International Journal of Theoretical and Applied Finance
Vol. 3, No. 2 (2000) 161–181

MARKOV MARKET MODEL CONSISTENT WITH CAP SMILE

P. BALLAND
Merrill Lynch International, 25 Ropemaker Street, London EC2Y 9LY, UK
E-mail: *Philippe_Balland@ml.com*

L. P. HUGHSTON
Department of Mathematics, King's College London,
The Strand, London WC2R 2LS, UK
E-mail: *Lane_Hughston@yahoo.com*

Received 18 December 1998

New interest rate models have emerged recently in which distributional assumptions are made directly on financial observables. In these "Market Models" the Libor rates have a log-normal distribution in the corresponding forward measure, and caps are priced according to the Black–Scholes formula. These models present two disadvantages. First, Libor rates do not in reality have a log-normal distribution since the implied volatility of a cap depends typically on the strike. Second, these models are difficult to use for pricing derivatives other than caps. In this paper, we extend these models to allow for a broader class of Libor rate distributions. In particular, we construct multi-factor Market Models that are consistent with an initial cap smile surface, and have the useful feature of exhibiting Markovian Libor rates. We show that these Markov Market Models can be used relatively easily to price complex Libor derivatives, such as Bermudan swaptions, captions or flexi-caps, by construction of a tree of Libor rates.

Keywords: Interest rate term structure, HJM theory, market models, BGM models, implied libor trees.

1. Introduction

Brace *et al.* [2], Jamshidian [13], and Musiela and Rutkowski [15], following earlier work by Sandmann and Sondermann [17], Goldys *et al.* [8], Musiela [14], Sandmann *et al.* [18], and others, have shown that there exists at least one HJM term structure model, called the Market Model, in which a given set of forward Libor rates with non-overlapping accrual intervals can be assumed log-normal under their respective forward measures. This paper offers an extension of this result. Given a set of N predictable functionals $\boldsymbol{\sigma}_i$ satisfying some mild conditions, we show in Sec. 2 that there exists at least one HJM term structure model in which a given set of N forward Libor rates with non-overlapping accrual periods have the functionals $\boldsymbol{\sigma}_i$ as their respective local volatility vectors. We derive this result by use of martingale techniques. In this generalisation of the BGM model, the local volatility vectors

$\boldsymbol{\sigma}_i$ of the forward Libor rates are allowed to be path-dependent. We note that this result can also be derived via the method of Brace, Gatarek and Musiela based on stochastic differential equations (cf. [2]). In Sec. 3, we construct multi-factor Markov Market Models for which the forward Libor rates are deterministic functions of a Markov process. These arbitrage-free models can be calibrated against an initial volatility smile surface for caps. We construct the Markov Market Models using the method of "undetermined numeraire" introduced in [9] and by use of discrete martingale techniques. A construction based on stochastic differential equations as in [2] is in principle possible, but in practice difficult, since the marginal distributions of the Libor rates are exogenous to the model. The Markov Market Models are constructed by specifying the marginal distributions of the Libor rates, as opposed to specifying the volatility of the forward Libor rates as in [2].

In Sec. 4, we show how to implement one-factor Markov Market Models in a single tree, thus allowing the pricing of Bermudan swaptions and other Libor derivatives under different smile assumptions for the Libor rates. In Secs. 5 and 6, we review two path-dependent one-factor Market Models with log-normal and square-root volatility respectively. We derive for both models analytical formulae for caplet prices.

In what follows we shall use bold characters to denote n-dimensional vectors and plain characters to denote scalars. The scalar product between the two vectors $\mathbf{A}$ and $\mathbf{B}$ is denoted $\mathbf{AB}$. We work in a standard setting whereby future uncertainty is modelled by a probability space $(\Omega, \Im_\infty, P)$ with a filtration $\Im = \{\Im_t : t \geq 0\}$ where $t = 0$ represents the present. We shall assume the filtration to be generated by a standard n-dimensional Brownian vector $\mathbf{W}_t$ with independent components. We write E_t^Q for conditional expectation under a probability measure Q. All probability measures considered here are equivalent on null sets, in the usual sense, to the measure P, and thus admit a Radon derivative with respect to P.

We denote by P_{tb} the value at time t of a discount bond maturing at time $b \geq t$. We assume that for each given value of the maturity date the process P_{tb} is a continuous $(\Im, P)$-semimartingale. The one-parameter family of discount bond price processes $\{P_{tb} : t \leq b\}$ is called an interest rate system. Following Geman, El-Karoui and Rochet (cf. [11, pp. 291–304]), we define interest rate systems that are complete and arbitrage-free as follows.

Definition 1.1. An interest rate system $\{P_{tb} : t \leq b\}$ is complete and arbitrage-free over the interval $[0, T]$ if and only if there exists a unique probability measure $Q(P_{\cdot T})$ such that the ratio of any discount bond price P_{tb} to P_{tT} is a $Q(P_{\cdot T})$-martingale on $[0, b \wedge T]$.

With the addition of some further technical conditions to ensure good behaviour for the instantaneous forward rates, Heath, Jarrow and Morton [10] gave a general characterisation of the interest rate systems that are arbitrage free and complete. For the purpose of this paper we adopt a slightly more general point of view, and simply define an HJM model on $[0, T]$ to be an interest rate system that is arbitrage-free and complete on $[0, T]$.

The probability measure $Q(P_{\cdot T})$ is the T-forward probability measure or terminal measure introduced by Jamshidian in the context of Gaussian interest rate models (cf. [11, pp. 161–190]). We show in Lemma 1.1 that the existence of the T-forward probability measure implies the existence, for each maturity b, of a b-forward probability measure $Q(P_{\cdot b})$ associated with the numeraire P_{tb}.

Lemma 1.1. *Let* $\{P_{tb} : t \leq b\}$ *be an HJM model over* $[0, T]$. *For any real* b, *we define the probability measure* $Q(P_{\cdot b})$ *according to the scheme*

$$E_t^{Q(P_{\cdot T})}\left[\frac{dQ(P_{\cdot b})}{dQ(P_{\cdot T})}\right] = \frac{P_{0T}P_{tb}}{P_{0b}P_{tT}}. \tag{1.1}$$

Then the ratio of any discount bond price P_{ta} *to* P_{tb} *is a* $Q(P_{\cdot b})$*-martingale over the interval* $[0, a \wedge b]$.

Proof. Equation (1.1) defines a probability measure $Q(P_{\cdot b})$ that is null-equivalent to $Q(P_{\cdot T})$. Consider s and t in $[0, a \wedge b]$ such that $s \leq t$. Then we observe that

$$\begin{aligned} E_s^{Q(P_{\cdot b})}\left[\frac{P_{ta}}{P_{tb}}\right] &= E_s^{Q(P_{\cdot T})}\left[\frac{dQ(P_{\cdot b})}{dQ(P_{\cdot T})}\frac{P_{ta}}{P_{tb}}\right]\frac{P_{0b}P_{sT}}{P_{0T}P_{sb}} \\ &= E_s^{Q(P_{\cdot T})}\left[\frac{P_{tb}}{P_{tT}}\frac{P_{ta}}{P_{tb}}\right]\frac{P_{sT}}{P_{sb}} = \frac{P_{sa}}{P_{sb}}. \end{aligned} \tag{1.2}$$

Hence the ratio of the discount bond price P_{ta} to P_{tb} is a $Q(P_{\cdot b})$-martingale over the interval indicated. This shows that $Q(P_{\cdot b})$ is the b-forward measure. □

Consider a partition of the interval $[0, T]$ into $N + 1$ non-overlapping intervals $[t_i, t_{i+1}]$ such that $t_0 = 0$ and $t_{N+1} = T$. For simplicity, we abbreviate P_{tt_i} to P_{ti} in appropriate contexts. We denote by L_{ti} the forward Libor rate for the accrual period $[t_i, t_{i+1}]$. This forward Libor rate is the par rate associated with a forward rate agreement (FRA) that sets at time t_i and pays at time t_{i+1} the interest earned over the designated accrual period $[t_i, t_{i+1}]$. This par rate is

$$L_{ti} = \frac{1}{\delta_i}\left(\frac{P_{ti} - P_{ti+1}}{P_{ti+1}}\right), \tag{1.3}$$

where δ_i is the accrual length associated with $[t_i, t_{i+1}]$ and the appropriate day count convention.

