^1 \times \mathbb{P}^1$. We can identify the first set of lines (the α -lines) with the projective unprimed spinors, and the second set of lines (the β -lines) with the projective primed spinors. The quadric \mathbb{Q}^2 is ruled in such a manner that two lines of the same type do not intersect, whereas two lines of the opposite type intersect at a point in \mathbb{Q}^2 —i.e. the null direction they together determine.

In the case of a general quantum space-time, we consider the space of complex vectors at a point of \mathcal{H}^N , and examine the corresponding space of directions, which has the structure of a complex projective space \mathbb{P}^{r^2-1} . The vanishing of the chronometric form $g_{ab\dots c}z^az^b \dots z^c$ identifies the space of complex null directions as a hypersurface of degree r in \mathbb{P}^{r^2-1} which we shall call \mathfrak{N}^{r^2-2} .

The points of \mathfrak{N}^{r^2-2} correspond to ‘weakly’ null directions. The strongly null directions in \mathfrak{N}^{r^2-2} , corresponding to those for which the associated null vectors are of minimal rank and hence of the form $z^{AA'} = \alpha^A \beta^{A'}$, constitute a subvariety $\mathbb{Q}^{2r-2} \subset \mathfrak{N}^{r^2-2}$ defined by the mutual intersection of a system of quadrics, given by the equation $g_{ab\dots c}z^az^b = 0$. In this case we have $\mathbb{Q}^{2r-2} = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$, and we can identify the two systems of $(r-1)$ -planes by which \mathbb{Q}^{2r-2} is foliated, which we refer to as α -planes and β -planes, as the spaces of projective unprimed and primed hyperspinors, respectively.

The various null directions of intermediate degree correspond to points in \mathfrak{N}^{r^2-2} lying on the linear spaces spanned by the joins of k points in \mathbb{Q}^{2r-2} ($k = 2, 3, \dots, r$). The degree of nullness, as defined in §3, is given by the integer $d = k - 1$.

In the case $r = 3$, for example, the space of directions at a point in \mathcal{H}^9 is \mathbb{P}^8 , and the null directions constitute a cubic hypersurface $\mathfrak{N}^7 \subset \mathbb{P}^8$. The null directions of the first degree (i.e. the totally null directions) lie in the doubly foliated surface $\mathbb{Q}^4 = \mathbb{P}^2 \times \mathbb{P}^2$ in \mathfrak{N}^7 . The points of \mathfrak{N}^7 all lie on the ‘first join’ of \mathbb{Q}^4 with itself; in other words, any point of \mathfrak{N}^7 lies on a line joining two points of \mathbb{Q}^4 . The space $\mathfrak{N}^7 \setminus \mathbb{Q}^4$ then consists of null directions that are strictly of the second degree. (A null direction is strictly of the second degree if it is of the second degree but not also of the first degree.) Note that any point of \mathbb{P}^8 can be represented as the join of three points in \mathbb{Q}^4 .

In the case $r = 4$, the space of directions at a point in \mathcal{H}^{16} is \mathbb{P}^{15} , and the null directions constitute a quadric hypersurface $\mathfrak{N}^{14} \subset \mathbb{P}^{15}$. The null directions of the first degree lie on the doubly foliated surface $\mathfrak{N}_{(1)}^6 = \mathbb{P}^3 \times \mathbb{P}^3$ in \mathfrak{N}^{14} . The null directions of the second degree lie on the first join of \mathbb{Q}^6 with itself: $\mathfrak{N}_{(2)} = J_1(\mathbb{Q}^6)$. The null directions of the third degree lie in $\mathfrak{N}_{(3)} = J_2(\mathbb{Q}^6)$ and constitute the general elements of \mathfrak{N}^{14} .

It is interesting to note a distinction between the Minkowskian case $r = 2$ and higher dimensional quantum space-times. In Minkowski space, the space of complex null directions at a given point corresponds to a nondegenerate quadric \mathbb{Q}^2 in \mathbb{P}^3 , which is doubly ruled in the sense that $\mathbb{Q}^2 = \mathbb{P}^1 \times \mathbb{P}^1$. In fact, any nondegenerate quadric in \mathbb{P}^3 has this structure, and by an automorphism of \mathbb{P}^3 one can transform any such quadric into another.

For $r > 2$, however, this is generally not the case. For example, in the case $r = 3$ the general cubic hypersurface in \mathbb{P}^8 does not contain within it a doubly foliated subvariety $\mathbb{Q}^4 = \mathbb{P}^2 \times \mathbb{P}^2$. The space of cubic hypersurfaces in \mathbb{P}^8 has (complex) dimension 164. If we factor out by the group of projective automorphisms of \mathbb{P}^8 , which is of dimension 80, then we are left with the 84-dimensional moduli-space for cubic forms in \mathbb{P}^8 . The chronometric form of our quantum space-time geometry represents a single point in this moduli-space, and as such constitutes a special cubic surface in \mathbb{P}^8 . Such a hypersurface is in fact completely characterised by the existence of an embedded hyperspinorial subvariety of the form $\mathbb{Q}^4 = \mathbb{P}^2 \times \mathbb{P}^2$. Any two cubic hypersurface in \mathbb{P}^8 admitting a hyperspinorial subvariety can be transformed into one another by an automorphism of \mathbb{P}^8 , and thus represent the same point in the moduli-space. A similar observation applies for all $r > 2$.

