Petrov Classification and Rational Quartic Curve

Dorje C. Brody* **and Lane P. Hughston**†

*Blackett Laboratory, Imperial College, London SW7 2BZ, UK

†Department of Mathematics, King's College London, The Strand, London WC2R 2LS, UK

Abstract. The classification of gravitational fields into algebraic types has been one of the most fruitful ideas in general relativity. Less well known is the fact that the Petrov scheme is intimately related to geometric properties of the rational quartic curve in complex projective four-space. This same geometry turns out to be highly useful in the analysis of the quantum state-space for elementary spin-two systems.

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In view of the need to clarify a number of important conceptual and physical issues in quantum theory concerning, e.g., quantum information and entanglement, the study of the geometrical structure of the space of pure states for various quantum systems has been attracting considerable interest. Such investigations have been motivated also in part by the point of view that, by 'geometrising' quantum theory we may be in a better position to develop a proper theory of quantum gravity (cf. Kibble 1979). With this in mind, we present here a simple but nevertheless intriguing example of an explicit close relationship between certain geometrical aspects of classical gravitation and a system of corresponding structures arising in connection with spin measurements on quantum systems.

It is known, in the theory of general relativity, that a nonvanishing Weyl spinor ψ^{ABCD} at a point in spacetime can be of five different types, denoted $\{1,1,1,1\}$, $\{2,1,1\}$, $\{2,2\}$, $\{3,1\}$, and $\{4\}$, according to the degree of degeneracy of its principle spinors (Penrose 1960, Penrose & Rindler 1986). This classification scheme can be given a useful characterisation in terms of the geometry of the *rational quartic curve* $\mathcal{R}$ in $\mathcal{P}^4$ (Telling 1936). According to this characterisation, the 'null' Weyl spinors, i.e., those of type $\{4\}$, constitute a locus $\mathcal{R}$ given by the Veronese embedding $\mathcal{R}:\mathcal{P}^1\to\mathcal{P}^4$, where the point with homogeneous coordinates $\xi^A\in\mathcal{P}^1$ is mapped to the point with homogeneous coordinates

$$\psi^{ABCD}=\xi^A\xi^B\xi^C\xi^D$$

in $\mathcal{P}^4$. The resulting curve $\mathcal{R}$ meets a generic prime (cubic-hyperplane) at four points.

The Petrov scheme can be directly interpreted in terms of properties of $\mathcal{R}$. Let us denote by $\mathcal{W}$ the space of projective Weyl spinors, i.e., $\mathcal{P}^4$ with the curve $\mathcal{R}$ singled out. Three special linear spaces are associated with any point of $\mathcal{R}$: the *tangent line*, the *osculating plane*, and the *osculating prime*.

The tangent line at $\alpha^A\alpha^B\alpha^C\alpha^D \in \mathcal{R}$ consists of spinors of the form $\alpha^{(A}\alpha^B\alpha^C\beta^{D)}$. The osculating plane consists of spinors of the form $\alpha^{(A}\alpha^B\beta^C\gamma^{D)}$. The osculating prime consists of spinors of the form $\alpha^{(A}\beta^B\gamma^C\delta^{D)}$. Four osculating primes of $\mathcal{R}$ pass through a generic point $\psi^{ABCD} \in \mathcal{W}$, corresponding to the principal spinors of ψ^{ABCD}, which meet $\mathcal{R}$ tangentially at four points. The unique 3-hyperplane through these four points is the *polar prime* ψ_{ABCD}. The fundamental polarity $\psi^{ABCD} \to \psi_{ABCD}$ arising here is *symmetric*, in the sense that

$$\phi_{ABCD}\psi^{ABCD} = \psi_{ABCD}\phi^{ABCD},$$

so ψ^{ABCD} lies on the polar prime of ϕ^{ABCD} iff ϕ^{ABCD} lies on the polar prime of ψ^{ABCD}. We say that two points are conjugate with respect to $\mathcal{R}$ if each lies on the polar prime of the other. It follows that the quadratic form

$$\mathcal{I} = \psi_{ABCD}\psi^{ABCD},$$

which plays an important role in general relativity, vanishes iff the Weyl spinor is self-conjugate with respect to $\mathcal{R}$. A necessary and sufficient condition for this is that the cross ratio of the principal spinors satisfy the *anharmonic* condition.