We denote by $\wp$ the space of $\Im$-predictable processes defined on $[0, T]$, and by $\wp^n$ the space of $\Im$-predictable n-dimensional vector processes. Let $\wp'$ be a subset of $\wp$. A functional $\boldsymbol{\sigma} : [0, T] \times \wp' \to \wp^n$ is called a predictable functional $\boldsymbol{\sigma}$. Such a functional maps a predictable process into a predictable vector process (cf. [16, p. 349]). Suppose that $\mathbf{A}$ is a predictable vector process in $\wp^n$. Then a simple example of a predictable functional is the functional that maps a predictable process

X to the vector process $\mathbf{Y}$ defined by $\mathbf{Y}_t = X_t \mathbf{A}_t$ for t in $[0.T]$. This functional is defined for all X such that $\mathbf{Y}$ is a predictable process.

Definition 1.2. A predictable functional $\boldsymbol{\sigma}$ is *volatility-realizable* iff for any $x \geq 0$ and for any probability measure Q equivalent to P, the following stochastic differential equation has a unique strong non-negative solution $X(t)$ defined on $[0, T]$ satisfying the initial condition $X(0) = x$:

$$dX(t) = \boldsymbol{\sigma}(t, X) d\mathbf{W}_t^Q \,, \tag{1.4}$$

and

$$E^Q \left[\int_0^T \boldsymbol{\sigma}(s, X)^2 ds \right] < \infty \,, \tag{1.5}$$

where $\mathbf{W}_t^Q$ is a n-dimensional Q-Brownian vector.

Now consider N predictable functionals $\boldsymbol{\sigma}_i$ with $1 \leq i \leq N$. We say that this set of functionals is volatility-realizable if and only if each functional $\boldsymbol{\sigma}_i$ is volatility-realizable.

Lemma 1.2. *Let $m_i(0)$ with $1 \leq i \leq N$ be N positive real numbers, and let $\boldsymbol{\sigma}_i$ with $1 \leq i \leq N$ be N predictable functionals satisfying the volatility realizability conditions. Then there exists a unique set of $N+1$ probability measures Q_i for $0 \leq i \leq N$, with $Q_N = P$, defined by the following system of equations:*

$$\frac{dQ_{i-1}}{dQ_i} = \frac{1 + \delta_i m_i(t_i)}{1 + \delta_i m_i(0)} \qquad (1 \leq i \leq N) \,, \tag{1.6}$$

where

$$m_i(t) = m_i(0) + \int_0^t \boldsymbol{\sigma}_i(s, m_i) d\mathbf{W}_s^{Q_i} \,. \tag{1.7}$$

Proof. Using the conditions satisfied by the volatility-realizable functional $\boldsymbol{\sigma}_N$, we define a P-martingale m_N on $[0, T]$ such that

$$dm_N = \boldsymbol{\sigma}_N(t, m_N) d\mathbf{W}_t \,, \tag{1.8}$$

where $\mathbf{W}_t$ is a P-Brownian vector. This martingale takes only positive values. Now suppose we write $P = Q_N$. Then we can define a probability measure Q_{N-1} such that

$$\frac{dQ_{N-1}}{dQ_N} = \frac{1 + \delta_N m_N(t_N)}{1 + \delta_N m_N(0)} \,. \tag{1.9}$$

From the volatility realizability condition, we deduce that there exists a Q_{N-1}-martingale m_{N-1} satisfying

$$dm_{N-1}(t) = \boldsymbol{\sigma}_{N-1}(t, m_{N-1}) d\mathbf{W}_t^{Q_{N-1}} \,, \tag{1.10}$$

where $\mathbf{W}_t^{Q_{N-1}}$ is a Q_{N-1}-Brownian vector. Since the above Q_{N-1}-martingale is positive, we can define a probability measure Q_{N-2} equivalent to P as follows:

$$\frac{dQ_{N-2}}{dQ_{N-1}} = \frac{1 + \delta_{N-1} m_{N-1}(t_{N-1})}{1 + \delta_{N-1} m_{N-1}(0)} . \tag{1.11}$$

Finally, we prove Lemma 1.2 by induction on the index i. □

We note that Lemma 1.2 is proved by use of a backward substitution method. Lemma 1.2 can thus be extended to cover the case where the volatility functionals are of the following form (cf. 13):

$$\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i(t, m_i, m_{i+1}, \ldots, m_N) .$$

In the next section, we prove that there exists at least one HJM model defined on $[0, T]$ in which the forward Libor rates satisfy

$$L_{ti} = m_i(t) , \tag{1.12}$$

where the processes m_i are as in Lemma 1.2. These HJM models are called "Market Models". In these models, the primary variables are forward Libor rates rather than zero coupon bonds.

2. Market Models

Having introduced the necessary preliminary material, we propose a straightforward extension of the results of [2, 13, 15]. Consider an initial discount curve $\{d(a) : 0 \leq a \leq T\}$, together with the specification of N predictable functionals $\{\boldsymbol{\sigma}_i : 1 \leq i \leq N\}$ satisfying the volatility realizability condition. We show in this section that there exists an HJM model defined on $[0, T]$ denoted $\{P_{tb} : t \leq b\}$ that satisfies $P_{0b} = d(b)$, and that is such that the forward Libor rates L_{ti} defined by (1.1) satisfy

$$dL_{ti} = \boldsymbol{\sigma}_i(t, L_i) d\mathbf{W}_t^{Q(P_{\cdot i+1})} \quad (1 \leq i \leq N,\ 0 \leq t \leq t_i) , \tag{2.1}$$

where $Q(P_{\cdot i+1})$ is the measure associated with the numeraire $P_{t\,i+1}$ as defined in Lemma 1.1.

By use of (1.3) and the initial discount curve d, we calculate the initial forward Libor rates L_{0i} for $i = 1, \ldots, N$. Using Lemma 1.2, we construct $N + 1$ probability measures Q_i for $0 \leq i \leq N$ and N Q_i-martingales M_i for $1 \leq i \leq N$ such that $Q_N = P$ and

$$\frac{dQ_{i-1}}{dQ_i} = \frac{1 + \delta_i M_i(t_i)}{1 + \delta_i M_i(0)} \quad (1 \leq i \leq N) , \tag{2.2}$$

where

$$M_i(t) = L_{0i} + \int_0^{t \wedge t_i} \boldsymbol{\sigma}_i(s, M_i) d\mathbf{W}_s^{Q_i} . \tag{2.3}$$

Let m_t be the continuous non-negative semimartingale defined, for t in the range $[t_{i-1}, t_i]$, by

$$m_t = \frac{1}{1 + \frac{\delta_{i-1}(t_i - t)}{t_i - t_{i-1}} M_{i-1}(t_{i-1})} \prod_{j=i}^{N} \frac{1}{1 + \delta_j M_j(t)}\,, \tag{2.4}$$

for $1 \leq i \leq N + 1$. A financial interpretation can be found in [13]. Then we define $N + 1$ processes P_{ti} according to the scheme

$$\begin{aligned} P_{ti} &= m_t \prod_{j=i}^{N} (1 + \delta_j M_j(t)) \quad && (0 \leq t \leq t_i,\, 1 \leq i \leq N)\,, \\ P_{t\,N+1} &= m_t && (0 \leq t \leq T)\,. \end{aligned} \tag{2.5}$$

The processes P_{ti} are strictly positive, satisfying $P_{ti} = 1$ and $P_{0i} = d(t_i)$, and thus define a model on $[0, t]$ for the prices of zero coupon bonds that mature at times $t_1, \ldots, t_N, T$. This model is consistent, by construction, with the initial discount curve. The associated forward Libor rates L_{ti} satisfy

$$L_{ti} = \frac{1}{\delta_i} \left(\frac{P_{ti} - P_{t\,i+1}}{P_{t\,i+1}} \right) = M_i(t) \quad (0 \leq t \leq T,\, 1 \leq i \leq N)\,. \tag{2.6}$$

Now we show that this model is arbitrage free on $[0, T]$, or equivalently that the ratio of P_{ti} to $P_{tT} = P_{t\,N+1}$ is a P-martingale for all i. Let us first prove the following result.

