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Relaxation of quantum states under energy perturbations

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The energy-based stochastic extension of the Schrödinger equation is perhaps the simplest mathematically rigorous and physically plausible model for the reduction of the wave function. In this article, we apply a new simulation methodology for the stochastic framework to analyse the dynamics of a particle confined to a square-well potential. We consider the situation when the width of the well is expanded instantaneously. Through this example we are able to illustrate in detail how a quantum system responds to an energy perturbation, and the mechanism, according to the stochastic evolutionary law, by which the system relaxes spontaneously into one of the stable eigenstates of the Hamiltonian. We examine, in particular, how the expectation value of the Hamiltonian and the probability distribution for the position of the particle change in time. An analytic expression for the typical time-scale of relaxation is derived. We also consider the small perturbation limit, and discuss the relation between the stochastic framework and the quantum adiabatic theorem.

Keywords: state-vector reduction; perturbation theory; quantum adiabatic theorem; square-well potential; nonlinear filtering; quantum-state diffusion

1. Introduction

The purpose of this paper is to present a theory of stochastic relaxation for perturbed quantum systems. The model for relaxation that we consider emerges as a natural consequence of the standard energy-based extension of the stochastic Schrödinger equation. In the present investigation, we carry out, in particular, a detailed analysis of the quantum dynamics of relaxation arising in the example of the perturbation induced by the sudden expansion of a one-dimensional potential well in which a particle is trapped.

The motivation for this work can be briefly sketched as follows. Suppose we consider a quantum system for which the Hamiltonian undergoes a sudden change $\hat{h} \to \hat{H}$ at time $t = 0$. If the initial wave function of the system is given by $\psi_0(x)$, the probability that the wave function $\psi_t(x)$ at a later time $t > 0$ will be found in the nth energy eigenstate $\chi_n(x)$ of the new Hamiltonian $\hat{H}$ is determined by the transition amplitude between $\psi_0(x)$ and $\chi_n(x)$. Just after the application of the perturbation, the wave function can be represented by an expansion of $\psi_0(x)$ in terms of the eigenfunctions $\chi_n(x)$ of $\hat{H}$. According to the standard quantum framework, the state then undergoes a unitary evolution generated by $\hat{H}$. The unitary law has the effect that it necessarily constrains the wave function $\psi_t(x)$ to remain in a state of superposition of the eigenmodes of $\hat{H}$, and that this superposition will linger indefinitely in time. Indeed, it is generally necessary to append to quantum theory an additional postulate to the effect that only as a result of another kind of sudden perturbation—namely, an operation of energy measurement—will the wave function of the system be able to 'jump' into one or another of the eigenstates of the new Hamiltonian.

There are many situations, however, in which it is natural to presume that, after the passage of some time, the system spontaneously relaxes into one or another of the eigenstates of the new Hamiltonian, irrespective of whether a measurement is made. Many natural phenomena are of this character: after perturbation, there follows relaxation. Quantum theory, as such, does not account for this possibility satisfactorily. There is arguably an implicit assumption often made in practice that, even in the absence of specific acts of measurement, quantum systems of any significant size or complexity will settle into a stable eigenstate, typically an eigenstate of energy.

In contrast, by use of the stochastic extension to the Schrödinger equation, it is possible to model the dynamics of the wave function in such a manner that, after the system is perturbed, the wave function spontaneously relaxes to one of the eigenstates of the new Hamiltonian. Stochastic extensions of the Schrödinger equation were first introduced to model the collapse of the wave function arising in the measurement process in quantum mechanics. In the model we consider here, the idea is that relaxation phenomena are constantly at work. It is a remarkable fact that it is possible to construct a stochastic differential equation for the wave function with the property that the probability laws thus arising result in *statistical* predictions that are in agreement with those of standard quantum theory, an essential requirement for a reasonable theory of relaxation.

The structure of the paper is as follows. In § 2 we review the basic set-up for a free particle in a potential well when the well is subjected to a sudden expansion. We argue that standard quantum mechanics does not give a completely satisfactory account of the matter. In § 3 we review the formalism of the standard energy-based stochastic extension of the Schrödinger equation, and propose the use of stochastic relaxation as a basis for the description of the dynamics of a quantum system following a perturbation. In § 4 we review the construction of the solution to the stochastic system and describe its properties. Here we make use of a novel nonlinear filtering technique introduced in Brody & Hughston (2002*b*), and show directly that the resulting state undergoes relaxation. In §§ 5 and 6 we indicate how the solution thus obtained can be used for the efficient construction of simulations. More precisely, we show that, according to the stochastic evolutionary law, the system energy fluctuates randomly and eventually relaxes to one of the eigenvalues of the new Hamiltonian. A number of graphic illustrations are provided for the behaviour of the energy, as well as the probability density of the location of the particle in the well, for various realizations of the collapse process. In § 7 we present an analysis of the time-scale associated with the eventual occurrence of relaxation. This analysis then forms the basis of a discussion of a stochastic version of the quantum adiabatic theorem, presented in §§ 8 and 9.

2. Free expansion in a potential well

As a concrete example of the phenomenon of relaxation, we examine in this paper the sudden expansion of a one-dimensional potential well in which a particle of mass μ is trapped, and study the subsequent evolution of the state of the system. Before the expansion, the width of the well is L. The energy spectrum of the particle is then given by the formula

$$\epsilon_n = \frac{\pi^2\hbar^2 n^2}{2\mu L^2}, \tag{2.1}$$

where $n = 1, 2, \ldots, \infty$, for which the corresponding eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right), \quad 0 \leqslant x \leqslant L. \tag{2.2}$$

We assume for simplicity that the particle is initially in one of the energy eigenstates associated with the potential.

Let us suppose that at $t = 0$ the Hamiltonian $\hat{h}$ is changed in such a way that the width of the potential is increased from L to αL, where $\alpha \geqslant 1$. Then, for any $t > 0$, the wave function of the system can be expressed as a superposition of the normalized stationary states of the new Hamiltonian $\hat{H}$. These are given by

$$\chi_n(x) = \sqrt{\frac{2}{\alpha L}}\sin\left(\frac{n\pi}{\alpha L}x\right), \quad 0 \leqslant x \leqslant \alpha L, \tag{2.3}$$

for which the associated eigenvalues are

$$E_n = \frac{\pi^2\hbar^2 n^2}{2\mu\alpha^2 L^2}, \tag{2.4}$$

where $n = 1, 2, \ldots, \infty$. It follows from the standard unitary dynamics of the Schrödinger equation that after the expansion has taken place the system will be in an indefinite state of energy, and will remain so.

If the particle is initially in the nth eigenstate $\phi_n(x)$ with energy ϵ_n, then the probability π_{nm} that, after the expansion has taken place, the particle will be found in the mth eigenstate $\chi_m(x)$ of the new Hamiltonian, with energy E_m, is

$$\pi_{nm} = \left(\int_0^L \phi_n(x)\chi_m(x)\,\mathrm{d}x\right)^2. \tag{2.5}$$

A short calculation shows that

$$\pi_{nm} = \frac{4\alpha^3 n^2}{\pi^2(m^2 - \alpha^2 n^2)^2}\sin^2\left(\frac{\pi m}{\alpha}\right). \tag{2.6}$$

Clearly, if such a change of state occurs, then energy is not strictly conserved. The conservation of energy is maintained in expectation, however. Indeed, we have the identity

$$\sum_{m=1}^{\infty} \pi_{nm} E_m = \epsilon_n, \tag{2.7}$$

which is valid for all n and for all α, by virtue of which we can confirm that the expectation value of the Hamiltonian is a constant of the motion (see Bender *et al.* 2000).