8. Generalised Klein representation

To proceed further it will be useful if we set the foregoing material in a geometric context that emphasises the conformal properties of the chronometric form. To this end we let \mathbb{T}^α denote the complex vector space of dimension $2r$ given by the pair $(\mathbb{S}^A, \mathbb{S}^{A'})$. Let us write $Z^\alpha = (\omega^A, \pi_{A'})$ for a typical element of \mathbb{T}^α . Such an element will be referred to as a hypertwistor. For a brief introduction to the theory of hypertwistors (also called ‘generalised twistors’) see Hughston (1978). Let $\mathbb{T}_\alpha = (\mathbb{S}_A, \mathbb{S}^{A'})$ denote the space of dual hypertwistors. A natural pseudo-Hermitian structure can be introduced on the geometry of hypertwistors by means of the complex conjugation operation that maps $(\omega^A, \pi_{A'}) \in \mathbb{T}^\alpha$ to $(\bar{\pi}_A, \bar{\omega}^{A'}) \in \mathbb{T}_\alpha$. The corresponding pseudo-Hermitian form is then given by

$$Z^\alpha \bar{Z}_\alpha = \omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'}, \quad (8.1)$$

and it is straightforward exercise to verify that the inner product $Z^\alpha \bar{Z}_\alpha$ is invariant under the action of the group $U(r, r)$.

The space \mathbb{P}^{2r-1} of projective hypertwistors is a natural starting point for analysing the conformal geometry of complex quantum space-time, which can be regarded as the Grassmannian variety \mathbb{V}^{r^2} of projective $(r-1)$ -planes in \mathbb{P}^{2r-1} . More precisely, \mathbb{V}^{r^2} can be understood as the complex quantum space-time $\mathbb{C}\mathcal{H}^{r^2}$ introduced earlier, together with some structure added at infinity. Thus \mathbb{V}^{r^2} is to be understood as a compactification of $\mathbb{C}\mathcal{H}^{r^2}$. The ‘finite’ points of \mathbb{V}^{r^2} correspond to the linear subspaces of \mathbb{P}^{2r-1} that are determined by a relation of the form

$$\omega^A = ix^{AA'} \pi_{A'} \quad (8.2)$$

for fixed $x^{AA'}$. The aggregate of such $(r-1)$ -planes constitute the points of $\mathbb{C}\mathcal{H}^{r^2}$. The $(r-1)$ -planes for which $x^{AA'}$ is Hermitian then constitute the points of the real space \mathcal{H}^{r^2} .

The conformal structure of quantum space-time is implicit in the various possibilities arising for the intersections of $(r-1)$ -planes in hypertwistor space. A pair of $(r-1)$ -planes in \mathbb{P}^{2r-1} in general will not intersect. This general lack of intersection corresponds to the nonvanishing of the chronometric form for the corresponding quantum space-time points. In this connection we note that the chronometric form $\Delta = g_{ab\dots c} r^a r^b \dots r^c$ introduced earlier for the pairs of real quantum space-time points is also well-defined for pairs of complex quantum space-time points. Now an $(r-1)$ -plane in \mathbb{P}^{2r-1} is represented by a simple skew hypertwistor $P^{\alpha\beta\dots\gamma}$ of rank r . If $r = 2$, we recover the

fact that ordinary space-time points correspond to projective lines in \mathbb{P}^3 , which in turn correspond to simple antisymmetric twistors of rank two. By a *simple* skew hypertwistor we mean one of the form

$$P^{\alpha\beta\cdots\gamma} = A^{[\alpha} B^\beta \cdots C^{\gamma]} \quad (8.3)$$

for some collection $A^\alpha, B^\alpha, \dots, C^\alpha$ of r hypertwistors (all of which lie on the given plane). Suppose that the simple skew hypertwistors $P^{\alpha\beta\cdots\gamma}$ and $Q^{\alpha\beta\cdots\gamma}$ represent, respectively, the $(r-1)$ -planes P and Q in \mathbb{P}^{2r-1} . Then a necessary and sufficient condition for the vanishing of the chronometric form for the corresponding quantum space-time points is

$$\varepsilon_{\alpha\beta\cdots\gamma\rho\sigma\cdots\tau} P^{\alpha\beta\cdots\gamma} Q^{\rho\sigma\cdots\tau} = 0, \quad (8.4)$$

where $\varepsilon_{\alpha\beta\cdots\gamma\rho\sigma\cdots\tau}$ is the totally skew hypertwistor of rank $2r$. We note that (8.4) is symmetric (*resp.*, antisymmetric) under the interchange of $P^{\alpha\beta\cdots\gamma}$ and $Q^{\alpha\beta\cdots\gamma}$ if r is even (*resp.*, odd). The vanishing of the form (8.4) is the condition that the projective planes P and Q contain a point in common. Equivalently, this means that the skew hypertwistors $P^{\alpha\beta\cdots\gamma}$ and $Q^{\rho\sigma\cdots\tau}$ contain at least one hypertwistor as a common factor. Thus we have deduced the following result.