Lemma 2.1. *For $0 \leq i \leq N$, the processes P_{ti} satisfy*

$$E_t \left[\frac{dQ_i}{dQ_N} \right] = \frac{P_{0T} P_{t\,i+1}}{P_{0\,i+1} P_{tT}} \quad (0 \leq t \leq t_{i+1})\,. \tag{2.7}$$

Proof. Equation (2.7) clearly holds for $i = N$. Suppose now that (2.7) holds for $i = j + 1$. Then, by use of (2.2) and (2.7) we have

$$\begin{aligned} E_t \left[\frac{dQ_j}{dQ_N} \right] &= E_t \left[\frac{dQ_{j+1}}{dQ_N} \frac{dQ_j}{dQ_{j+1}} \right] \\ &= E_t^{Q_{j+1}} \left[\frac{dQ_j}{dQ_{j+1}} \right] E_t \left[\frac{dQ_{j+1}}{dQ_N} \right] \\ &= \frac{1 + \delta_{j+1} L_{t\,j+1}}{1 + \delta_{j+1} L_{0\,j+1}} \frac{P_{0T} P_{t\,j+2}}{P_{0\,j+2} P_{tT}} \\ &= \frac{P_{0T} P_{t\,j+1}}{P_{0\,j+1} P_{tT}}\,. \end{aligned} \tag{2.8}$$

Hence Eq. (2.6) holds for $i = j$. By induction, it holds for all i. □

As a consequence of Lemma 2.1, we conclude that the ratio of P_{tj} to P_{tT} is a P-martingale up to t_j and for $j = 1, \ldots, N$. Hence, we have proved that the model defined by Eq. (2.5) is arbitrage free. Comparing Eqs. (1.1) and (2.7), we conclude the probability measure Q_j is a martingale measure associated with the numeraire P_{tj+1}.

The ratio P_{ti}/P_{tT} is a non-negative P-martingale for $1 \leq i \leq N+1$. Using Eq. (2.5) we derive the following representation of these ratios in the form:

$$\frac{P_{ti}}{P_{tT}} = \frac{P_{0i}}{P_{0T}} \exp\left(-\int_0^t \mathbf{\Omega}_{si} d\mathbf{W}_s - \frac{1}{2}\int_0^t \mathbf{\Omega}_{si}^2 ds\right) , \tag{2.9}$$

where the predictable vector processes $\mathbf{\Omega}_{ti}$ are given by

$$\mathbf{\Omega}_{ti} = \sum_{k=i}^{N} \frac{\delta_k}{1+\delta_k L_{tk}} \boldsymbol{\sigma}_k(t, L_k) \, . \tag{2.10}$$

By linear interpolation on the maturity index, we can construct a one-parameter family of predictable vector processes $\{\Lambda_{tb} : t \leq b\}$ such that $\Lambda_{tt_i} = \Omega_{ti}$ for $0 \leq i \leq N+1$ and such that the one-parameter family of vectors $\{\Lambda_{tb} : t \leq b\}$ has maximal rank for all t in $[0,T]$, i.e. only the null vector is orthogonal to all vectors of this family. Finally, we define the following interest rate system on $[0,T]$:

$$P_{ta} = m_t \frac{P_{0a}}{P_{0T}} \exp\left(-\int_0^t \Lambda_{sa} d\mathbf{W}_s - \frac{1}{2}\int_0^t \Lambda_{sa}^2 ds\right) \quad (0 \leq t \leq T,\, t \leq a) \, . \tag{2.11}$$

This interest rate system defined on $[0,T]$ admits P as a T-forward probability measure. The measure P is in fact the unique T-forward probability measure since the vector family $\{\Lambda_{ta} : t \leq a\}$ has maximal rank at all time t in $[0,T]$. Hence, this interest rate system is complete and arbitrage-free on $[0,T]$. Moreover, the forward Libor rates associated with this HJM model satisfy (2.1). Summing up, we see that the following result has been established.

Proposition 2.1. *There exists at least one HJM model defined on* $[0,T]$ *for which the forward Libor rates satisfy*

$$dL_{ti} = \boldsymbol{\sigma}_i(t, L_i) d\mathbf{W}_t^{Q(P_{\cdot i+1})} \quad (1 \leq i \leq N,\, 0 \leq t \leq t_i) \, , \tag{2.12}$$

where $Q(P_{\cdot i+1})$ *is the martingale measure associated with the numeraire* $P_{t\,i+t}$.

Using the extension of Lemma 1.2 proposed in the previous section, we can prove that there exists also an HJM model such that

$$dL_{ti} = \boldsymbol{\sigma}_i(t, L_i, L_{i+1}, \ldots, L_N) d\mathbf{W}_t^{Q(P_{\cdot i+1})} \quad (1 \leq i \leq N,\, 0 \leq t \leq t_i) \, . \tag{2.13}$$

As a direct consequence of this result, we are able to make the following remark (cf. [13]).

Corollary 2.1. *The forward Libor rates* L_{ti} *defined by* $dL_{ti} = \sigma_i(t, L_i) dW_t^{Q(P_{\cdot i+1})}$ *satisfy the stochastic differential equations*

$$dL_{ti} = \boldsymbol{\sigma}_i(t, L_i) d\mathbf{W}_t - \sum_{j=i+1}^{N} \frac{\delta_j \boldsymbol{\sigma}_i(t, L_i) \boldsymbol{\sigma}_j(t, L_j)}{1+\delta_j L_{tj}} dt \quad (0 \leq t \leq t_i,\, 1 \leq i \leq N) \tag{2.14}$$

under the probability measure P.

We note the similarity between (2.14) and the HJM drift adjustment formula for instantaneous forward rates under the terminal measure (cf. [10]).