In what follows we take the point of view that, after the perturbation, the system evolves spontaneously into one of the stable eigenstates of the new Hamiltonian. Quantum theory in itself offers little clue as regards the mechanism according to which the system relaxes to the new stationary state. In particular, it could be argued that a weakness of the conventional measurement theory is that it does not offer a basis for the description of relaxation phenomena. The question of explaining such phenomena can, nevertheless, be addressed in a satisfactory way by use of the dynamics of the stochastic extension of the Schrödinger equation, as we shall demonstrate.

3. Energy-based stochastic dynamics

In what follows we shall model the dynamical mechanism governing the relaxation of the quantum system by the standard energy-based stochastic extension of the Schrödinger equation. This is given by the stochastic differential equation

$$\mathrm{d}\psi_t(x) = -\mathrm{i}\hat{H}\psi_t(x)\,\mathrm{d}t - \tfrac{1}{8}\sigma^2(\hat{H}-H_t)^2\psi_t(x)\,\mathrm{d}t + \tfrac{1}{2}\sigma(\hat{H}-H_t)\psi_t(x)\,\mathrm{d}W_t, \tag{3.1}$$

for which it is assumed that there is a prescribed initial wave function $\psi_0(x)$. Here, W_t denotes a standard Wiener process, and

$$H_t = \frac{\int \psi_t^*(x)\hat{H}\psi_t(x)\,\mathrm{d}x}{\int \psi_t^*(x)\psi_t(x)\,\mathrm{d}x} \tag{3.2}$$

is the random process corresponding to the expectation value of the Hamiltonian operator $\hat{H}$ in the random state $\psi_t(x)$. The volatility parameter σ appearing in (3.1) has the units

$$[\sigma] = [\text{energy}]^{-1}[\text{time}]^{-1/2}. \tag{3.3}$$

Dynamical equations of the type (3.1) for the evolution of the wave function, and various generalizations thereof, were introduced originally as simple models to characterize the collapse of the wave function when a measurement is carried out (Gisin 1984, 1989; Ghirardi *et al.* 1986, 1990; Diosi 1988). See, for example, Percival (1998), Pearle (2000), Bassi & Ghirardi (2003) and references cited therein for a more comprehensive account of the literature. Similar dynamical equations arise in quantum optics, and in the *a posteriori* analysis of continuous quantum measurements. The idea that the wave function should proceed to energy eigenstates was proposed by Bedford & Wang (1975, 1977). The specific energy-based form of the dynamics (3.1), which has been studied by Gisin (1989), Percival (1994, 1995), Hughston (1996), Adler & Horwitz (2000), Adler (2002) and Brody & Hughston (2002*a*), amongst others, is the most parsimonious of these state-reduction models and in many respects the most attractive as the basis for a fundamental model. In this paper, we carry the physical application of the stochastic theory a step further and propose (3.1) as an elementary model for the relaxation of the state of a quantum system when its Hamiltonian has been perturbed. We leave open here the question of whether the volatility parameter σ governing the time-scale of relaxation is phenomenological, i.e. varying according to the structure of the system, or universal, e.g. Planckian. Before presenting the solution to (3.1) and applying the results to perturbation theory, we briefly sketch some of the basic mathematical and physical properties associated with the dynamics (3.1). For further details, see Adler *et al.* (2001) and references cited therein.

Let us note first that the coefficient of $\mathrm{d}W_t$ in the third term of the right-hand side of (3.1) is given by the difference of the Hamiltonian operator and its expectation, acting on the wave function. Thus, if the system enters into an eigenstate of the Hamiltonian, this coefficient becomes zero, and the random fluctuations generated by W_t make no further contribution to the dynamics of $\psi_t(x)$. This property also applies to the second term, which together with the third term 'drives' the system into a state of lower energy uncertainty. Starting from an arbitrary initial state, the system is randomly driven into states with lower energy variance, until it asymptotically reaches an eigenstate of the Hamiltonian, in which the variance vanishes.

Given the dynamical equation (3.1) for the wave function and the corresponding process (3.2) for the expected energy, we can determine the stochastic equation satisfied by H_t. This is given by

$$\mathrm{d}H_t = \sigma V_t \, \mathrm{d}W_t, \tag{3.4}$$

where $V_t = \langle\psi_t|(\hat{H} - H_t)^2|\psi_t\rangle / \langle\psi_t|\psi_t\rangle$ is the process associated with the variance of the energy. The variance process satisfies

$$\mathrm{d}V_t = -\sigma^2 V_t^2 \, \mathrm{d}t + \sigma\beta_t \, \mathrm{d}W_t, \tag{3.5}$$

where $\beta_t = \langle\psi_t|(\hat{H} - H_t)^3|\psi_t\rangle / \langle\psi_t|\psi_t\rangle$. Integrating (3.4) and (3.5), we find that the energy process can be expressed in the form

$$H_t = H_0 + \sigma \int_0^t V_s \, \mathrm{d}W_s \tag{3.6}$$

and the variance process can be written as

$$V_t = V_0 - \sigma^2 \int_0^t V_s^2 \, \mathrm{d}s + \sigma \int_0^t \beta_s \, \mathrm{d}W_s. \tag{3.7}$$

Owing to elementary properties of the stochastic integrals (3.6) and (3.7), it then follows as a standard exercise in stochastic analysis that the following *conditional expectation relations* hold for the energy H_t and its variance V_t:

$$\mathbb{E}[H_u \mid \{H_s\}_{0\leqslant s\leqslant t}] = H_t \tag{3.8}$$

and

$$\mathbb{E}[V_u \mid \{V_s\}_{0\leqslant s\leqslant t}] \leqslant V_t, \tag{3.9}$$

for $t \leqslant u$. Here, the operation $\mathbb{E}[\,\cdot \mid \{X_s\}_{0\leqslant s\leqslant t}]$ denotes the conditional expectation given the history of the process X_s from time 0 up to time t. Therefore, the conditional expectation of the energy process at any time $u \geqslant t$, given its history up to time t, is given by its value at time t. We thus say that H_t satisfies the *martingale* condition. It follows that H_t is on average conserved, whereas the variance V_t tends, on average, to decrease, corresponding to the spontaneous reduction of the system to an eigenstate. The reductive character of the dynamics (3.1) is indicated by the fact that V_t satisfies the *supermartingale* condition (3.9).

4. Stochastic relaxation

Despite its nonlinearity, equation (3.1) can be solved exactly to yield an analytic solution that characterizes the state of the system at any time in terms of a pair of state variables (Brody & Hughston 2002*b*). The method of obtaining the solution, which is of interest in its own right, makes use of the classical techniques of nonlinear filtering. We consider a process of the form $\xi_t = \sigma t H + B_t$, where B_t is a Brownian motion and σ is a parameter that determines the characteristic relaxation time-scale. The random variable H takes the value E_m with probability π_m, where π_m is the transition probability from the given initial quantum state to the energy eigenstate with energy E_m. Intuitively, one can think of ξ_t as representing the value of a phase, scaled by the constant σ, together with a random noise term, whose strength, relative to H, decreases inverse proportionally in time.