Proposition 8.1. *A necessary and sufficient condition for a pair of quantum space-time points to be weakly null separated is that the corresponding $(r-1)$ -planes in \mathbb{P}^{2r-1} should intersect.*

More generally, the degree d of null separation for a pair of quantum space-time events is given by $d = r - m - 1$, where m is the dimensionality of the intersection of the corresponding $(r-1)$ -planes in \mathbb{P}^{2r-1} . The possible degrees of null separation are given by $d = 1, 2, \dots, r-1$. If we interpret (as usual) the case of no intersection as an intersection of dimension -1 , then a non-null separation between the corresponding quantum space-time points can be interpreted as a ‘separation of degree r ’. Thus separations of degree less than r are all null, whereas a separation of degree r is non-null. The degree of separation of a pair of points, we recall, is given by the rank of the separation matrix $r^{AA'} = x^{AA'} - y^{AA'}$. Equivalently, given two skew hypertwistors $P^{\alpha\beta\cdots\gamma}$ and $Q^{\alpha\beta\cdots\gamma}$, each with r indices, let us form the dual hypertwistor by $Q_{\alpha\beta\cdots\gamma} = \varepsilon_{\alpha\beta\cdots\gamma\rho\sigma\cdots\tau} Q^{\rho\sigma\cdots\tau}$. Then d is given by the maximum number of index contractions we can make between $P^{\alpha\beta\cdots\gamma}$ and $Q_{\alpha\beta\cdots\gamma}$ without obtaining the result zero. If a single index contraction gives zero, this corresponds to the case where $P^{\alpha\beta\cdots\gamma}$ is proportional to $Q^{\alpha\beta\cdots\gamma}$. Thus $d = 0$ (separation of degree zero) can be interpreted as the ‘degenerate’ case where the two quantum space-time points coincide.

9. Quantum infinity

As indicated in the previous section, for any skew hypertwistor $P^{\alpha\beta\cdots\gamma}$ of rank r we define its dual $P_{\alpha\beta\cdots\gamma}$ by the relation

$$P_{\alpha\beta\cdots\gamma} = \varepsilon_{\alpha\beta\cdots\gamma\rho\sigma\cdots\tau} P^{\rho\sigma\cdots\tau}. \quad (9.1)$$

Here $\varepsilon_{\alpha\beta\cdots\gamma\rho\sigma\cdots\tau}$ is the totally skew hypertwistor of rank $2r$, which is unique up to an overall scale factor. Depending on whether r is even or odd, we have the following interchange relations:

$$\varepsilon_{\alpha\beta\cdots\gamma\rho\sigma\cdots\tau} = \pm \varepsilon_{\rho\sigma\cdots\tau\alpha\beta\cdots\gamma}. \quad (9.2)$$

Thus if r is even, then once the scale of the totally skew hypertwistor is fixed we obtain a *symmetric inner product* on the space of skew hypertwistors of rank r , which we can denote symbolically by

$$\langle P, Q \rangle = \varepsilon_{\alpha\beta\cdots\gamma\rho\sigma\cdots\tau} P^{\alpha\beta\cdots\gamma} Q^{\rho\sigma\cdots\tau}. \quad (9.3)$$

On the other hand if r is odd then the inner product (9.3) is a symplectic structure. In this respect the cases for even r and odd r are quite distinct. We shall return to this issue later when we specialise to the symmetric case.

Recall that under complex conjugation the skew hypertwistor $P^{\alpha\beta\cdots\gamma}$ becomes $\bar{P}_{\alpha\beta\cdots\gamma}$. If $P^{\alpha\beta\cdots\gamma}$ is simple, thus corresponding to an $(r-1)$ -plane P in \mathbb{P}^{2r-1} , then we say that P is a *real plane* if $\bar{P}_{\alpha\beta\cdots\gamma}$ is proportional to $P_{\alpha\beta\cdots\gamma}$. The real $(r-1)$ -planes of \mathbb{P}^{2r-1} thus defined correspond to the real points of quantum space-time.