To proceed further, we note that a point of $\mathcal{W}$ lies on $\mathcal{R}$ if and only if $Q^{ABCD} = 0$, where

$$Q^{ABCD} = \psi^{(AB}{}_{EF}\psi^{CD)EF}.$$

This relation states that $\mathcal{R}$ is the common intersection of a four-dimensional net of quadrics in $\mathcal{W}$. It follows that for points of type $\{3, 1\}$, or less degenerate, there is a quadratic map Q from $\mathcal{W} - \mathcal{R}$ to points in $\mathcal{W}$. A necessary and sufficient condition for a point $\psi^{ABCD} \in \mathcal{W}$ to be conjugate to its image Q^{ABCD} under Q is the vanishing of the invariant

$$\mathcal{J} = \psi_{AB}{}^{CD}\psi_{CD}{}^{EF}\psi_{EF}{}^{AB}.$$

It is a classical result that $\mathcal{J}$ vanishes iff the cross ratio of the principal spinors satisfies the *harmonic* condition, which implies that ψ^{ABCD} lies on a *chord* of $\mathcal{R}$, and is thus of the form

$$\psi^{ABCD} = \alpha^A\alpha^B\alpha^C\alpha^D + \beta^A\beta^B\beta^C\beta^D.$$

The hypersurface $\mathcal{J} = 0$ is called the *chordal primal* of $\mathcal{W}$.

Now we are able to give a more detailed description of the subspaces of $\mathcal{W}$ corresponding to the different algebraic types for the Weyl spinor. Let us use the notation [x] to represent the *closure* of $\{\mathrm{x}\} \in \mathcal{W}$, denoting a degeneracy of at least type $\{\mathrm{x}\}$. Thus, $[3, 1]$ means $\{3, 1\}$ or $\{4\}$, whereas $[4] = \{4\}$. Spinors of type $\{4\}$ correspond to points of $\mathcal{R}$ itself. The spinors of type $[3,1]$ constitute a sextic 2-surface $\mathfrak{M} \in \mathcal{W}$, characterised by the vanishing of $\mathcal{I}$ and $\mathcal{J}$. We note that $\mathfrak{M}$ is the ruled by the tangent lines of $\mathcal{R}$.

The spinors of type $[2,1,1]$ constitute a sextic primal $\mathfrak{D} \in \mathcal{W}$ given by $\mathcal{I}^3 = 6\mathcal{J}^2$. This relation holds iff ψ^{ABCD} lies on an osculating plane of $\mathcal{R}$. We note that $\mathcal{P}^4 - \mathfrak{M}$ is foliated by a one-dimensional system of sextic primals with the common intersection $\mathfrak{M}$, given by $\alpha\mathcal{I}^3 = \beta\mathcal{J}^2$, where α, β (not both vanishing) are homogeneous coordinates for $\mathcal{P}^1$. The spinors of type $[2,2]$ lie on a two-dimensional quartic subsurface $\mathfrak{K}$ of $\mathfrak{D}$, given by

$$\psi_{(ABC}{}^{K}\psi_{DE}{}^{LM}\psi_{F)KLM} = 0,$$

which can be interpreted as the common intersection of a six-dimensional net of cubic primals. The surface $\mathfrak{K}$ is generated by intersections of pairs of osculating 2-planes. The spinors of type $\{2,2\}$ are distinguished as being the points of $\mathcal{W} - \mathcal{R}$ that map into themselves under Q.

The discussion so far has referred exclusively to 'classical' gravitation, in the sense being applicable to the geometry of the Weyl spinor at a point in spacetime. It is interesting therefore that *all* of the structure indicated above carries over explicitly to the geometry of the state-space for an elementary spin-two quantum system, and is essential to the proper analysis of spin measurements on such a system.

In this case the Hilbert space is five dimensional, and the quantum state space is again given by $\mathcal{W}$, but endowed with additional structure. In particular, the complex conjugation operation in quantum mechanics gives rise to an antilinear map from points to primes and vice versa on $\mathcal{W}$. Further, the rational quartic is required to be self-conjugate under this operation in the sense that the complex conjugate of the osculating prime of any point on $\mathcal{R}$ is another point on $\mathcal{R}$.

Now suppose we consider a measurement of the spin of the system. The choice of a point on $\mathcal{R}$ determines a spin-axis, and the complex conjugate of this point corresponds to the same axis but with the orientation reversed. It then follows that the chosen state on $\mathcal{R}$ has $S_z = 2$ with respect to the corresponding z-axis, and the complex conjugate states has $S_z = -2$. The $S_z = 1$ state is obtained by intersecting the tangent line of the $S_z = 2$ state with the osculating prime of the corresponding $S_z = -2$ state. The $S_z = -1$ state is obtained by intersecting the osculating prime of the $S_z = 2$ state with the tangent line of the $S_z = -2$ state. Finally, the $S_z = 0$ state is the point obtained by intersecting the osculating planes of the $S_z = 2$ and $S_z = -2$ states, as illustrated in Figure 1.

The analysis above indicates how by the systematic use of some elementary algebraic projective geometry—in this case, the theory of the rational quartic curve—we can strengthen our understanding of both classical gravitation and the quantum theory of spin-two systems. We take this to be an encouraging basis for the pursuit of further research in this line.

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[1] Kibble, T. W. B. 1979 *Commun. Math. Phys.* **65**, 189.
[2] Penrose, R. 1960 *Ann. Phys.* **10**, 171.
[3] Penrose, R. and Rindler, W. 1986 *Spinors and Space-time*, Vol. 2, Cambridge: Cambridge University Press.
[4] Telling, H. G. 1936 *The Rational Quartic Curve in Space of Three and Four Dimensions*, London: Cambridge University Press.