Proof. Define the density martingale ρ_{ti} according to the scheme

$$\rho_{ti} = E_t^{Q_{i+1}}\left[\frac{dQ_i}{dQ_{i+1}}\right] = \frac{1+\delta_{i+1}L_{t\,i+1}}{1+\delta_{i+1}L_{0\,i+1}}\,. \tag{2.15}$$

Then, we deduce that

$$\frac{d\rho_{ti}}{\rho_{ti}} = \frac{\delta_{i+1}}{1+\delta_{i+1}L_{t\,i+1}}\boldsymbol{\sigma}_{i+1}(t,L_{i+1})d\mathbf{W}_t^{Q_{i+1}}\,. \tag{2.16}$$

It follows from the Girsanov theorem that

$$d\mathbf{W}_t^{Q_i} = d\mathbf{W}_t^{Q_{i+1}} - \frac{\delta_{i+1}\boldsymbol{\sigma}_{i+1}(t,L_{i+1})}{1+\delta_{i+1}L_{t\,i+1}}dt\,. \tag{2.17}$$

Adding these equations, and using $\mathbf{W}_t^{Q_N} = \mathbf{W}_t$, we get

$$dL_{ti} = \boldsymbol{\sigma}_i(t,L_i)d\mathbf{W}_t - \sum_{j=i+1}^{N}\frac{\delta_j\boldsymbol{\sigma}_i(t,L_i)\boldsymbol{\sigma}_j(t,L_j)}{1+\delta_j L_{tj}}dt\,. \tag{2.18}$$

□

The ith volatility functional $\boldsymbol{\sigma}_i$ depends on time and on the ith forward Libor rate path L_i. It is possible as we mentioned earlier to choose this volatility functional also to depend on the paths of the successive forward Libor rates, i.e. on $L_{i+1},\ldots,L_N$. Doing so may allow calibration of the model to cap and swaption smiles that are "compatible". Further discussion of this topic is, however, beyond the scope of the present paper. In Secs. 5 and 6, we give two examples of volatility functionals. In Secs. 3 and 4, we show how to construct a Libor tree from caplet smile.

As pointed out in [13] and [15], the system of stochastic differential Eq. (2.14) is not easy to solve numerically, and a Monte Carlo method has to be used in general. See [7]. As a consequence, the pricing of derivatives on forward Libor rates with American features such as Bermudan swaptions or flexi-caps is difficult if not impractical. See [7, 13, 15].

In the next two sections we show how to construct Market Models with Markovian Libor rates for which a single tree of Libor rates can be constructed. In these models, derivatives on forward Libor rates such as Bermudan swaptions and flexi-caps can be priced using a tree of Libor rates which can be calibrated to an initial caplet smile surface.

3. Markov Market Models

Here we consider Market Models for which the N Libor rates are driven by N Markov processes defined under the T-forward probability measure. Markov Market Models with low dimension, i.e. for which the correlation matrix of the Markov processes has a low rank, can be implemented efficiently using standard tree techniques.

In the next section, we review a tree implementation of a Markov Market Model driven by N Markov processes that are perfectly correlated. Markov Market Models with high dimension can be implemented using the stochastic mesh introduced in [4].

In this section, we show how the Markov Market Models can be calibrated against an initial Libor smile surface thus allowing the pricing of complex Libor derivatives under various correlation and smile assumptions.

Denote by $C_t(t_i, t_{i+1}, K)$ the price at time t of a caplet set at time t_i and paying $\max(L_{t_i i} - K, 0)$ at time t_{i+1}. We shall construct a family of forward Libor rates that is arbitrage free, Markovian and consistent with an initial set of arbitrage-free caplet prices given by $C_0(t_i, t_{i+1}, K)$ for $1 \le i \le N$, where K is the strike. Note that the term and strike structure of caplet prices can be implied from the term and strike structure of cap prices, providing, of course, that the cap prices cannot be arbitraged. If the caplet prices present no smile then each Libor rate $L_{t_i i}$ is log-normally distributed under the measure $Q(P_{\cdot i+1})$. This case corresponds to the Market Model introduced in [2, 13, 15]. Let us first introduce some notation and take note of some preliminary results.

Definition 3.1. The family $\{Z_i : 1 \le i \le N\}$ of P-Markov processes is *probability-realizable* if the cumulative distributions of the random variables $Z_{t_i i}$ under P are continuous functions that are strictly increasing on $\mathfrak{R}$.

Consider N non-zero vectors $\{\boldsymbol{\rho}_i : 1 \le i \le N\}$. Then a simple example of a family of Markov processes that is probability-realizable is the family defined by $Z_{ti} = \boldsymbol{\rho}_i \mathbf{W}_t$.

To proceed we first derive a set of four lemmas for later use. In what follows $\chi\{S\}$ denotes the indicator function for the event S. The first lemma relates the caplet prices and the forward probability distributions of the corresponding Libor rates. This result was first derived by Breeden and Litzenberger [3] and then used by Dupire [5] in the context of equity options.

Lemma 3.1. *Under the assumption of no arbitrage, there exists a martingale measure* $Q(P_{\cdot i+1})$ *associated with the numeraire* $P_{t\,i+1}$ *as defined in Lemma* 1.1. *Denote by* $D_i(x)$ *the cumulative distribution of* $L_{t_i i}$ *under the measure* $Q(P_{\cdot i+1})$. *Then providing the initial caplet prices are smooth functions of the strike, we have*:

$$D_i(x) = E_0^{Q(P_{\cdot i+1})}[\chi\{L_{t_i i} < x\}] = 1 + \frac{\partial}{\partial K}\left(\frac{C_0(t_i, t_{i+1}, K)}{P_{0\,i+1}}\right)_{K=x}. \tag{3.1}$$

These functions are increasing under the assumption of no arbitrage.

Proof. Since we assume no arbitrage, the ratio of $C_t(t_i, t_{i+1}, K)$ to $P_{t\,i+1}$ is a $Q(P_{\cdot i+1})$-martingale on $[0, t_{i+1}]$. It follows that

$$\begin{aligned}\frac{C_0(t_i, t_{i+1}, K)}{P_{0\,i+1}} &= E_0^{Q(P_{\cdot i+1})}[(L_{t_i i} - K)^+] \\ &= \int_K^{+\infty} (x - K) E^{Q(P_{\cdot i+1})}[\chi\{L_{t_i i} \in dx\}] \,. \end{aligned} \tag{3.2}$$

Differentiating this relation with respect to K, we obtain

$$\frac{\partial}{\partial K}\left(\frac{C_0(t_i,t_{i+1},K)}{P_{0\,i+1}}\right) = -E^{Q(P_{\cdot i+1})}[\chi\{L_{t_i i} > K\}]\,, \tag{3.3}$$

from which it follows that

$$D_i(x) = 1 + \frac{\partial}{\partial K}\left(\frac{C_0(t_i,t_{i+1},K)}{P_{0\,i+1}}\right)_{K=x}\,. \tag{3.4}$$

Since a cumulative probability distribution must be increasing, we conclude from Eq. (3.4) that the caplet prices must be convex functions of the strike, since otherwise there would be an arbitrage opportunity. This arbitrage could be realised by being long a butterfly (cf. [5]). □

Given a cap smile surface, we can derive a caplet smile surface providing that the cap prices are consistent with the no arbitrage assumption. From a caplet smile surface, we obtain the distribution of the Libor rates $L_{t_i i}$ under the probability $Q(P_{\cdot i+1})$ using the previous lemma. We note however that the distribution of the Libor rates under a unique probability distribution cannot be implied without further assumptions.

In the next lemma, we show how to construct a random variable X that has a prescribed cumulative probability distribution D under a given probability measure Q. This lemma will be used to prove Lemma 3.4 below.

Lemma 3.2. *Let D be a continuously differentiable probability distribution. Let Y be a random variable with distribution C under probability measure Q such that C is a strictly increasing continuous function defined on $\mathfrak{R}$. Then the random variable X defined by*

$$X = D^{-1}(C(Y)) \tag{3.5}$$

has probability distribution D under Q. The function D^{-1} is the inverse of D.