The points at infinity in the compactified quantum space-time $\mathcal{V}^{r^2} = \mathbb{C}\mathcal{H}_{\sharp}^{r^2}$ can be described as follows. In the hypertwistor space \mathbb{P}^{2r-1} we choose a real $(r-1)$ -plane I represented by a simple skew hypertwistor $I^{\alpha\beta\cdots\gamma}$. The point I in \mathcal{V}^{r^2} corresponding to the $(r-1)$ -plane I in \mathbb{P}^{2r-1} will be called the *point at infinity*. The cone in \mathcal{V}^{r^2} consisting of all points that are chronometrically null-separated from I will be called *null infinity*. (There will be no ambiguity if we use the symbol I to denote both the point I in \mathcal{V}^{r^2} and the corresponding $(r-1)$ -plane in \mathbb{P}^{2r-1} .) It should be evident that null infinity has a rich structure, with various domains that can be classified according to their degree of null separation from the point I .

The ‘finite’ points of \mathcal{V}^{r^2} are those for which the chronometric separation from I is non-null, i.e. those points P for which $\langle P, I \rangle \neq 0$ with respect to the inner product (9.3).

Proposition 9.1. *In the case of two finite quantum space-time points the chronometric form Δ is given by the following ratio:*

$$\Delta(P, Q) = \frac{\varepsilon_{\alpha\beta\dots\gamma\rho\sigma\dots\tau} P^{\alpha\beta\dots\gamma} Q^{\rho\sigma\dots\tau}}{(\varepsilon_{\alpha\beta\dots\gamma\rho\sigma\dots\tau} P^{\alpha\beta\dots\gamma} I^{\rho\sigma\dots\tau})(\varepsilon_{\alpha\beta\dots\gamma\rho\sigma\dots\tau} Q^{\alpha\beta\dots\gamma} I^{\rho\sigma\dots\tau})}. \quad (9.4)$$

Equivalently we can write $\Delta = \langle P, Q \rangle / \langle P, I \rangle \langle Q, I \rangle$. If P and I are not null separated, then we can choose the scales of $P^{\alpha\beta\dots\gamma}$ and $I^{\alpha\beta\dots\gamma}$ such that $\langle P, I \rangle = 1$, without loss of generality, and similarly for $Q^{\alpha\beta\dots\gamma}$ and $I^{\alpha\beta\dots\gamma}$. This leads to a further simplification in formula (9.4).

In general, even in the absence of such a simplification, we note that $\Delta(P, Q)$ is independent of the scale of $P^{\alpha\beta\dots\gamma}$ and $Q^{\alpha\beta\dots\gamma}$. On the other hand, $\Delta(P, Q)$ does depend on the scale of $\varepsilon_{\alpha\beta\dots\gamma\rho\sigma\dots\tau}$ and the scale of $I^{\alpha\beta\dots\gamma}$. It has an epsilon ‘weight’ of -1 and an I ‘weight’ of -2 (cf. Hughston & Hurd 1982). However, if we form the ratio associated with four hypertwistors $P, Q, R,$ and S , given by

$$\frac{\Delta(P, Q)}{\Delta(R, S)} = \frac{\|p - q\|^r}{\|r - s\|^r}, \quad (9.5)$$

where $p, q, r,$ and s are the quantum space-time points corresponding to $P, Q, R,$ and S , respectively, then we obtain an expression that is absolute—that is to say, a geometric invariant. This is because $\Delta(P, Q)$ has the ‘dimensionality’ of time raised to the power r ; whereas the ratio (9.5) arises as a comparison of two such time intervals, and thus is dimensionless. The basic chronometric geometry, with infinity chosen as indicated above, admits no absolute or ‘preferred’ unit of time: in this geometry only ratios of time intervals have an absolute meaning.

10. Cosmological infinity

There is, however, no reason *a priori* why such a ‘minimal’ structure should prevail at infinity. Other choices are in principle available for $I^{\alpha\beta\dots\gamma}$, and these have the effect of giving \mathcal{V}^{r^2} the structure of a cosmological model. In the case $r = 2$, for example, if $I^{\alpha\beta}$ is chosen to be real and non-simple, then the quadratic form $\varepsilon_{\alpha\beta\gamma\delta} I^{\alpha\beta} I^{\gamma\delta}$, which has an epsilon weight of one and an I -weight of two, has the dimensionality of inverse squared-time. Hence in this case there *is* a preferred unit of time.