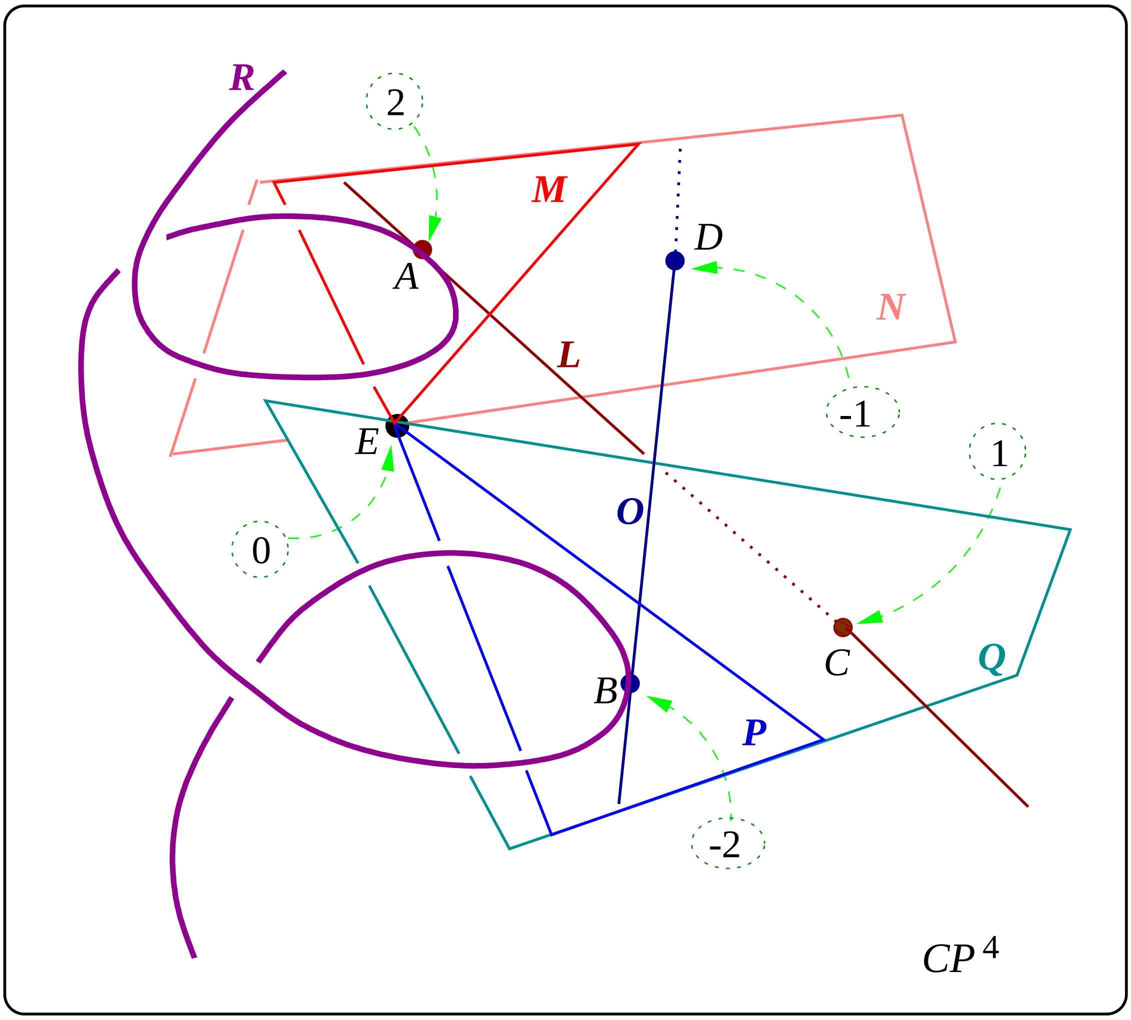

Figure 1. *Geometry of the rational quartic curve.* At a generic point A on the curve $\mathcal{R}$ we draw the tangent line L, which lies in the osculating 2-plane M at A, which lies in the osculating 3-prime N at A. If B is another point on $\mathcal{R}$, then the tangent line O to R at B lies in the osculating 2-plane P at B, which lies in the osculating 3-prime Q at B. The tangent to $\mathcal{R}$ at A meets the osculating prime Q at C. The tangent to $\mathcal{R}$ at B meets the osculating prime N at D. The osculating plane M at A intersects the osculating plane P at B at the point E. For a quantum spin-2 system, the curve $\mathcal{R}$ is self-conjugate, i.e., the complex conjugate prime of a point on $\mathcal{R}$ is the osculating prime of another point on $\mathcal{R}$. The choice of a point on $\mathcal{R}$ determines a spin axis. If A and B are conjugate, then the eigenstates of the spin operator S_z with eigenvalues $2, 1, 0, -1, -2$ are the points A, C, E, D, B respectively.