The random variable H, which is assumed to be independent of B_t, is to be thought of as the value of the energy to which the system relaxes after the passage of sufficient time. Given the trajectory of ξ_t up to time t, we determine the best estimate for the value of H in the sense of least squares. Because the standard error in the value of B_t grows like the square root of time, it follows that as time passes the true value of H is gradually revealed. The estimate will be denoted H_t, which is obtained by taking the conditional expectation of H, given the history of ξ_t up to that time. Because ξ_t is a Markov process, this implies that H_t is the expectation of H conditional on the value ξ_t, that is,

$$H_t = \mathbb{E}[H \mid \xi_t]. \tag{4.1}$$

The proof that H_t is the best estimate for H given the history of ξ_t is as follows. Suppose Y_t is any process that at time t can be expressed as a functional of the history of ξ_s for $0 \leqslant s \leqslant t$. Then the choice of Y_t that minimizes the expected mean square error $\mathbb{E}[(H - Y_t)^2 \mid \{\xi_s\}_{0\leqslant s\leqslant t}]$ given the history of ξ_t is the process H_t defined by (4.1). This can be deduced by a straightforward variational argument.

Because H_t is given, for each t, by a conditional expectation with respect to ξ_t, it follows that H_t is a function of ξ_t. In order to determine (4.1) explicitly, we require the conditional probability $\mathbb{P}(H = E_m \mid \xi_t)$ for the random variable H. By use of the Bayes law for conditional probability, we find that

$$\mathbb{P}(H = E_m \mid \xi_t) = \frac{\pi_m \rho(\xi_t \mid H = E_m)}{\sum_{n=1}^{\infty} \pi_n \rho(\xi_t \mid H = E_n)}. \tag{4.2}$$

Here, $\rho(\xi_t \mid H = E_m)$ denotes the conditional density function for the continuous random variable ξ_t given that $H = E_m$. In deriving (4.2), we make use of the fact that the *unconditional* probability $\mathbb{P}(H = E_m)$ is just π_m for the given initial state. We have also used the relation

$$\rho(\xi_t) = \sum_{n=1}^{\infty} \pi_n \rho(\xi_t \mid H = E_n). \tag{4.3}$$

Since B_t is a Brownian motion, it is by definition normally distributed with mean zero and variance t. Thus the conditional probability density for ξ_t is

$$\rho(\xi_t \mid H = E_m) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(\xi_t - \sigma E_m t)^2\right). \tag{4.4}$$

Inserting this expression into (4.2), we deduce that

$$\mathbb{P}(H = E_m \mid \xi_t) = \frac{\pi_m \exp(\sigma E_m \xi_t - \frac{1}{2}\sigma^2 E_m^2 t)}{\sum_{n=1}^{\infty} \pi_n \exp(\sigma E_n \xi_t - \frac{1}{2}\sigma^2 E_n^2 t)}. \tag{4.5}$$

Since the energy process is given by the expectation

$$H_t = \sum_{m=1}^{\infty} E_m \mathbb{P}(H = E_m \mid \xi_t), \tag{4.6}$$

it follows that

$$H_t = \frac{\sum_{m=1}^{\infty} \pi_m E_m \exp(\sigma E_m \xi_t - \frac{1}{2}\sigma^2 E_m^2 t)}{\sum_{m=1}^{\infty} \pi_m \exp(\sigma E_m \xi_t - \frac{1}{2}\sigma^2 E_m^2 t)}. \tag{4.7}$$

We note that in (4.7) the process H_t for the conditional expectation of the random variable H is expressed in terms of a function of t and ξ_t. As a consequence, the dynamics of the process ξ_t can be expressed as a diffusion equation of the form

$$\mathrm{d}\xi_t = \sigma H_t \, \mathrm{d}t + \mathrm{d}W_t, \tag{4.8}$$

where W_t is a standard Brownian motion with respect to the filtration generated by the history of H_t. The existence of a Brownian motion W_t satisfying (4.8) follows as a consequence of well-established line of argument in nonlinear filtering theory (see Liptser & Shiryaev 1974), the details of which in the present context are set out in Brody & Hughston (2002*b*).

In order to establish directly that the Hamiltonian process (4.7) necessarily relaxes to one of the energy eigenvalues, one can argue as follows. Suppose that the random variable H happens to take the value E_j. By definition, such an event occurs with probability π_j. More precisely, we condition on the outcome of the event $H = E_j$ and analyse the evolution of process (4.7). Then, writing $\xi_t = \sigma E_j t + B_t$, we obtain, for the corresponding realization of H_t, the expression

$$H_t^j = \frac{\pi_j E_j + \sum_{m \neq j}^{\infty} \pi_m E_m \exp(\sigma(E_m - E_j)B_t - \frac{1}{2}\sigma^2 (E_m - E_j)^2 t)}{\pi_j + \sum_{m \neq j}^{\infty} \pi_m \exp(\sigma(E_m - E_j)B_t - \frac{1}{2}\sigma^2 (E_m - E_j)^2 t)}, \tag{4.9}$$

where the superscript j in H_t^j indicates the conditioning on the event $\{H = E_j\}$.

Because the exponential terms appearing in both the numerator and the denominator have the property that, as $t \to \infty$, the probability that these terms remain positive approaches zero, it follows that the energy process H_t^j converges asymptotically to the designated eigenvalue E_j. Indeed, for any normally distributed random variable B_t with mean zero and variance t, it is an elementary fact that if $\nu \neq 0$, then

$$\lim_{t \to \infty} \mathbb{P}(\exp(\nu B_t - \tfrac{1}{2}\nu^2 t) > x) = 0 \tag{4.10}$$

for any given $x > 0$. The closed-form solution (4.9) shows how the expectation value of the system Hamiltonian evolves in time. In particular, it shows how the system organizes itself spontaneously to move into a stable stationary state of the new Hamiltonian. Since (4.9) is expressed in terms of an analytic function of the Brownian motion B_t and the time t, there arises the possibility of efficiently simulating the evolution of H_t.

5. Wave-function dynamics

We return now to the analysis of the dynamics of the state $\psi_t(x)$ of a particle trapped in a potential well following a sudden expansion of the width of the well. In this case, the solution to the stochastic differential equation (3.1) is

$$\psi_t(x) = \frac{\sum_{m=1}^{\infty} \pi_m^{1/2} \exp(-\mathrm{i}E_m t + \frac{1}{2}\sigma E_m \xi_t - \frac{1}{4}\sigma^2 E_m^2 t)\chi_m(x)}{(\sum_{m=1}^{\infty} \pi_m \exp(\sigma E_m \xi_t - \frac{1}{2}\sigma^2 E_m^2 t))^{1/2}}. \tag{5.1}$$

Here, $\chi_m(x)$ denotes the normalized eigenfunction of the Hamiltonian $\hat{H}$ with energy E_m, and the choice of the initial state $\psi_0(x)$ is implicit in the probability π_m, given by

$$\pi_m = \left(\int_0^L \psi_0(x)\chi_m(x)\,\mathrm{d}x\right)^2. \tag{5.2}$$

The solution (5.1) can be verified by taking the stochastic differential of the right-hand side and using the Ito rules, and then making the substitution (4.8).

The convergence of the infinite sum in the numerator and the denominator of (5.1) for finite t may not be immediately evident. To check that these sums converge, we substitute a particular realization of the path for ξ_t, say, $\xi_t = \sigma E_j t + B_t$, when the random variable H happens to take the value E_j. Then, on account of the fact that $E_m \sim m^2$, we find that the summands decay, for each fixed j, like $\sim \exp(-m^4)$, and convergence is ensured.