To pursue this point further, we recall that \mathcal{V}^4 has the structure of a quadric Ω in \mathbb{P}^5 . More specifically, for $r = 2$ the space of skew rank two twistors is \mathbb{C}^6 , which is projectively \mathbb{P}^5 , and \mathcal{V}^4 is the locus defined by the homogeneous quadratic equation $\varepsilon_{\alpha\beta\gamma\delta} X^{\alpha\beta} X^{\gamma\delta} = 0$. Infinity

in \mathcal{V}^4 can then be defined by the intersection of \mathcal{V}^4 in \mathbb{P}^5 with the projective 4-plane I given by the equation $\varepsilon_{\alpha\beta\gamma\delta}I^{\alpha\beta}X^{\gamma\delta} = 0$. If $I^{\alpha\beta}$ is simple, then I^4 is tangent to \mathcal{V}^4 , and the intersection is a cone—the null cone at infinity. On the other hand, if $I^{\alpha\beta}$ is not simple, then the intersection $I^4 \cap \mathcal{V}^4$ is a 3-quadric. The resulting geometry, if $I^{\alpha\beta}$ is real, is that of de Sitter space, and the parameter $\lambda = (\varepsilon_{\alpha\beta\gamma\delta}I^{\alpha\beta}I^{\gamma\delta})^2$ has the interpretation of being the associated *cosmological constant*. The de Sitter group then consists of those projective transformations of \mathbb{P}^5 that preserve both Ω and the point I . In fact, with the incorporation of some additional structure at infinity, the entire class of Robertson-Walker cosmological models can be represented in this way (Penrose & Rindler 1984, Hurd 1985, 1995, Penrose 1995).

For general r a similar situation arises—in other words, the choice of structure at infinity gives rise in general to a chronometric metric that is not flat, thus giving \mathcal{V}^{r^2} the character of a cosmological model. The key point is that, whereas in the case of a standard four-dimensional cosmological model based on Einstein's theory the existence of structure at infinity has a bearing on the geometry of space-time alone, in the case of a hypercosmology the structure at infinity also has implications for microscopic physics. In particular, whereas in the four-dimensional de Sitter cosmology the relevant structure at infinity contains the information of a single dimensional constant (the cosmological constant), in the higher-dimensional situation there will in general be a number of such dimensional constants emerging as geometrical invariants of the theory. Thus within the same overall geometric framework one has the scope for introducing structure (or what amounts to the same thing—the breaking of symmetry) both on a global or cosmological scale, as well as on those scales of distance, time, and energy associated with the phenomenology of elementary particles.

In order to prepare the groundwork necessary as a basis for investigating this idea in more detail we must now introduce the particular structure in \mathcal{V}^{r^2} needed to make its relation to ordinary four-dimensional space-time apparent.

11. Segré embedding and symmetry breaking mechanism

Let us therefore consider the mechanisms for symmetry breaking at our disposal in the case of a standard 'flat' quantum space-time endowed with a canonical reality structure and null infinity. We shall demonstrate that the breaking of symmetry in quantum space-time is intimately linked to the notion of quantum entanglement.

In practical terms the breaking of symmetry can be represented by an ‘index decomposition’. The point is that if the dimension r of the hyperspin space is not a prime number, then a natural method of breaking the symmetry arises by consideration of the decomposition of r into factors. The specific assumption that we shall make at this juncture will be that the dimension of the hyperspin space \mathbb{S}^A is *even*. Then we write $r = 2n$, where $n = 1, 2, \dots$, and set $\mathbb{S}^A = \mathbb{S}^{\mathbf{A}i}$, where \mathbf{A} is a standard two-component spinor index, and i will be called an *internal* index ($i = 1, 2, \dots, n$). Thus we can write $\mathbb{S}^{\mathbf{A}i} = \mathbb{S}^{\mathbf{A}} \otimes \mathbb{H}^i$, where $\mathbb{S}^{\mathbf{A}}$ is a standard spin space of dimension two, and \mathbb{H}^i is a complex vector space of dimension n . The breaking of the symmetry then amounts to the fact that we can identify the hyperspin space with the tensor product of these two spaces.

We shall assume, moreover, that as far as the internal space is concerned, there is a canonical anti-linear isomorphism between the complex conjugate of the internal space \mathbb{H}^i and the dual space \mathbb{H}_i . In other words, if $\psi^i \in \mathbb{H}^i$, then we can write $\bar{\psi}_i$ for the complex conjugate of ψ^i , where $\bar{\psi}_i \in \mathbb{H}_i$. Therefore, \mathbb{H}^i is a complex Hilbert space—and indeed although for the moment we consider for technical simplicity the case for which n is finite, one should have in mind also the infinite dimensional situation.

For the other hyperspin spaces we write $\mathbb{S}_A = \mathbb{S}_{\mathbf{A}i}$, $\mathbb{S}^{A'} = \mathbb{S}^{\mathbf{A}'i}$, and $\mathbb{S}_{A'} = \mathbb{S}_{\mathbf{A}'i}$, respectively. These equivalences preserve the duality between \mathbb{S}^A and \mathbb{S}_A , and between $\mathbb{S}^{A'}$ and $\mathbb{S}_{A'}$; and at the same time are consistent with the complex conjugation relations between \mathbb{S}^A and $\mathbb{S}^{A'}$, and between \mathbb{S}_A and $\mathbb{S}_{A'}$. Hence if $\alpha^{\mathbf{A}i} \in \mathbb{S}^{\mathbf{A}i}$ then under complex conjugation we have $\alpha^{\mathbf{A}i} \rightarrow \bar{\alpha}^{\mathbf{A}'i}$, and if $\beta_{\mathbf{A}i} \in \mathbb{S}_{\mathbf{A}i}$ then $\beta_{\mathbf{A}i} \rightarrow \bar{\beta}_{\mathbf{A}'i}$.

