Beyond Hazard Rates: a New Framework for Credit-Risk Modelling

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Summary. A new approach to credit risk modelling is introduced that avoids the use of inaccessible stopping times. Default events are associated directly with the failure of obligors to make contractually agreed payments. Noisy information about impending cash flows is available to market participants. In this framework the market filtration is modelled explicitly, and is assumed to be generated by one or more independent market information processes. Each such information process carries partial information about the values of the market factors that determine future cash flows. For each market factor, the rate at which true information is provided to market participants concerning the eventual value of the factor is a parameter of the model. Analytical expressions that can be readily used for simulation are presented for the price processes of defaultable bonds with stochastic recovery. Similar expressions can be formulated for other debt instruments, including multi-name products. An explicit formula is derived for the value of an option on a defaultable discount bond. It is shown that the value of such an option is an increasing function of the rate at which true information is provided about the terminal payoff of the bond. One notable feature of the framework is that it satisfies an overall dynamic consistency condition that makes it suitable as a basis for practical modelling situations where frequent recalibration may be necessary.

Key words: Credit risk; credit derivatives; incomplete information; information-based asset pricing; market filtration; Bayesian inference; Brownian bridge process.

1 Introduction and Summary

Models for credit risk management and, in particular, for the pricing of credit derivatives are usually classified into two types: structural and reduced-form.

For general overviews of these approaches see, e.g., Jeanblanc & Rutkowski [25], Hughston & Turnbull [18], Bielecki & Rutkowski [4], Duffie & Singleton [12], Schönbucher [35], Bielecki et al. [3], Lando [29], and Elizalde [13].

There is a divergence of opinion in the literature as to the relative merits of the structural and reduced-form methodologies. Both approaches have strengths, but there are also shortcomings. Structural models attempt to account at some level of detail for the events leading to default (see, e.g., Merton [33], Black & Cox [5], Leland & Toft [30], Hilberink & Rogers [17], Jarrow et al. [23], Hull & White [19]). One general problem of the structural approach is that it is difficult in a structural model to deal systematically with the multiplicity of situations that can lead to default. For this reason structural models are sometimes viewed as unsatisfactory as a basis for a practical modelling framework, particularly when multi-name products such as $n^{\rm th}$-to-default swaps and collateralised debt obligations are involved.

Reduced-form models are more commonly used in practice on account of their tractability and because fewer assumptions are required about the nature of the debt obligations involved and the circumstances that might lead to default (see, e.g., Flesaker et al. [14], Jarrow & Turnbull [22], Duffie et al. [10], Jarrow et al. [20], Lando [28], Madan & Unal [31], Duffie & Singleton 1999, Madan & Unal [32], Jarrow & Yu [24]). Most reduced-form models are based on the introduction of a random time of default, modelled as the time at which the integral of a random intensity process first hits a certain critical level, this level itself being an independent random variable. An unsatisfactory feature of such intensity models is that they do not adequately take into account the fact that defaults are typically associated directly with a failure in the delivery of a contractually agreed cash flow—for example, a missed coupon payment. Another drawback of the intensity approach is that it is not well adapted to the situation where one wants to model the rise and fall of credit spreads—which can in reality be due in part to changes in the level of investor confidence.

The purpose of this paper is to introduce a new class of reduced-form credit models in which these problems are addressed. The modelling framework that we develop is broadly in the spirit of the incomplete-information approaches of Kusuoka [27], Duffie & Lando [9], Cetin et al. [6], Gieseke [15], Gieseke & Goldberg [16], Jarrow & Protter [21], and others. In our approach, no attempt is made as such to bridge the gap between the structural and the intensity-based models: rather, by abandoning the intensity-based approach we are able to formulate a class of reduced-form models that exhibit a high degree of intuitively natural behaviour and analytic tractability. Our approach is to build an economic model for the information that market participants have about future cash flows.

For simplicity we assume that the underlying default-free interest rate system is deterministic. The cash-flows of the debt obligation—in the case of a coupon bond, the coupon payments and the principal repayment—are modelled by a collection of random variables, and default is identified as the event of the first such payment that fails to achieve the terms specified in the contract. We assume that partial information about each such cash flow is available at earlier times to market participants; however, the information of the actual value of the payout is obscured by a Gaussian noise process that is conditioned to vanish once the time of the required cash flow is reached. We proceed under these assumptions to derive an exact expression for the bond price process.