Proof. We denote by I_D the open set in $\mathfrak{R}$ in which the derivative of D is strictly positive.

$$I_D = \left\{z \in \mathfrak{R} : \frac{dD}{dx}(z) > 0\right\} \tag{3.6}$$

The cumulative distribution D is thus constant on $\mathfrak{R} \setminus I_D$ and the inverse of D maps (0,1) onto I_D. The random variable X defined by (3.5) belongs thus to I_D. Hence, the probability that X belongs to any subset of $\mathfrak{R} \setminus I_D$ is zero under the measure Q. We define $x_0 = \inf\{z : z \in I_D\}$. For a real $x > x_0$, we define

$$d(x) = \sup\{z : D(z) < D(x)\}\,. \tag{3.7}$$

We can prove that $d(x)$ satisfies $d(x) \leq x$, $(d(x), x) \subset \mathfrak{R} \setminus I_D$ and $D(d(x)) = D(x)$. We now show that the probability that X belongs to the interval $(-\infty, x)$ under Q

is given by $D(x)$ for arbitrary x. If $x \leq x_0$ then this probability is equal to zero as well as $D(x)$. Otherwise, we have

$$E^Q[\chi\{X < x\}] = E^Q[\chi\{X < d(x)\}] + E^Q[\chi\{d(x) < X < x\}]\,, \tag{3.8}$$

where the last expectation is equal to zero as previously explained. Since the function D is strictly increasing on I_D, we have

$$E^Q[\chi\{X < d(x)\}] = E^Q[\chi\{D(X) < D(x)\}] = E^Q[\chi\{C(Y) < D(x)\}]\,. \tag{3.9}$$

The function C^{-1} is strictly increasing on (0,1) and $C^{-1}(C(Y)) = Y$. Therefore, we obtain

$$E^Q[\chi\{C(Y) < D(x)\}] = C(C^{-1}(D(x)))\,. \tag{3.10}$$

It follows that $E^Q[\chi\{X < x\}] = D(x)$. □

Since the N processes Z_i are Markov processes, we have the following result.

Lemma 3.3. *Let Z_i be P-Markov processes for $1 \leq i \leq N$. Let g be a function of N variables such that $E[g(Z_{b1}, \ldots, Z_{bN})] < \infty$ for some real $b > 0$. Then there exists a function f of $N+1$ variables such that*

$$E_a[g(Z_{b1}, \ldots, Z_{bN})] = f(a, Z_{a1}, \ldots, Z_{aN}) \qquad (a \leq b)\,. \tag{3.11}$$

The following lemma is crucial to the proof of Proposition 3.1. It is instructive to consider the difference between Lemma 3.4 and Lemma 1.2. Lemma 3.4 defines the forward Libor rates $M_i(t) = E_t^{Q_i}[M_i]$ as Markovian Q_i-martingales that have the prescribed distributions D_i under Q_i at time t_i, while Lemma 1.2 defines these Q_i-martingales as the strong solutions of a set of stochastic differential equations.

Lemma 3.4. *Given N continuously differentiable distributions D_i vanishing on $(-\infty, 0)$ for $1 \leq i \leq N$, there exist $N+1$ equivalent probability measures Q_i for $0 \leq i \leq N$, with $Q_N = P$, N random variables $M_i \in \Im_{t_i}$, and N functions g_i for $1 \leq i \leq N$, such that*

$$\frac{dQ_{i-1}}{dQ_i} = \frac{1 + \delta_i M_i}{1 + \delta_i E^{Q_i}[M_i]}\,, \tag{3.12}$$

$$M_i = g_i(Z_{t_i i})\,, \tag{3.13}$$

$$E^{Q_i}[\chi\{M_i < x\}] = D_i(x)\,, \tag{3.14}$$

where $1 \leq i \leq N$ and $\{Z_i : 1 \leq i \leq N\}$ is a probability-realizable family of P-Markov processes.

Proof. First we set $Q_N = P$. Then following Lemma 3.2, we define the random variable $M_N \in \Im_{t_N}$ by the formulae:

$$g_N(x) = D_N^{-1}(E^{Q_N}[\chi\{Z_N(t_N) < x\}]) \text{ and } M_N = g_N(Z_{t_N}N)\,, \tag{3.15}$$

The random variable M_N has cumulative distribution D_N under P and satisfies Eq. (3.14). Next, we define the probability measure Q_{N-1} as follows:

$$\frac{dQ_{N-1}}{dQ_N} = \frac{1 + M_N\delta_N}{1 + E[M_N]\delta_N}\,. \tag{3.16}$$

We have thus constructed a measure Q_{N-1}, a random variable M_N, and a function such that Eqs. (3.12), (3.13) and (3.14) hold.

Suppose now that we have constructed the probability measures $Q_{i-1}, \ldots, Q_{N-1}$, the random variables $M_i, \ldots, M_N$ and the functions $g_i, \ldots, g_N$ such that Eqs. (3.12), (3.13) and (3.14) hold. Let us construct Q_{i-2}, M_{i-1} and g_{i-1} such that Eqs. (3.12), (3.13) and (3.14) hold. We define the random variable M_{i-1} as follows:

$$g_{i-1}(x) = D_{i-1}^{-1}(E^{Q_{i-1}}[\chi\{Z_{t_{i-1}}(i-1) < x\}])\,, \tag{3.17}$$

$$M_{i-1} = g_{i-1}(Z_{t_{i-1}}(i-1))\,. \tag{3.18}$$

By application of Lemma 3.2, we conclude that this random variable satisfies (3.14). This shows that M_{i-1} satisfies (3.13) and (3.14). By use of Eq. (3.12), we construct the probability measure Q_{i-2}. Finally, we have constructed the probability measure Q_{i-2}, the random variable M_{i-1} and the function g_{i-1} such that Eqs. (3.12), (3.13) and (3.14) hold. By induction, we prove Lemma 3.4. □

Using Lemma 3.3, we can prove by induction that there exist N functions f_i such that

$$E_t^{Q_i}[M_i] = f_i(t, Z_{ti}, \ldots, Z_{tN})\,. \tag{3.19}$$

Having established the necessary preliminary lemmas, we can now prove the following result.

Proposition 3.1. *There exists at least one HJM model that is consistent with the initial set of caplet prices $C_0(t_i, t_{i+1}, K)$ for $1 \le i \le N$, providing that the caplet prices are arbitrage-free, are C^2 functions of the strike, and are such that for all negative K, they satisfy the relation $\partial_K C_0(t_i, t_{i+1}, K) = -1$. Among these HJM models there is at least one that satisfies for*

$$L_{t_i i} = g_i(Z_{t_i i}) \text{ and } L_{ti} = f_i(t, Z_{ti}, \ldots, Z_{tN})\,, \tag{3.20}$$

for $1 \le i \le N$, where g_i, f_i are real functions and $\{Z_i : 1 \le i \le N\}$ is a probability-realizable family of P-Markov processes.

Proof. Using Lemma 3.1, we derive the cumulative probability distribution $D_i(x)$ of $L_{t_i i}$ from caplet prices or caplet implied volatilities as follows:

$$D_i(x) = 1 + \frac{\partial}{\partial K}\left(\frac{C_0(t_i, t_{i+1}, K)}{P_{0\,i+1}}\right)_{K=x}\,. \tag{3.21}$$

We have assumed that

$$\partial_K C_0(t_i, t_{i+1}, K) = -1 \quad (k \le 0)\,. \tag{3.22}$$

Therefore, the Libor rates are almost surely positive. Note that we only need to impose that $(1 + \delta_i L_{ti})$ remains strictly positive. Since the caplet prices are C^2 functions of the strike, we conclude that the forward cumulative distributions D_i of the Libor rates are continuously differentiable functions.