Let us now consider the random dynamics of the probability density function $\rho_t(x) = \psi_t^*(x)\psi_t(x)$ for finding the particle at the location x in the interval $[0, \alpha L]$. We would like to obtain the probability distribution for the particle in the case where the final state of the system is given by the eigenstate $\chi_j(x)$ for some given value of j. The probability for this particular realization to occur is given by π_j. By rearrangement of terms, and writing

$$\omega_{mj} = E_m - E_j \tag{5.3}$$

for the energy-level difference ($\hbar = 1$), we obtain

$$\rho_t^j(x) = \frac{|\sum_m \pi_m^{1/2} \exp(-\mathrm{i}\omega_{mj} t + \frac{1}{2}\sigma\omega_{mj} B_t - \frac{1}{4}\sigma^2\omega_{mj}^2 t)\chi_m(x)|^2}{\sum_m \pi_m \exp(\sigma\omega_{mj} B_t - \frac{1}{2}\sigma^2\omega_{mj}^2 t)} \tag{5.4}$$

for the probability density. Thus $\rho_t(x)$ is a measure-valued process. That is to say, at each given time t, it is a smooth density function over $[0, \alpha L]$, but the form of the function evolves randomly in time until it relaxes to the final distribution $\chi_j^2(x)$.

6. Simulation of the energy and the probability density

With these formulae at hand, we consider the simulation of the random trajectory for the expectation value of the Hamiltonian governed by the dynamics (4.7). We also consider the simulation of the corresponding probability-density function (5.4) for the position of the particle in the potential well. The quantum system is defined by specifying the value of the mass μ of the particle, the width L of the well, the volatility parameter σ governing the stochastic dynamics, and the well-expansion factor α. For simplicity, we examine the case where the system is initially in its ground state $\phi_1(x)$, although other initial conditions, including mixed initial states, can be treated analogously. Then the energy process (4.9), corresponding to the particular realization $\psi_\infty(x) = \chi_j(x)$ of the terminal state, can be written in the form

$$H_t^j = \frac{\sum_m \pi_m E_m \exp(\sigma\omega_{mj}B_t - \frac{1}{2}\sigma^2\omega_{mj}^2 t)}{\sum_m \pi_m \exp(\sigma\omega_{mj}B_t - \frac{1}{2}\sigma^2\omega_{mj}^2 t)}. \tag{6.1}$$

The associated probability density for the position of the particle in the interval $[0, \alpha L]$ is given by

$$\rho_t^j(x) = \frac{(\sum_m \pi_m^{1/2} \exp(\frac{1}{2}\sigma\omega_{mj}B_t - \frac{1}{4}\sigma^2\omega_{mj}^2 t)\chi_m(x)\cos(\omega_{mj}t))^2}{\sum_m \pi_m \exp(\sigma\omega_{mj}B_t - \frac{1}{2}\sigma^2\omega_{mj}^2 t)} + \frac{(\sum_m \pi_m^{1/2} \exp(\frac{1}{2}\sigma\omega_{mj}B_t - \frac{1}{4}\sigma^2\omega_{mj}^2 t)\chi_m(x)\sin(\omega_{mj}t))^2}{\sum_m \pi_m \exp(\sigma\omega_{mj}B_t - \frac{1}{2}\sigma^2\omega_{mj}^2 t)}. \tag{6.2}$$

The process H_t^j defined by (6.1) can be thought of as a kind of 'Brownian bridge' that interpolates between the two energy levels ϵ_1 and E_j. That is, initially the system has energy ϵ_1, and then it progresses along a random trajectory to reach the designated terminal level E_j. A similar remark applies to the density process $\rho_t^j(x)$ of (6.2), which interpolates between the two functions $\phi_1^2(x)$ and $\chi_j^2(x)$. It should be evident that, unlike H_t, the process H_t^j does not conserve energy.

To obtain a better feeling for the dynamics of H_t^j, it will be useful to examine the associated stochastic differential equation. In particular, if we take the stochastic differential of (6.1), then after a rearrangement of terms we obtain

$$\mathrm{d}H_t^j = \sigma^2 V_t^j (E_j - H_t^j)\,\mathrm{d}t + \sigma V_t^j\,\mathrm{d}B_t, \tag{6.3}$$

where the non-negative process V_t^j is given by

$$V_t^j = \frac{\sum_m \pi_m (E_m - H_t^j)^2 \exp(\sigma\omega_{mj}B_t - \frac{1}{2}\sigma^2\omega_{mj}^2 t)}{\sum_k \pi_m \exp(\sigma\omega_{mj}B_t - \frac{1}{2}\sigma^2\omega_{mj}^2 t)}. \tag{6.4}$$

We learn from (6.3) that H_t^j is a *mean-reverting* process with mean level E_j and reversion rate $\sigma^2 V_t^j$. In fact, one can integrate (6.3) by means of the standard technique used in the case of the classical Ornstein–Uhlenbeck process (see, for example, Doob 1942) to obtain

$$H_t^j = E_j + (H_0^j - E_j)\exp\left[-\sigma^2\int_0^t V_s^j\,\mathrm{d}s\right] + \sigma\int_0^t \exp\left[-\sigma^2\int_u^t V_s^j\,\mathrm{d}s\right]V_u^j\,\mathrm{d}B_u, \tag{6.5}$$

which shows clearly how the information of the initial condition is damped away at the rate $\sigma^2\bar{V}_t^j$, where

$$\bar{V}_t^j = \frac{1}{t}\int_0^t V_s^j\,\mathrm{d}s. \tag{6.6}$$

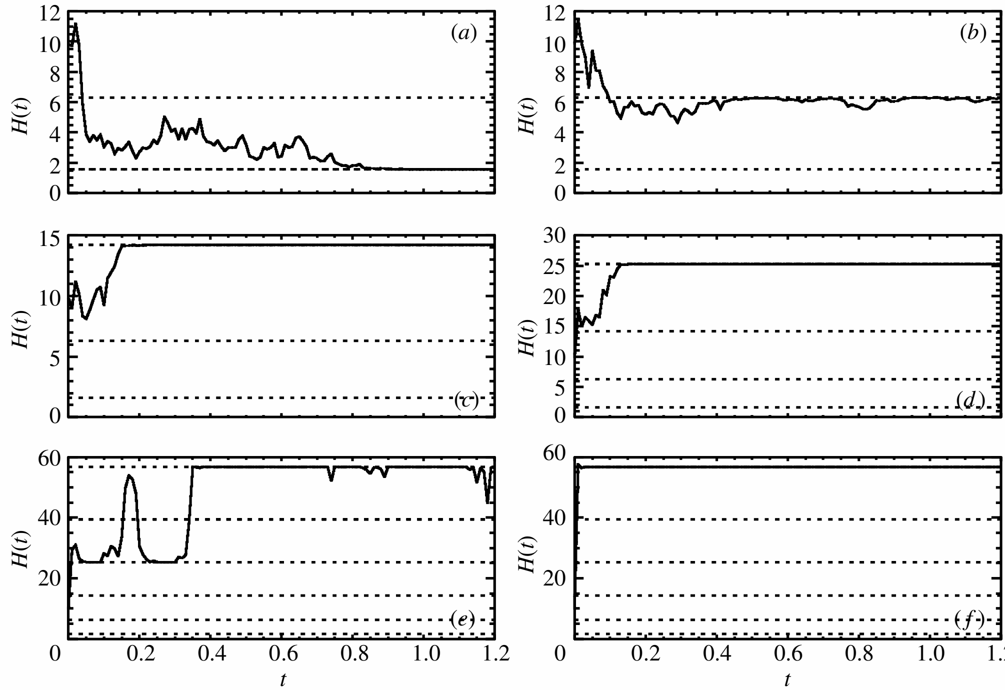

Figure 1. Several realizations of the process H_t for the expectation value of the Hamiltonian are illustrated. The initial condition is chosen to be $\psi_0(x) = \phi_1(x)$ and the final conditions are given by $\psi_\infty(x) = \chi_1(x)$, $\chi_2(x)$, $\chi_3(x)$, $\chi_4(x)$, $\chi_5(x)$, $\chi_6(x)$, respectively. The horizontal lines indicate the energy levels E_j, where we have set $\alpha = 2.5$. For convenience, the vertical axis is expressed in units of the characteristic energy $\varepsilon = \hbar^2/2\mu L^2$, and the horizontal axis is expressed in units of the corresponding characteristic time-interval $1/\sigma^2\varepsilon^2$. When $j = 5$, we have $\pi_5 = 0$. As a result, if $E_j = E_5$ is chosen for the value of the random variable, the energy does not reduce to that eigenvalue. (*a*)–(*f*) $n = 1$, $\alpha = 2.5$ with (*a*) $j = 1$, (*b*) $j = 2$, (*c*) $j = 3$, (*d*) $j = 4$, (*e*) $j = 5$ and (*f*) $j = 6$.