In the case of a quantum space-time vector $r^{AA'}$ we have a corresponding induced structure indicated by the identification

$$r^{AA'} = r^{\mathbf{A}\mathbf{A}'i}{}_j. \quad (11.1)$$

When the quantum space-time vector is real, the weak Hermitian structure on $r^{AA'}$ is manifested in the form of a standard *weak* Hermitian structure on the spinor index pair, together with a *strong* Hermitian structure on the internal index pair. (In the case of a strong Hermitian structure it is assumed that there is a canonical isomorphism between the complex conjugate of the given complex vector space and the dual of that vector space.) In other words, if we define

$$\overline{r^{\mathbf{A}\mathbf{A}'i}{}_j} = \bar{r}^{\mathbf{A}'\mathbf{A}}{}_i{}^j, \quad (11.2)$$

then the Hermitian condition on the quantum space-time vector $r^{AA'}$ is given by

$$\bar{r}^{A'A}{}_j{}^i = r^{AA'}{}_j{}^i. \quad (11.3)$$

One striking consequence of these relations is that we can interpret each point in quantum space-time as being a *space-time valued operator*. Ordinary classical space-time then ‘sits’ inside the quantum space-time in a canonical manner—namely, as the locus of those points of quantum space-time that factorise into the product of a space-time point $x^{AA'}$ and the identity operator on the internal space:

$$x^{AA'}{}_j{}^i = x^{AA'}\delta^i{}_j. \quad (11.4)$$

Thus, in those situations for which special relativity amounts to a satisfactory theory, we can regard the relevant events as taking place on or in the immediate neighbourhood of this embedding of the Minkowski space \mathfrak{M}^4 in \mathbb{R}^{4n^2} .

This picture can be presented in somewhat more geometric terms as follows. The hypertwistor space \mathbb{P}^{2r-1} in the case $r = 2n$ admits a Segré embedding of the form $\mathbb{P}^3 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{4n-1}$. Many such embeddings are possible, though they are all equivalent to one another under the action of the overall symmetry group $U(2n, 2n)$. If the symmetry is broken and one such embedding is selected out, then following the conventions discussed earlier we can introduce homogeneous coordinates and write

$$Z^\alpha = Z^{\alpha i}. \quad (11.5)$$

Here the bold Greek letter α denotes an ordinary twistor index ($\alpha = 0, 1, 2, 3$) and i denotes an internal index ($i = 1, 2, \dots, n$). The Segré embedding consists of those points in \mathbb{P}^{4n-1} for which we have a product decomposition of the associated hypertwistor given by $Z^{\alpha i} = Z^\alpha \psi^i$.

The idea of symmetry breaking that we are putting forward here is closely related to the notion of disentanglement in standard quantum mechanics (cf. Brody & Hughston 2001). That is to say, at the unified level the degrees of freedom associated with space-time symmetry are quantum mechanically entangled with the internal degrees of freedom associated with microscopic physics. The phenomena responsible for the breakdown of symmetry are thus analogous to the mechanisms of decoherence through which quantum entanglements are gradually diminished.

The compactified complexified quantum space-time $\mathbb{C}\mathcal{H}_\#^N = \mathcal{V}^{4n^2}$ can be regarded as the aggregate of projective $(2n - 1)$ -planes in \mathbb{P}^{4n-1} .

Now generically a \mathbb{P}^{2n-1} in \mathbb{P}^{4n-1} will not intersect the Segré variety $\mathcal{G}^{n+2} = \mathbb{P}^3 \times \mathbb{P}^{n-1}$. Such a generic $(2n-1)$ -plane corresponds to a generic point in \mathcal{V}^{4n^2} . The $(2n-1)$ -planes that correspond to the points of compactified complexified Minkowski space $\mathbb{C}\mathcal{H}_\#^4$ can be constructed as follows. For each line L in \mathbb{P}^3 we consider the subvariety $\mathcal{G}_L^n \subset \mathcal{G}^{n+2}$ where $\mathcal{G}_L^n = \mathbb{P}_L^1 \times \mathbb{P}^{n-1}$. For any algebraic variety $V^j \subset \mathbb{P}^l$ ($j \leq l-1$) we define the *span* of V^j to be the projective plane spanned by the points of V^j . More precisely, we say a point X in the ambient space \mathbb{P}^l lies in the span of the variety V^j if and only if there exist m points in V^j for some $m \geq 2$ with the property that X lies in the $(m-1)$ -plane spanned by those m points. The dimension k of the span of V^j satisfies $j \leq k \leq l$; however, the value of k depends on the geometry of V^j .