Using Lemma 3.4, we construct $N + 1$ probability measures $Q_0, \ldots, Q_N$ with $Q_N = P$ and N random variables $M_1, \ldots, M_N$ such that

$$\frac{dQ_{i-1}}{dQ_i} = \frac{1 + \delta_i M_i}{1 + \delta_i E^{Q_i}[M_i]}\,, \tag{3.23}$$

$$M_i = g_i(Z_{t_i i})\,, \tag{3.24}$$

$$E^{Q_i}[\chi\{M_i < x\}] = D_i(x)\,, \tag{3.25}$$

for $1 \leq i \leq N$. We define $M_i(t) = E_t^{Q_i}[M_i]$. We note that $E^{Q_i}[M_i] = L_{0i}$ and that

$$M_i(t) = f_i(t, Z_{ti}, \ldots, Z_{tN})\,. \tag{3.26}$$

Since $M_i(t)$ is a non-negative Q_i-martingale, $M_i(t)$ can be represented as follows:

$$M_i(t) = L_{0i} \exp\left(\int_0^t \sigma_{si} dW_s^{Q_i} - \frac{1}{2}\int_0^t \sigma_{si}^2 ds\right)\,, \tag{3.27}$$

where σ_{ti} is a predictable process and $1 \leq i \leq N$. Applying Proposition 2.1 to the set of N predictable functionals $(X_t)_t \mapsto (\sigma_{ti} X_t)_t$ which satisfies the volatility realizability condition, we deduce that there exists at least one HJM model that is such that $L_{ti} = M_i(t)$ and $Q(P_{\cdot i+1}) = Q_i$. This HJM model is therefore consistent with the initial caplet prices and satisfies (3.18). That concludes the proof of Proposition 3.1. □

We have thus constructed Market Models in which the forward Libor rates are deterministic functions of a probability-realizable family of P-Markov processes.

4. Implied Libor Trees for Markov Market Models

In this section we review a tree implementation of a one-factor Markov Market Model for which forward Libor rates are deterministic functions of a single Brownian motion. p-factor Markov Market Models can be implemented using N Brownian motions having a correlation matrix with rank p.

The algorithm presented below allows the pricing of complex Libor derivatives under different Libor volatility smile assumptions.

We first construct a trinomial tree that approximates the values of the Brownian motion W. This trinomial tree is constructed using a mesh with time mesh τ_k in $[0, T]$. This mesh contains the Libor setting dates t_i defined in the Sec. 1. Note that there are more time meshes than time grids for reasons of accuracy.

We denote by $\tilde{W}(t)$ the trinomial process approximating the P-Brownian motion $W(t)$ in the limit of zero mesh-size. The trinomial process is defined for all time meshes τ_k and thus, for all time grids t_i. We label the values of this trinomial process at a given time mesh using an integer j with the convention that increasing j corresponds to increasing values of the trinomial process.

We denote by ν_i the implied volatility for an at-the-money caplet with setting date t_i and payment date t_{i+1}. We populate each tree layer corresponding to a grid date t_i with the following P-log-normal random variable:

$$\hat{L}_{t_i i}(j) = L_{0i} \exp\left(\nu_i \tilde{W}_{t_i}(j) - \frac{1}{2}\nu_i^2 t_i\right). \tag{4.1}$$

By forward induction, we compute for each time grid t_i and each state j the probability $\pi_{t_i}(j)$ that the trinomial process with initial value zero takes the value $\tilde{W}_{t_i}(j)$ at time t_i.

The algorithm consists in modifying the log-normal random variables $\hat{L}_{t_i i}$ to obtain Libor rates $L_{t_i i}$ that have the distribution D_i under the corresponding forward measures Q_i and implied from the ith caplet smile.

We initialise the algorithm by use of Lemma 3.2, with

$$L_{t_N N}(j) = D_N^{-1}(C_N(\hat{L}_{t_N N}(j))), \tag{4.2}$$

where D_N is the cumulative distribution of the Nth Libor rate under the probability measure $Q_N = P$, and C_N is the cumulative distribution of the P-log-normal random variable $\hat{L}_{t_N N}$ under P. The cumulative distribution C_N can be obtained using the tree. Indeed, we have

$$C_N(\hat{L}_{t_N N}(j)) = \Pr\{\tilde{W}_{t_N} < \tilde{W}_{t_N}(j)\} = \sum_{k<j} \pi_{t_N}(k). \tag{4.3}$$

We calculate the Nth forward Libor rate $L_{t_N} = E_t[L_{t_N N}]$, by backward induction and finally, we calculate the Radon Nikodym derivative

$$E_t\left[\frac{dQ_{N-1}}{dQ_N}\right](j) = \frac{1 + \delta_N L_{tN}(j)}{1 + \delta_N L_{0N}}, \tag{4.4}$$

where E is expectation under the probability measure $Q_N = P$. We obtain the previously set Libor rates by use of the following algorithm initialised with $i = N-1$.

Step 1. Having obtained the value of $E_t[dQ_i/dQ_N]$ for all states j and time $t = t_i$, we calculate for each state j the cumulative distribution $\hat{C}_i(x)$ of $\hat{L}_{t_i i}$ under the measure Q_i, and the cumulative distribution $C_i(x)$ of $\hat{L}_{t_i i}$ under the measure $Q_N = P$. The values of these cumulative probability distributions at the nodes of the tree are given by

$$C_i(\hat{L}_{t_i i}(j)) = \sum_{k<j} \pi_{t_i}(k), \tag{4.5}$$

$$\hat{C}_i(\hat{L}_{t_i i}(j)) = \sum_{k<j} \pi_{t_i}(k) E_{t_i}\left[\frac{dQ_i}{dQ_N}\right](k). \tag{4.6}$$

Step 2. Following Lemma 3.2, for each state j we define

$$L_{t_i i}(j) = D_i^{-1}(\hat{C}_i(\hat{L}_{t_i i}(j)))\,. \tag{4.7}$$

Equation (4.7) can be solved numerically by interpolating the increasing discrete curve $(D_i(\hat{L}_{t_i i}(j)), \hat{L}_{t_i i}(j))$ at the abscissae $\hat{C}_i(\hat{L}_{t_i i}(j))$.

Step 3. We then use the relation

$$E_{t_i}\left[\frac{dQ_{i-1}}{dQ_N}\right] = E_{t_i}\left[\frac{dQ_i}{dQ_N}\right]\frac{1+\delta_i L_{t_i i}}{1+\delta_i L_{0i}}\,. \tag{4.8}$$

By backward induction, we calculate $E_t[dQ_{i-1}/dQ_N](j)$ for all states and time step $t = t_i$.

Step 4. Go back to Step 1.

The algorithm just presented provides a tree of forward Libor rates under the terminal measure Q_N. Once the tree of forward Libor rates is constructed, the price of complex Libor derivatives can be calculated using the terminal measure P as pricing measure and the discount bond price P_{tT} as numeraire. Numerical results obtained using this algorithm may be reported elsewhere. It is also possible to check that for any strike K the price of a caplet with setting date t_i is given approximately by the Black–Scholes formula, by calculating

$$C_0(t_i, t_{i+1}, K) = P_{0i+1}\sum_j \pi_{t_i}(j)E_{t_i}\left[\frac{dQ_i}{dQ_N}\right](j)(L_{t_i i}(j) - K)^+\,. \tag{4.9}$$

5. Market Model with Log-normal Volatility

In Sec. 2 we constructed arbitrage-free Market Models that are specified via the volatility functionals of forward Libor rates. In this section and the next, we give examples of one-factor Market Models defined via the volatility of the forward Libor rates. These Market Models are path-dependent, and a tree implementation is not possible. A "recombining" Monte Carlo method, such as the stochastic mesh method described in [4], must be used to price complex Libor derivatives using such models. The number of state variables of a one-factor path-dependent Market Model is N, and thus the amount of work involved is the same as if we were using a N-factor Markov Market Model.