In the case of a defaultable discount bond admitting two possible payouts (either the full principal, or some partial recovery payment), we derive an exact expression for the value of an option on the bond. Remarkably, this turns out to be a formula of the Black-Scholes type. In particular, the parameter σ that governs the rate at which the true value of the impending cash flow is revealed to market participants, against the background of the obscuring noise process, turns out to play the role of a volatility parameter in the associated option pricing formula; this interpretation is reinforced with the observation that the option price is an increasing function of this parameter.

The structure of the paper is as follows. In Section 2 we introduce the notion of an "information process" as a mechanism for modelling the market filtration. In the case of a credit-risky discount bond, the information process carries partial information about the terminal payoff, and it is assumed that the market filtration is generated by this process. The price process of the bond is obtained by taking the conditional expectation of the payout, and the properties of the resulting formula are analysed in the case of a binary payout. In Section 3 these results are extended to the situation of a defaultable discount bond with stochastic recovery, and a proof is given for the Markov property of the information process. In Section 4 we work out the dynamics of the price of a defaultable bond, and we show that the bond price process satisfies a diffusion equation. An explicit construction is presented for the Brownian motion that drives this process. This is given in Equation (20). We also work out an expression for the volatility of the bond price, given in Equation (28). In Section 5 we simulate the resulting price processes, and show that for suitable values of the information flow-rate parameter introduced in Equation (4) the default of a credit-risky discount bond occurs in effect as a "surprise". In Section 6 we establish a decomposition formula similar to that obtained by Lando [28] in the case of a bond with partial recovery. In Section 7 we show that the framework has a general "dynamical consistency" property. This has important implications for applications of the resulting models. In deriving these results we make use of special orthogonality properties of the Brownian bridge process (see, e.g., Yor [36, 37]). In Section 8 and Section 9 we consider options on defaultable bonds, and work out explicit formulae for the price processes of such options. In doing so we introduce a novel change-of-measure technique that enables us to calculate explicitly various expectations involving Brownian bridge processes. This technique is of some interest in its own right. One of the consequences of the fact that the bond price process is a diffusion in this framework is that explicit formulae can be worked out for the delta-hedging of option positions. In Section 10 we extend the general theory to the situation where there are multiple cash flows. We look in particular at the case of a coupon bond and derive an explicit formula for the price process of such a bond. In Section 11–Section 13 we consider complex debt instruments, and construct appropriate information-based valuation formulae for credit default swaps and portfolios of credit-risky instruments.

2 The Information-Based Approach

The object of this paper is to build an elementary modelling framework in which matters related to credit are brought to the forefront. Accordingly, we assume that the background default-free interest-rate system is deterministic: this assumption allows us to focus attention on credit-related issues; it also permits us to derive explicit expressions for certain types of credit derivative prices. The general philosophy is that we should try to sort out credit-related matters first, before attempting to incorporate a stochastic default-free system of interest rates.

As a further simplifying feature we take the view that default events are directly linked with anomalous cash flows. Thus default is not something that happens "in the abstract", but rather is associated with the failure of an agreed cash flow to materialise at the required time. In this way we improve the theory by eradicating the superfluous use of inaccessible stopping times.

Our theory is based on modelling the flow of partial information to market participants about impending debt payments. As usual, we model the financial markets with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$. The probability measure $\mathbb{Q}$ is understood to be the risk-neutral measure, and $\{\mathcal{F}_t\}$ is understood to be the market filtration. All asset-price processes and other information-providing processes accessible to market participants are adapted to $\{\mathcal{F}_t\}$. Contrary to the usual practice, we shall model $\{\mathcal{F}_t\}$ explicitly, rather than simply regarding it as being given.