Now the linear span of the points in \mathcal{G}_L^n , for any given L , is a $(2n-1)$ -plane. This is the \mathbb{P}_L^{2n-1} in \mathbb{P}^{4n-1} that represents the point in $\mathbb{C}\mathcal{H}_\#^4$ corresponding to the line L in \mathbb{P}^3 . The aggregate of such special $(2n-1)$ -planes, defined by their intersection properties with the Segré variety \mathcal{G}^{n+2} , constitutes a submanifold of \mathcal{V}^{4n^2} , and this submanifold is compactified complexified Minkowski space. Thus we see that once the symmetry of quantum space-time \mathcal{H}^{4n^2} has been broken in the particular way we have discussed, then ordinary Minkowski space \mathfrak{M}^4 can be identified as a submanifold.

12. Causality and quantum states

The embedding of Minkowski space in the quantum space-time \mathcal{H}^{4n^2} given by (11.4) implies a corresponding embedding of the Poincaré group in the hyper-Poincaré group. Indeed, if in \mathfrak{M}^4 the standard Poincaré group consists of transformations of the form

$$x^{\mathbf{A}\mathbf{A}'} \longrightarrow l_{\mathbf{B}}^{\mathbf{A}} \bar{l}_{\mathbf{B}'}^{\mathbf{A}'} x^{\mathbf{B}\mathbf{B}'} + b^{\mathbf{A}\mathbf{A}'}, \quad (12.1)$$

then the hyper-Poincaré transformations in \mathcal{H}^{4n^2} are of the form

$$x^{\mathbf{A}\mathbf{A}'i}_j \longrightarrow l_{\mathbf{B}}^{\mathbf{A}} \bar{l}_{\mathbf{B}'}^{\mathbf{A}'} x^{\mathbf{B}\mathbf{B}'i}_j + b^{\mathbf{A}\mathbf{A}'i}_j \delta_j^i. \quad (12.2)$$

On the other hand, with the identification $A = \mathbf{A}i$, the general hyper-Poincaré transformation in the broken symmetry phase can be expressed in the form

$$x^{\mathbf{A}\mathbf{A}'i}_j \longrightarrow L_{\mathbf{B}k}^{\mathbf{A}i} \bar{L}_{\mathbf{B}'j}^{\mathbf{A}'l} x^{\mathbf{B}\mathbf{B}'k}_l + b^{\mathbf{A}\mathbf{A}'i}_j. \quad (12.3)$$

Thus the embedding of the Poincaré group as a subgroup of the hyper-Poincaré group is given explicitly by

$$L_{\mathbf{B}j}^{\mathbf{A}i} = l_{\mathbf{B}}^{\mathbf{A}} \delta_j^i \quad \text{and} \quad b^{\mathbf{A}\mathbf{A}'i}_j = b^{\mathbf{A}\mathbf{A}'i}_j \delta_j^i. \quad (12.4)$$

Bearing these relations in mind, we now consider the problem of constructing a certain class of maps from the general even-dimensional quantum space-time \mathcal{H}^{4n^2} to Minkowski space \mathfrak{M}^4 . It will be shown that under rather general and reasonable physical assumptions such maps necessarily take the form

$$x^{\mathbf{AA}'i} \longrightarrow x^{\mathbf{AA}'} = \rho_i^j x^{\mathbf{AA}'i}_j, \quad (12.5)$$

where ρ_i^j is a *density matrix*, that is to say, a positive semi-definite Hermitian matrix with unit trace. Thus the maps under consideration can be regarded as quantum expectations.

Theorem 12.1. *Let $\rho : \mathcal{H}^{4n^2} \rightarrow \mathfrak{M}^4$ satisfy the following conditions: (i) ρ is linear and maps the origin of \mathcal{H}^{4n^2} to the origin of \mathfrak{M}^4 ; (ii) ρ is Poincaré invariant; and (iii) ρ is causal. Then ρ is given by a density matrix on the internal space.*

Proof. The general linear map from \mathcal{H}^{4n^2} to \mathfrak{M}^4 preserving the origin is given by

$$x^{\mathbf{AA}'i} \longrightarrow x^{\mathbf{AA}'} = \rho_{\mathbf{BB}'i}^{\mathbf{AA}'j} x^{\mathbf{BB}'i}_j, \quad (12.6)$$

where $\rho_{\mathbf{BB}'i}^{\mathbf{AA}'j}$ is weakly Hermitian. Now suppose that we subject \mathcal{H}^{4n^2} to a Poincaré transformation of the form (12.2), and require the corresponding transformation of \mathcal{H}^4 to be of the form (12.1). If ρ satisfies these conditions then we shall say that the map ρ is Poincaré invariant. It should be evident from this definition that Poincaré invariance holds if and only if