Using the notation introduced in the previous sections, we consider N strictly positive bounded functions $\alpha_i(t)$. Using Lemma 1.2, we define N probability measures Q_i, with, $Q_N = P$ and N Q_i-martingales $M_i(t)$ according to the scheme

$$\frac{dQ_{i-1}}{dQ_i} = \frac{1+\delta_i M_i(t_i)}{1+\delta_i M_i(0)}\,, \tag{5.1}$$

$$M_i(t) = L_{0i} + \int_0^t \alpha_i(s)M_i(s)dW_s^{Q_i}\,, \tag{5.2}$$

where $1 \leq i \leq N$. Here $W_t^{Q_i}$ is a Q_i-Brownian motion. We note that $M_i(t)$ is strictly positive. Therefore the set of N predictable functions $(X_t)_t \mapsto (\alpha_i(t) X_t)_t$ satisfies the volatility realizability condition. Applying Proposition 2.1, we conclude that there exists an HJM model defined on $[0, T]$ for which the forward Libor rates L_{ti} are equal to $M_i(t)$. Under these assumptions, the initial price of a caplet denoted $C_0(t_i, t_{i+1}, K)$, with setting date t_i and payment date t_{i+1}, is given by

$$C_0(t_i, t_{i+1}, K) = P_{0i+1} E_0^{Q_i}[\max(0, L_{t_i i} - K)] \,. \tag{5.3}$$

Since $L_{t_i i}$ is the exponential of a Q_i-normal random variable we conclude that the price is given by the Black–Scholes formula. This Market Model was first introduced in [2, 13, 15].

6. Market Model with Square-Root Volatility

In this section, we analyze a Market Model for which the forward Libor rates have a square-root volatility. In particular, we derive an efficient algorithm for pricing caplets based on a Gaussian quadrature scheme. We note, however, that a "recombining" Monte Carlo method, together with Eq. (2.10), must be used to price other Libor derivatives.

Using Lemma 1.2, we define a set of $N + 1$ probability measures $Q_0, \ldots, Q_N$ and a set of N Q_i-martingales $M_1, \ldots, M_N$ according to the scheme

$$\frac{dQ_{i-1}}{dQ_i} = \frac{1 + \delta_i M_i(t_i)}{1 + \delta_i M_i(0)} \,, \tag{6.1}$$

$$M_i(t) = L_{0i} + \int_0^t \alpha_i(s) \sqrt{M_i(s)} \, dW_s^{Q_i} \,, \tag{6.2}$$

with $Q_N = P$. Here $W_t^{Q_i}$ is again a Q_i-Brownian motion, and α_i is a bounded strictly positive function defined on $[0, t_i]$. Let us now show that Proposition 2.1 is applicable.

Lemma 6.1. *The Q_i-martingale $M_i(t)$ exists, and is given by the time-change of the square of a Bessel process with dimension zero.*

Proof. We recall that the square of a Bessel process in $(\Omega, \Im, Q_i)$ with dimension δ and with initial value L_{0i} is the unique solution of

$$Z_t = L_{0i} + 2 \int_0^t \sqrt{Z_s} \, dW_s^i + \delta t \,. \tag{6.3}$$

Properties of such processes, which are necessary positive, are discussed in [16, p. 421]. Now we define the strictly increasing functions ϕ_i by

$$\phi_i(t) = \frac{1}{4} \int_0^t \alpha_i(s)^2 ds \,. \tag{6.4}$$

We denote the inverse of $\phi_i(t)$ by $f_i(t)$. Define the process $m_i(t)$ by

$$dm_i(t) = \frac{1}{2}\alpha_i(t)dW_t^i\,. \tag{6.5}$$

From the Dubins–Schwarz theorem (cf. [13, p. 173]) we conclude that $m_i(t) = B_i(\phi_i(t))$, where $B_i(t)$ is a Q_i-Brownian motion adapted to the filtration $\hat{\Im} \equiv \{\Im_{f_i(t)} : t \geq 0\}$. Consider the $\hat{\Im}$-adapted square Bessel process $Z_i(t)$ with zero dimension defined by

$$Z_i(t) = L_{0i} + 2\int_0^t \sqrt{Z_i(s)}\,dB_i(s)\,. \tag{6.6}$$

It follows that

$$Z_i(\phi_i(t)) = L_{0i} + \int_0^t \alpha_i\sqrt{Z_i(\phi_i(s))}\,dW_s^i\,. \tag{6.7}$$

The process $Z_i(\phi_i(t))$ is $\Im$-adapted and satisfies the stochastic Eq. (6.2). Since this Equation admits a unique strong solution by standard results (cf. [16, p. 371]), we conclude that $M_i(t) = Z_i(\phi(t))$. Hence $M_i(t)$ is the time-change of the square of a Bessel process of dimension zero. □

Using Proposition 2.1, we conclude that the assumption $L_{ti} = M_i(t)$ does not imply an arbitrage, and that there exists an HJM model in which the forward Libor rates L_{ti} are equal to $M_i(t)$. Since $M_i(t)$ is the time-change of the square of a Bessel process with dimension zero, we know the probability density of $M_i(t)$ and thus of $L_i(t)$.

Proposition 6.1. *For $t > 0$, the random variable $L_i(t)$ has the following distribution under Q_i:*

$$\Pr{}^{Q_i}\{L_i(t) > Y \mid L_i(0)\} = \int_{\sqrt{Y}}^{+\infty} p_i(t,x)dx \quad (Y > 0)\,, \tag{6.8}$$

with

$$p_i(t,y) = \frac{\sqrt{L_{0i}}}{2\phi_i(t)\sqrt{y}}\exp\left[-\frac{(\sqrt{y}-\sqrt{L_{0i}})^2}{2\phi_i(t)}\right]K_1\left[\frac{\sqrt{yL_{0i}}}{\phi_i(t)}\right], \tag{6.9}$$

where $K_i(x) = e^{-x}I_i(x)$, and I_i is the modified Bessel function of index i.

Recall that I_1 has the following integral representation (cf. [1, p. 270]):

$$I_1(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos t}\cos t\,dt\,. \tag{6.10}$$

We note that zero is an absorbing point for L_i and as a consequence, the probability distribution of $L_i(t)$ has a mass at zero (cf. [16, p. 423]). The probability that $L_i(t)$ is zero is indeed non-vanishing, given by

$$\Pr{}^{Q_i}\{L_i(t) = 0 \mid L_i(0)\} = 1 - \int_{0^+}^{+\infty} P_i(t,x)\,dx > 0\,. \tag{6.11}$$

After calculation, we obtain

$$\Pr^{Q_i}\{L_i(t)=0\}=\exp\left(\frac{-L_i(0)}{2\phi_i(t)}\right).$$

Proof. The result is obtained directly using Lemma 6.1 and the probability density function of the square of a Bessel process of dimension zero (cf. [16, p. 422]). □

Equation (6.8) can then be used, for example, to calculate the price of a caplet. Indeed, at $t=0$ the price of a caplet with setting date t_i and payment date t_{i+1} is

$$\begin{aligned} C_0(t_i,t_{i+1},K) &= P_{0i+1}E_0^{Q_i}[(L_i(t_i)-K)^+] \\ &= P_{0i+1}\int_K^{+\infty}(y-K)p_i(t_i,y)dy\,. \end{aligned} \tag{6.12}$$

After simplification, we get

$$\begin{aligned} &C_0(t_i,t_{i+1},K) \\ &\quad = 4\phi_i(t_i)P_{0i+1}A_i\int_{B_i}^{+\infty}\exp(-(z-A_i)^2)K_1(2zA_i)(z^2-B_i^2)\,dz\,, \end{aligned} \tag{6.13}$$

where $A_i=\sqrt{L_{0i}/2\phi_i(t_i)}$ and $B_i=\sqrt{K/2\phi_i(t_i)}$. We recall in this connection that $2\phi_i(t)=\frac{1}{2}\int_0^t\alpha_i(s)^2ds$ and $K_1(x)=e^{-x}I_1(x)$.