Alternatively, we can write (6.5) in the form

$$H_t^j - E_j = \left(H_0^j - E_j + \sigma \int_0^t \exp\left[\sigma^2 \int_0^u V_s^j \,\mathrm{d}s \right] V_u^j \,\mathrm{d}B_u \right) \exp\left[-\sigma^2 \int_0^t V_s^j \,\mathrm{d}s \right], \tag{6.7}$$

which expresses the difference $H_t^j - E_j$ between H_t^j and its terminal value as the product of a martingale and a positive decreasing process.

With expressions (6.1) and (6.2) at our disposal, we proceed to simulate some realizations of these processes. For convenience, we take advantage of the fact that there is a natural energy unit ε determined by the problem, which is given by

$$\varepsilon = \frac{\hbar^2}{2\mu L^2}. \tag{6.8}$$

Thus, when we plot figures, we can express energies in units of ε, and we can express

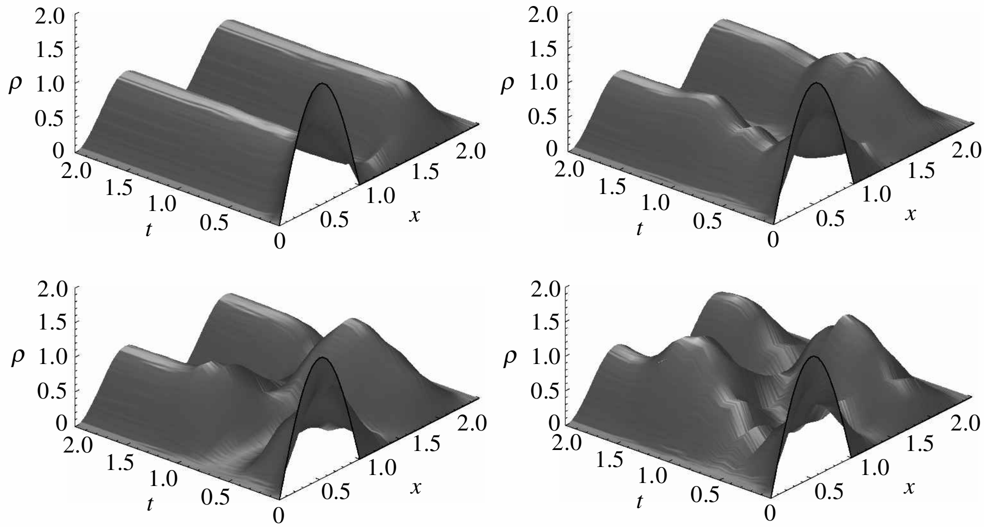

Figure 2. Several realizations of the probability distributions $\rho_t(x)$ for the particle in the well are illustrated. The initial and final conditions are $\psi_0(x) = \phi_1(x)$ and $\psi_\infty(x) = \chi_2(x)$, respectively, and we set $\alpha = 2.5$. The x-axis is measured in units of L, and the time axis is measured in units of $4\mu^2 L^4/\hbar^4\sigma^2$. Most realizations have the property that the initial state swiftly relaxes into the terminal state.

times in units of

$$\frac{1}{\sigma^2\varepsilon^2} = \frac{4\mu^2 L^4}{\hbar^4\sigma^2}. \tag{6.9}$$

In the analysis that follows, energy and time will be expressed in these units.

In figure 1, the energy process H_t^j is shown for several values of j ranging from 1 to 6, where initially the system is in the ground state of the old potential, with energy $\epsilon_1 = \pi^2$. In these examples, the width of the potential is expanded at $t = 0$ by a factor of $\alpha = 2.5$. The most likely transition to occur, given the initial state $\phi_1(x)$, when the width is expanded by a factor of 2.5, is the first excited state $\chi_2(x)$. In this example, it follows as a consequence of (2.6) that $\pi_{15} = 0$, and hence the transition into the fifth energy level does not occur. Instead, there is a transition to the fourth level.

Several realizations of the corresponding probability distribution for the particle are shown in figure 2, where the terminal wave function is the first excited state $\chi_2(x)$. The probability of this event to be realized is as large as approximately 0.43. In most examples, the initial state swiftly changes into the terminal state, although some interesting behaviour can occasionally be observed when the value of the Brownian motion B_t grows large.

In figure 3, the numerical average of the energy process, corresponding to the expectation of H_t^j, is shown for a range of terminal states in the case where the expansion factor is $\alpha = 2.5$. Each graph represents the average of one thousand simulations for each energy level. The behaviour observed in the fifth energy level reflects the fact that, in this case, the terminal state is given by the fourth eigenstate $\chi_4(x)$. This is because E_4 is the closest energy eigenvalue to E_5 in the present example. To see this, we note that if we write $\xi = \exp(\sigma\omega_{45}B_t - \frac{1}{2}\sigma^2\omega_{45}^2 t)$, then, as a consequence

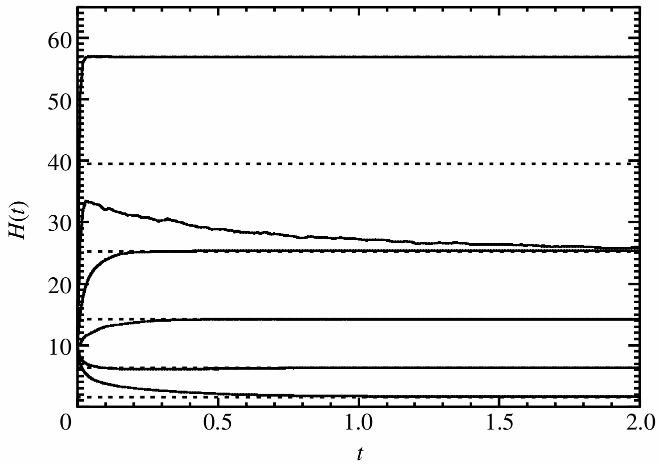

Figure 3. The expectation of H_t^j is shown for a variety of realizations $j = 1, 2, 3, 4, 5, 6$ ($n = 1$, $\alpha = 2.5$), when the initial state is $\phi_1(x)$. The curious behaviour for $j = 5$ arises in this example because $\pi_5 = 0$, and the state never collapses to $\chi_5(x)$. Instead, the state reduced to the eigenstate having the energy closest to E_5, which, in the present case, is the fourth eigenstate.

of (6.1), it follows that H_t^5 can be written in the form

$$H_t^5 = \frac{\xi(\pi_4 E_4 + \sum_m' \pi_m E_m \exp(\sigma(\omega_{m5} - \omega_{45})B_t - \frac{1}{2}\sigma^2(\omega_{m5}^2 - \omega_{45}^2)t))}{\xi(\pi_4 + \sum_m' \pi_m \exp(\sigma(\omega_{m5} - \omega_{45})B_t - \frac{1}{2}\sigma^2(\omega_{m5}^2 - \omega_{45}^2)t))}, \tag{6.10}$$

where we use the notation

$$\sum_m{}' = \sum_{m \neq 4}.$$

Thus the ξ dependence cancels, and because $\omega_{m5}^2 - \omega_{45}^2 > 0$ for all $m \neq 4$, we see that, as $t \to \infty$, the exponents in the denominator and the numerator go to zero and we are left with the leading term E_4. In general, if the terminal value for the energy is chosen to be E_j when the initial state $\psi_0(x)$ is orthogonal to $\chi_j(x)$, i.e. $\pi_j = 0$, then the system necessarily relaxes into the eigenstate whose eigenvalue is the closest to E_j (S. L. Adler 2001, personal communication).