$$\rho_{\mathbf{BB}'i}^{\mathbf{AA}'j} \left(l_{\mathbf{C}}^{\mathbf{B}} \bar{l}_{\mathbf{C}'}^{\mathbf{B}'} x^{\mathbf{CC}'i}_j + b^{\mathbf{BB}'} \delta_j^i \right) = l_{\mathbf{B}}^{\mathbf{A}} \bar{l}_{\mathbf{B}'}^{\mathbf{A}'} \left(\rho_{\mathbf{CC}'i}^{\mathbf{BB}'j} x^{\mathbf{CC}'i}_j \right) + b^{\mathbf{AA}'} \quad (12.7)$$

for all $l_{\mathbf{B}}^{\mathbf{A}} \in SL(2n, \mathbb{C})$, all $b^{\mathbf{AA}'} \in \mathbb{V}^{\mathbf{AA}'}$, and all $x^{\mathbf{AA}'i}_j \in \mathcal{H}^{4n^2}$. Thus we have

$$\rho_{\mathbf{BB}'i}^{\mathbf{AA}'j} l_{\mathbf{C}}^{\mathbf{B}} \bar{l}_{\mathbf{C}'}^{\mathbf{B}'} = l_{\mathbf{B}}^{\mathbf{A}} \bar{l}_{\mathbf{B}'}^{\mathbf{A}'} \rho_{\mathbf{CC}'i}^{\mathbf{BB}'j} \quad (12.8)$$

for all $l_{\mathbf{B}}^{\mathbf{A}}$, and

$$\rho_{\mathbf{BB}'i}^{\mathbf{AA}'j} \delta_j^i b^{\mathbf{BB}'} = b^{\mathbf{AA}'} \quad (12.9)$$

for all $b^{\mathbf{AA}'}$. Now (12.8) implies that ρ is of the form

$$\rho_{\mathbf{BB}'i}^{\mathbf{AA}'j} = \delta_{\mathbf{B}}^{\mathbf{A}} \delta_{\mathbf{B}'}^{\mathbf{A}'} \rho_i^j \quad (12.10)$$

for some ρ_i^j . Then (12.9) implies that ρ must satisfy the trace condition $\rho_i^i = 1$. Finally we require that if $x^{\mathbf{AA}'i}$ and $y^{\mathbf{AA}'i}$ are quantum space-time points such that the interval $r^{\mathbf{AA}'i} = x^{\mathbf{AA}'i} - y^{\mathbf{AA}'i}$ is future-pointing then $r^{\mathbf{AA}'} = x^{\mathbf{AA}'} - y^{\mathbf{AA}'}$ must also be future pointing, where $r^{\mathbf{AA}'} = \rho_i^j r^{\mathbf{AA}'i}$. This is the requirement that ρ should be a *causal* map. However, this condition immediately implies that ρ must be positive semi-definite. The argument is as follows. If $r^{\mathbf{AA}'i}$ is future-pointing then it is necessarily of the form

$$r^{\mathbf{AA}'i} = \alpha^{\mathbf{A}i} \bar{\alpha}_j^{\mathbf{A}'} + \beta^{\mathbf{A}i} \bar{\beta}_j^{\mathbf{A}'} + \dots \quad (12.11)$$

Consider therefore the case for which $r^{\mathbf{AA}'i}$ is strongly null. Then we require that $\alpha^{\mathbf{A}i} \bar{\alpha}_j^{\mathbf{A}'} \rho_i^j$ should be future-pointing (or vanish) for any choice of $\alpha^{\mathbf{A}i}$. Thus in particular we require that $\alpha^{\mathbf{A}i} \bar{\alpha}_j^{\mathbf{A}'} \rho_i^j$ should be future-pointing if $\alpha^{\mathbf{A}i}$ is of the form $\alpha^{\mathbf{A}i} = \alpha^{\mathbf{A}} \psi^i$ for any choice of $\alpha^{\mathbf{A}}$ and ψ^i . This means that $\rho_i^j \psi^i \bar{\psi}_j \geq 0$ for all ψ^i , which shows that ρ_i^j must be positive semi-definite. Since we have already shown that the trace of ρ_i^j must be unity, it follows that ρ_i^j is a density matrix. \square

This theorem shows how the causal structure of quantum space-time is linked in a surprising way with the probabilistic structure of quantum mechanics. The concept of a quantum state emerges when we ask for consistent ways of ‘averaging’ over the geometry of quantum space-time in order to obtain a reduced description of phenomena in terms of the geometry of Minkowski space.

It is interesting to note that Theorem 12.1 has a formal resemblance to Gleason’s theorem in quantum mechanics, which states that a map from an observable to a real number (expectation value) must be given by a density matrix, if appropriate probabilistic conditions are imposed. In the present framework we see that a probabilistic interpretation of the map from a general quantum space-time to Minkowski space emerges as a consequence of elementary causality requirements. We can thus view the space-time events in \mathcal{H}^{4n^2} themselves as representing quantum observables, the expectations of which correspond to points of \mathfrak{M}^4 .

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