The integral in (6.13) can be simplified by use of a Frobenius series for the modified Bessel function of index one (cf. [1, p. 74]), given by

$$I_1(x)=\sum_{n=0}^{+\infty}\frac{x^{2n+1}}{2^{2n+1}n!(n+1)!}\,. \tag{6.14}$$

Substituting this expression for I_1 into Eq. (6.13), and permuting the integration and the summation, we get

$$\begin{aligned} &\frac{C_0(t_i,t_{i+1},K)}{P_{0i_1}} \\ &\quad = 2\phi_{t_i}(A^2e^{-A^2-B^2}I_0(2AB)+AB\,e^{-A^2-B^2}I_1(2AB)+(A^2-B^2)F_{AB})\,, \end{aligned} \tag{6.15}$$

where I_0 and I_1 are modified Bessel functions of index zero and one, respectively, and the function F_{AB} is defined by

$$F_{AB}=2A\int_B^{+\infty}\exp(-X^2-A^2)I_1(2AX)\,dX\,. \tag{6.16}$$

By use of this formula we obtain the following analytical expression for the price of an at-the-money caplet:

$$C_0(t_i,t_{i+1},K)=2\phi(t_i)P_{0i}A^2e^{-A^2-B^2}(I_0(2A^2)+I_1(2A^2))\,. \tag{6.17}$$

The integral for F_{AB} can be rewritten as follows:

$$F_{\mathrm{AB}}=2A\int_0^{+\infty}\exp(-X^2)G_{AB}(X)\,dX\,, \tag{6.18}$$

where

$$G_{AB}(X) = e^{-(A-B)^2-2X(B-A)}K_1(2AX+2AB)\,. \tag{6.19}$$

From (6.15) we see that approximating the price of the caplet is equivalent to approximating F_{AB}. When the caplet is out of the money ($B \geq A$) we note that $G_{AB}(X)$ goes to zero for large X. This function has bounded derivatives on $(0,\infty)$ when $B \geq A$. Hence we can evaluate the integral (6.18) by use of the Gaussian quadrature method based on Hermite polynomials, which gives

$$F_{AB} \approx 2Ae^{-(A-B)^2}\sum_{i=1}^{N}\omega_i e^{-2\xi_i(B-A)}K_1(2A\xi_i+2AB)\,, \tag{6.20}$$

where N is an integer that governs the accuracy of the approximation and N is usually taken to be less than 20. The coefficients ω_i, ξ_i depend only on N. These coefficients can be found in [19]. When N equals 20, the relative error in the above approximation is of the order 10^{-8}. When the caplet is in the money ($A > B$), the approximation does not perform very well. This is because the function G_{AB} and its derivative become exponentially large, and thus the error can become fairly large. Fortunately, we have the following put-call parity relation:

$$F_{\mathrm{BA}} = (1-F_{AB}) - e^{-(A-B)^2}K_0(2AB)\,. \tag{6.21}$$

This expression can be derived by taking the Frobenius series for $I_1(x)$ and integrating first over the interval $(0,B)$ and then over the interval $(B,+\infty)$. By use of (6.20) and (6.21), we then derive an approximation for the caplet price in the case where the caplet is in the money. This formula, together with (5.25), provides us with an approximation for F_{AB} for all A and B, and consequently with an approximation for the price of the caplet. We note that the approximation used in the case $N = 20$ is more accurate than achieved by use of Simpson's rule with 10 000

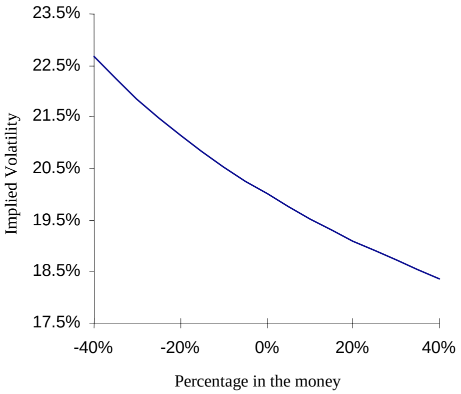

Fig. 1. Volatility smile implied by the square-root model for a caplet, with a 10% initial forward Libor rate and a 20% at-the-money Black volatility. This smile is not as pronounced as the smile implied by the Hull–White model, for example.

steps, and that the method is as fast as an analytical expression since it involves only summing 20 exponential terms.

We conclude with a plot showing the volatility smile implied by the square-root model for a forward Libor rate of initially 10% and an annual Black–Scholes volatility of 20%. We note that this volatility smile is not as pronounced as the volatility smile implied, for example, by the Hull–White interest rate model.

Acknowledgments

We would like to thank the referee for useful suggestions. We thank the participants of the 1998 European Finance Association conference in Fontainebleau and participants of the 1999 Risk conference on Pricing and Hedging Complex Derivatives, in London, where this work was presented, for helpful comments. We also express our gratitude to Alan Brace, Nicolas Rabeau, Michael Jones, Charles Liu, Guillaume Gimonet, and Eric Ben-Hamou for stimulating discussions. We are grateful to Phil Hunt for drawing our attention to an overlap of our work with [12].

References

[1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978).

[2] A. Brace, D. Gatarek and M. Musiela, *The market model of interest rate dynamics*, Math. Finance **7** (1997) 125–155.

[3] D. Breeden and R. Litzenberger, *The pricing of state contingent claims implicit in option prices*, J. Business **51** (1978) 621–651.

[4] M. Broadie and P. Glasserman, *A stochastic mesh method for pricing high-dimensional American options*, working paper, Columbia Univ. (1997).

[5] B. Dupire, *Pricing with a smile*, Risk **7** (1994) 18–20.

[6] B. Flesaker and L. P. Hughston, *International models for interest rates and foreign exchange*, Net Exposure **3** (1997) 55–79.

[7] P. Glasserman and X. Zhao, *Arbitrage-free discretization of log-normal forward Libor and swap rate models*, ICBI Global Derivatives Conference, Paris (1998).

[8] B. Goldys, M. Musiela and D. Sondermann, *Log-normality of rates and term structure models*, working paper, Univ. of New South Wales (1994).

[9] P. S. Hagan and D.E. Woodward, *Markov interest rate models*, preprint (1996).

[10] D. Heath, R. Jarrow and A. Morton, *Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation*, Econometrica **60** (1992) 77–80.

[11] L. P. Hughston (ed.), *Vasicek and Beyond, Approaches to Building and Applying Interest Rate Models*, Risk Publications (1996).

[12] P. Hunt, J. Kennedy and A. Pelsser, *Markov-functional interest rate models*, preprint (1998).

[13] F. Jamshidian, *Libor and swap markets models and measures*, Finance and Stochastics **1** (1997) 293–330.

[14] M. Musiela, *Nominal annual rates and log-normal volatility structures*, working paper, Univ. of New South Wales (1994).

[15] M. Musiela and M. Rutkowski, *Martingale methods in Financial Modelling*, Springer-Verlag (1997).

[16] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag (1994).

[17] K. Sandmann and D. Sondermann, *On the stability of log-normal interest rate systems and the pricing of Eurodollar futures*, working paper, Univ. of Bonn (1994).

[18] K. Sandmann, D. Sondermann and K. R. Miltersen, *Closed form term structure derivatives in a Health–Jarrow–Morton model with log-normal annually compounded interest rates*, in *Proc. of the Seventh Annual European Futures Research Symposium Bonn 1994*, Chicago Board of Trade (1994) 145–165.

[19] N. M. Seen, G. D. Byrne and E. M. Gelbard, *Gaussian quadratures*, Math. Computation **63** (1969) 661–671.