In figure 4, we sketch the dynamics of the ensemble average for the probability density function for the location of the particle, in the case where the terminal state is the first excited state $\chi_2(x)$. This result corresponds to the average of a thousand examples of the type presented in figure 2.

7. Relaxation time-scale

One of the advantages of the simulation methodology is that it opens up the possibility of a direct analysis of the time-scale τ_R over which relaxation typically occurs. Let us define the process M_{mj} by

$$M_{mj} = \exp(\sigma\omega_{mj}B_t - \tfrac{1}{2}\sigma^2\omega_{mj}^2 t), \tag{7.1}$$

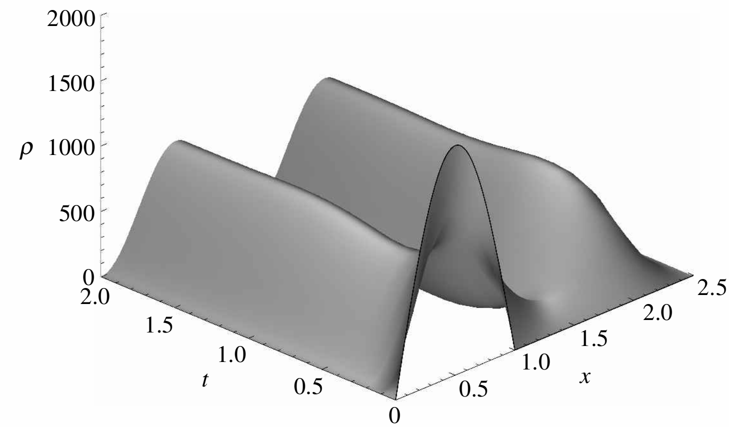

Figure 4. The average of $\rho_t(x)$ for a thousand runs. The initial and final conditions are given by $\psi_0(x) = \phi_1(x)$ and $\psi_\infty(x) = \chi_2(x)$, respectively, and we set $\alpha = 2.5$. The x-axis is measured in units of L, the time axis is measured in units of $4\mu^2L^4/\hbar^4\sigma^2$ and the magnitude of $\rho_t(x)$ is scaled by a factor of 1000. This plot illustrates how the density matrix is diagonalized.

where, for simplicity of notation, we suppress the time dependence of $M_{mj}(t)$. The energy process (6.1) conditional on the outcome E_j can then be written in the form

$$H_t^j = \frac{\pi_j E_j + \sum_{m\neq j}^{\infty} \pi_m E_m M_{mj}}{\pi_j + \sum_{m\neq j}^{\infty} \pi_m M_{mj}}, \tag{7.2}$$

since $M_{jj} = 1$. Clearly, if M_{mj} is sufficiently small, then H_t^j will approach its terminal value E_j and the system will have relaxed. This occurs when $t > \tau_{\mathrm{R}}$.

In order to study the scale of τ_{R}, we consider the probability that the process M_{mj} is smaller than $\mathrm{e}^{-\lambda}$ for some number λ. If $\omega_{mj} > 0$, this is given by

$$\begin{aligned}\mathbb{P}(M_{mj} < \mathrm{e}^{-\lambda}) &= \mathbb{P}(\sigma\omega_{mj}B_t - \tfrac{1}{2}\sigma^2\omega_{mj}^2 t < -\lambda) \\ &= \mathbb{P}\left(B_t < \tfrac{1}{2}\sigma\omega_{mj}t - \frac{\lambda}{\sigma\omega_{mj}}\right) \\ &= N\left(\tfrac{1}{2}\sigma|\omega_{mj}|\sqrt{t} - \frac{\lambda}{\sigma|\omega_{mj}|\sqrt{t}}\right).\end{aligned} \tag{7.3}$$

Here, $N(x)$ is the standard normal distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,\mathrm{d}y. \tag{7.4}$$

Note that in obtaining the second expression on the right-hand side of (7.3) we have used the fact that $\omega_{mj} > 0$, whereas if $\omega_{mj} < 0$ we would instead have the probability

$$\mathbb{P}(M_{mj} < \mathrm{e}^{-\lambda}) = \mathbb{P}\left(B_t > \tfrac{1}{2}\sigma\omega_{mj}t - \frac{\lambda}{\sigma\omega_{mj}}\right). \tag{7.5}$$

However, owing to the symmetry relation $N(-x) = 1 - N(x)$ satisfied by the normal distribution function, the final result (7.3) is unaltered.

Now, for relaxation, we would like the probability (7.3) to be sufficiently great, say

$$\mathbb{P}(M_{mj} < \mathrm{e}^{-\lambda}) \geqslant 0.95, \tag{7.6}$$

for a suitably large λ. The choice of λ corresponds to how far the reduction has to proceed for relaxation to have effectively set in. Because $N(x) \geqslant 0.95$ when $x \gtrsim 1.65$, we can put a bound on the time variable such that the probability (7.3) is greater than 0.95. This is given by

$$\tfrac{1}{2}\sigma|\omega_{mj}|\sqrt{t} - \frac{\lambda}{\sigma|\omega_{mj}|\sqrt{t}} > 1.65 \tag{7.7}$$

or, equivalently,

$$\sqrt{t} > \frac{3.3 + \sqrt{3.3^2 + 8\lambda}}{2\sigma|\omega_{mj}|}. \tag{7.8}$$

In particular, for a fixed value of j, we would like this relation to hold for all values of $m \neq j$. That is, we would like M_{mj} to become negligible for all $m \neq j$, which will ensure that $H_t^j \to E_j$. This can be guaranteed by choosing $m = j + 1$, since this choice maximizes the right-hand side of (7.8). For example, if we take $\lambda = 10$, then the relaxation time-scale obtained is of the order

$$\tau_{\mathrm{R}} \sim \frac{40\alpha^4}{\pi^4\sigma^2(2j+1)^2} \tag{7.9}$$

when the terminal state is $\chi_j(x)$. Thus, provided $t \geqslant \tau_{\mathrm{R}}$, we can be 95% confident that $M_{mj} < 10^{-5}$ for every $m \neq j$.

We note, incidentally, that the result in (7.9) is based on the assumption that the expansion factor α is strictly and sufficiently greater than one. In the exceptional case where $\alpha = 1$, we have $\tau_{\mathrm{R}} = 0$, which follows directly from (7.2), if one notes that $\pi_m = 0$ for every $m \neq j$, and thus $H_t^j = E_j$ irrespective of the values of M_{mj} and t.

When $\alpha \sim 1$, the foregoing analysis on the relaxation time-scale needs to be carried out more carefully. For this purpose, let us consider the case in which we have $\alpha = 1 + \epsilon$, where the value of $\epsilon \geqslant 0$ can be very small. Now, if the system is initially in the ground state $\phi_1(x)$, then, for $\epsilon \ll 1$, the transition probability $\pi_m(\epsilon)$ admits the expansion

$$\pi_m(\epsilon) \sim \frac{4\sin^2(\pi m)}{\pi^2(m^2-1)^2} + \left(\frac{16\sin^2(\pi m)}{\pi^2(m^2-1)^3} + \frac{4(3\sin^2(\pi m) - 2\pi m\sin(\pi m)\cos(\pi m))}{\pi^2(m^2-1)^2}\right)\epsilon + \cdots \tag{7.10}$$

for all m. In particular, $\pi_1(\epsilon)$ is of order one, whereas the next largest transition probability when $\alpha \sim 1$ is given, up to order ϵ^2, by $\pi_2(\epsilon) = \frac{16}{9}\epsilon^2$. The implication of this is that the condition for $\pi_m M_{m1} < \mathrm{e}^{-\lambda}$ to be valid for all $m \neq 1$ with 95% confidence level is automatically satisfied for all $t \geqslant 0$, and thus this is not a sufficient criterion to determine the relaxation time-scale.

Therefore, it is important in the small perturbation regime to determine how small ϵ can be for the $(p \times 100)\%$ confidence-level analysis to be viable. This can be determined if we replace λ in (7.3) by $\lambda + \ln \pi_m(\epsilon)$ and 0.95 in (7.6) by p. The latter is equivalent to replacing 1.65 in (7.7) by $N^{-1}(p)$, where $N^{-1}(x)$ is the inverse of the normal distribution function (7.4). The result gives us

$$\sqrt{t} > \frac{2N^{-1}(p) + \sqrt{(2N^{-1}(p))^2 + 8(\lambda + \ln \pi_m(\epsilon))}}{2\sigma|\omega_{mj}|}, \tag{7.11}$$

and we maximize the right-hand side of (7.11) over all $m \neq j$, when $j = 1$. The maximum is obtained by setting $m = 2$, which leads us to the time-scale for relaxation into ground state. This is given by

$$\tau_{\mathrm{R}} \sim \frac{1}{\sigma^2 \omega_{21}^2}\Big(N^{-1}(p) + \sqrt{2\lambda + 4\ln(\tfrac{4}{3}\epsilon) + (N^{-1}(p))^2}\Big)^2. \tag{7.12}$$

In particular, for a fixed ϵ, either λ or p must be sufficient large so that the right-hand side of (7.12) is real. Conversely, for fixed λ and p, the confidence-level analysis is viable only when ϵ is large enough to ensure the reality of the right-hand side of (7.12).

8. Towards the quantum adiabatic theorem

The analysis of the relaxation time-scale shows that, for an infinitesimal expansion of potential well, the initial eigenstate $\phi_n(x)$ will almost immediately relax into the corresponding eigenstate $\chi_n(x)$. This result suggests that when the potential well is expanded sufficiently slowly, then the dynamics of relaxation will force the system to *remain* in the nth eigenstate. In other words, the stochastic evolutionary law appears to give rise to an analogue of quantum adiabatic theorem. In what follows we shall explore this idea in greater depth.

For this purpose, we need to consider now the dynamics of a quantum system in the presence of a time-dependent Hamiltonian. Let us denote by $\hat{H}(t)$ a generic time-dependent Hamiltonian, and for each fixed time t we write $\chi_n(t,x)$ $(n = 1, 2, \dots)$ for the nth eigenstate of the operator $\hat{H}(t)$, with eigenvalue $E_n(t)$. For an adiabatic approximation we shall assume that $\hat{H}(t)$, $\chi_n(t,x)$ and $E_n(t)$ are continuous in t, and vary sufficiently slowly that we can write

$$\frac{\partial \chi_n(t,x)}{\partial t} \approx -\mathrm{i}E_n(t)\chi_n(t,x). \tag{8.1}$$

Suppose we consider the eigenfunction expansion

$$\psi_t(x) = \sum_k a_k(t)\chi_k(t,x) \tag{8.2}$$

for a general wave function. Then, for the Schrödinger equation, we obtain

$$\begin{aligned}\frac{\partial \psi_t(x)}{\partial t} &= -\mathrm{i}\hat{H}(t)\psi_t(x)\\ &= -\mathrm{i}\sum_k a_k(t)\hat{H}(t)\chi_n(t,x)\end{aligned}$$

$$= -\mathrm{i}\sum_k a_k(t)E_k(t)\chi_n(t,x)$$

$$= \sum_k a_k(t)\frac{\partial\chi_k(t,x)}{\partial t}, \tag{8.3}$$

by use of (8.1). Differentiating (8.2) with respect to t and comparing the result with (8.3), we deduce that

$$\sum_n \dot{a}_n(t)\chi_n(t,x) = 0 \tag{8.4}$$

in the adiabatic approximation. Multiplying this equation by $\chi_m(t,x)$ and integrating over x, we see as a consequence of the orthogonality of the eigenstates of $\hat{H}(t)$ that $\dot{a}_k(t) \approx 0$ in the adiabatic approximation. Thus, in this regime, the solution to the deterministic Schrödinger equation is

$$\psi_t(x) = \sum_n a_n\chi_n(t,x). \tag{8.5}$$

These relations arise as a consequence of the adiabatic approximation (8.1) in the deterministic theory. What we would like to consider next are the implications of the assumption (8.1) in the case of a stochastic evolutionary law. In the time-dependent case, the appropriate generalization of the stochastic extension of the Schrödinger equation is given by

$$\mathrm{d}\psi_t(x) = -\mathrm{i}\hat{H}(t)\psi_t(x)\,\mathrm{d}t - \tfrac{1}{8}\sigma^2(\hat{H}(t)-H_t)^2\psi_t(x)\,\mathrm{d}t + \tfrac{1}{2}\sigma(\hat{H}(t)-H_t)\psi_t(x)\,\mathrm{d}W_t, \tag{8.6}$$

where the process

$$H_t = \frac{\int \psi_t^*(x)\hat{H}(t)\psi_t(x)\,\mathrm{d}x}{\int \psi_t^*(x)\psi_t(x)\,\mathrm{d}x} \tag{8.7}$$

represents the expectation of the time-dependent operator $\hat{H}(t)$ in the state $\psi_t(x)$. We note first that (8.6) has the important property that, like (3.1), it preserves the norm of $\psi_t(x)$. This can easily be verified with a short calculation making use of the Ito rules. However, in the case of the energy process, we find that the dynamical law of H_t is given by

$$\mathrm{d}H_t = \dot{H}_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t, \tag{8.8}$$

instead of (3.4), where we have defined

$$\dot{H}_t = \int \psi_t^*(x)(\partial_t\hat{H}(t))\psi_t(x)\,\mathrm{d}x. \tag{8.9}$$

It is natural that the process for the expectation value of the Hamiltonian is no longer a martingale in the time-dependent case, but rather exhibits a drift, the sign of which depends on the expectation value of the time derivative of the Hamiltonian. A similar calculation in the case of the process for the variance of the energy shows that

$$\mathrm{d}V_t = -\sigma^2V_t^2\,\mathrm{d}t + \sigma\beta_t\,\mathrm{d}W_t + 2\left(\int \psi_t^*(x)(\hat{H}(t)-H_t)(\partial_t\hat{H}(t)-\dot{H}_t)\psi_t(x)\,\mathrm{d}x\right)\mathrm{d}t. \tag{8.10}$$

Now, for state reduction, a necessary requirement is that the variance process V_t should satisfy the supermartingale condition (3.9). In other words, V_t should be, on average, a decreasing process.

We observe, however, that the last term in (8.10) is given by the covariance of the two operators $\hat{H}(t)$ and $\partial_t\hat{H}(t)$, which can be positive or negative. If it is negative, then the variance process V_t is a supermartingale, whereas if it is positive, V_t can still be a supermartingale, provided the covariance is not too large. In particular, if we write

$$\Delta H_t = \sqrt{V_t} \tag{8.11}$$

and

$$\Delta\dot{H}_t = \sqrt{\mathrm{Var}[\partial_t\hat{H}(t)]}, \tag{8.12}$$

respectively, for the standard deviations of $\hat{H}(t)$ and $\partial_t\hat{H}(t)$, then, for the supermartingale condition, we require

$$2\rho_t\Delta\dot{H}_t\Delta H_t < \sigma^2(\Delta H_t)^4, \tag{8.13}$$

where ρ_t is the correlation between $\hat{H}(t)$ and $\partial_t\hat{H}(t)$. If we take into account the fact that the correlation necessarily lies in the range

$$-1 \leqslant \rho_t \leqslant 1, \tag{8.14}$$

we find, to ensure that V_t is a supermartingale, it suffices that the following relation should hold:

$$\frac{\Delta\dot{H}_t}{\Delta H_t} < \tfrac{1}{2}\sigma^2(\Delta H_t)^2. \tag{8.15}$$

In other words, we require the uncertainty in the change of $\hat{H}$ to be small compared with the uncertainty in $\hat{H}$ itself.

Intuitively, if the change of the Hamiltonian is sufficiently slow, then $\partial_t\hat{H}(t)$ will be small, and therefore we would expect $\Delta\dot{H}_t \ll \Delta H_t$ be valid in the adiabatic regime. In particular, relation (8.15) can be viewed as a necessary consequence of adiabatic motion, thus ensuring that the process for the squared energy uncertainty V_t is a supermartingale.

9. Stochastic evolution in the adiabatic approximation

Now we turn to the consideration of the stochastic equation (8.6) for the wave function in the case of a time-dependent Hamiltonian. To begin with, let us define the process Π_t^k by

$$\Pi_t^k = \Pi_0^k \exp\left(\sigma\int_0^t (E_k(s) - H_s)\,\mathrm{d}W_s - \tfrac{1}{2}\sigma^2\int_0^t (E_k(s) - H_s)^2\,\mathrm{d}s\right). \tag{9.1}$$

Then it is a straightforward exercise in stochastic calculus, using Ito's lemma, to deduce that

$$\mathrm{d}\Pi_t^k = \sigma(E_k(t) - H_t)\Pi_t^k\,\mathrm{d}W_t. \tag{9.2}$$

It follows that, for each value of k, the dynamical equation of the process

$$a_{kt} = (\Pi_t^k)^{1/2} \tag{9.3}$$

is given by

$$\mathrm{d}a_{kt} = -\tfrac{1}{8}\sigma^2(E_k(t) - H_t)^2 a_{kt}\,\mathrm{d}t + \tfrac{1}{2}(E_k(t) - H_t)a_{kt}\,\mathrm{d}W_t. \tag{9.4}$$

Therefore, if, as before, we introduce the expression

$$\psi_t(x) = \sum_k a_{kt}\chi_k(t,x), \tag{9.5}$$

where the functions $\chi_j(t,x)$ are the eigenfunctions of $\hat{H}(t)$, then we obtain

$$\mathrm{d}\psi_t(x) = \sum_k (\mathrm{d}a_{kt})\chi_k(t,x) + \sum_k a_{kt}(\partial_t\chi_k(t,x))\,\mathrm{d}t. \tag{9.6}$$

Therefore, substituting (9.4) into this formula and using (9.5), we obtain

$$\begin{aligned}\mathrm{d}\psi_t(x) = {} & -\mathrm{i}\hat{H}(t)\psi_t(x)\,\mathrm{d}t - \tfrac{1}{8}\sigma^2(\hat{H}(t) - H_t)^2\psi_t(x)\,\mathrm{d}t \\ & + \tfrac{1}{2}\sigma(\hat{H}(t) - H_t)\psi_t(x)\,\mathrm{d}W_t + \left(\sum_k a_{kt}(\partial_t\chi_k(t,x) + \mathrm{i}\hat{H}(t)\chi_k(t,x))\right)\mathrm{d}t.\end{aligned} \tag{9.7}$$

Thus far, our analysis is exact. Now, in the adiabatic regime, we have the relation (8.1), which implies that

$$\partial_t\chi_k(t,x) \approx -\mathrm{i}\hat{H}(t)\chi_k(t,x). \tag{9.8}$$

Therefore, substituting this relation into equation (9.7), we see that, *in the adiabatic approximation, the process $\psi_t(x)$ defined by (9.5) is the solution to the stochastic dynamics (8.6).*

We observe that, in the adiabatic approximation, the process Π_t^k has the interpretation that it represents the probability that $\psi_t(x)$ is in the eigenstate $\chi_k(t,x)$. On the other hand, Π_t^k is also a martingale, satisfying (9.2). Thus the *expected* probability that $\psi_t(x)$ is in the state $\chi_k(t,x)$, given information up to time s, is precisely the probability that $\psi_s(x)$ is in the state $\chi_k(s,x)$. It follows that if $\psi_t(x)$ is in the state $\chi_k(0,x)$ at time 0, it will be in the state $\chi_k(t,x)$ at time t with probability one.

10. Discussion

In the present investigation, we have examined primarily the case of the square-well potential. In conclusion, we point out, however, that many of the results obtained, namely, the wave function (5.1), the conditional Hamiltonian process (6.1), the probability distribution of the particle (6.2) and the bound on relaxation time (7.8), are independent of the specific model being considered. As long as the eigenvalues E_m and the eigenfunctions $\chi_m(x)$ of the Hamiltonian are known, either analytically or numerically, these results can be applied to study the details of the dynamical evolution of the system. In other words, the general approach outlined here, based on the nonlinear filtering methodology introduced in Brody & Hughston (2002*b*), can be applied to study a wide range of problems in perturbation theory, including, of course, a more generic time-dependent Hamiltonian. We hope to take up this line of investigation in greater detail elsewhere.

The point of view we have put forward in this paper is that the dynamical law (3.1) offers a simple but plausible characterization of the subsequent evolution of a quantum system after the Hamiltonian has been perturbed. Whether the reduction to energy eigenstates is a general feature of quantum dynamics is, of course, an open question, and, ultimately, an empirical matter. If a measurement of the system energy is carried out after a passage of time greater than the relaxation time-scale, then, according to the stochastic postulate, when the system is initially in the eigenstate $\phi_n(x)$, the eigenvalue E_m will be observed with probability π_{nm}. This is because the system is already in the eigenstate $\chi_m(x)$ with that probability. On the other hand, according to the unitarity postulate associated with the Schrödinger equation, the system is in a superposition of a myriad of eigenstates. Nevertheless, the outcome of the energy measurement will give E_m with probability π_{nm}. The question thus arising is whether one can distinguish the two theories by means of a suitable experiment. To this end, we note that, for time $t \geqslant \tau_{\mathrm{R}}$, the state of the system, according to the stochastic law, is given by a mixed-state density matrix, whose diagonal elements, in the energy basis, are given by $\pi_m = \pi_{nm}$; whereas, according to unitary law, the system is in a pure state, given by the superposition of the energy eigenstates, whose coefficients are given by $\sqrt{\pi_m}$. Therefore, if these pure and mixed states can be distinguished statistically in some way by means of an experiment, then one might be able to rule out at least one of the two postulates indicated here. In particular, the distinction between pure and mixed states might be achieved by means of an interference experiment. The design of such an experiment remains at present an open problem.

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