The real probability measure does not enter into the present investigation. We assume the absence of arbitrage and the existence of a pricing kernel (cf. Cochrane [8], and references cited therein). With these conditions the existence of a unique risk-neutral measure is ensured, even though the market may be incomplete. We assume further that the default-free discount-bond system, denoted $\{P_{tT}\}_{0\leq t\leq T<\infty}$, can be written in the form $P_{tT} = P_{0T}/P_{0t}$, where the function $\{P_{0t}\}_{0\leq t<\infty}$ is differentiable and strictly decreasing, and satisfies $0 < P_{0t} \leq 1$ and $\lim_{t\to\infty} P_{0t} = 0$. Under these assumptions it follows that if the integrable random variable H_T represents a cash flow occurring at T, then its value H_t at any earlier time t is given by

$$H_t = P_{tT}\mathbb{E}\left[H_T|\mathcal{F}_t\right]. \tag{1}$$

Now let us consider, as a first example, the case of a simple credit-risky discount bond that matures at time T to pay a principal of h_1 dollars, if there is no default. In the event of default, the bond pays h_0 dollars, where $h_0 < h_1$. When two such payoffs are possible we call the resulting structure a "binary" discount bond. In the case given by $h_1 = 1$ and $h_0 = 0$ we call the resulting defaultable debt obligation a "digital" bond.

Let us write p_0 for the probability that the bond will pay h_0, and p_1 for the probability that the bond will pay h_1. The probabilities are risk-neutral, and hence build in any risk adjustments required in expectations needed in order to deduce prices. Thus if we write B_{0T} for the price at time 0 of the credit-risky discount bond we have

$$B_{0T} = P_{0T}(p_0 h_0 + p_1 h_1). \tag{2}$$

It follows that, providing we know the market data B_{0T} and P_{0T}, we can infer the *a priori* probabilities p_0 and p_1:

$$p_0 = \frac{1}{h_1 - h_0}\left(h_1 - \frac{B_{0T}}{P_{0T}}\right), \quad p_1 = \frac{1}{h_1 - h_0}\left(\frac{B_{0T}}{P_{0T}} - h_0\right). \tag{3}$$

Given this setup we proceed to model the bond-price process $\{B_{tT}\}_{0 \le t \le T}$. We suppose that the true value of H_T is not fully accessible until time T; that is, we assume H_T is $\mathcal{F}_T$-measurable, but not $\mathcal{F}_t$-measurable for $t < T$. We assume, nevertheless, that partial information regarding the value of the principal repayment H_T is available at earlier times. This information will in general be imperfect—one is looking into a crystal ball, so to speak, but the image is cloudy. Our model for such cloudy information will be of a simple type that allows for analytic tractability. More precisely, we assume that the following $\{\mathcal{F}_t\}$-adapted process is accessible to market participants:

$$\xi_t = \sigma H_T t + \beta_{tT}. \tag{4}$$

We call $\{\xi_t\}$ a *market information process*. The process $\{\beta_{tT}\}_{0 \le t \le T}$ appearing here is a standard Brownian bridge on the time interval $[0, T]$. Thus $\{\beta_{tT}\}$ is a Gaussian process satisfying $\beta_{0T} = 0$ and $\beta_{TT} = 0$, and such that $\mathbb{E}[\beta_{tT}] = 0$ and $\mathbb{E}[\beta_{sT}\beta_{tT}] = s(T - t)/T$ for all s, t satisfying $0 \le s \le t \le T$. We assume that $\{\beta_{tT}\}$ is independent of H_T, and thus represents pure noise. Market participants do not have direct access $\{\beta_{tT}\}$; that is to say, $\{\beta_{tT}\}$ is not assumed to be adapted to $\{\mathcal{F}_t\}$. We can think of $\{\beta_{tT}\}$ as representing the rumour, speculation, misrepresentation, overreaction, and general disinformation often occurring, in one form or another, in connection with financial activity.

Clearly the choice (4) can be generalised to include a wider class of models enjoying similar qualitative features. The present analysis will stick with (4) for the sake of definiteness and tractability. Indeed, the choice of $\{\xi_t\}$ defined by (4) has many attractive features, and can be regarded as a convenient "standard" model for an information process. The motivation for the use of a bridge process to represent the noise is intuitively as follows. We assume that initially all available market information is taken into account in the determination of the price; in the case of a credit-risky discount bond, the relevant information is embodied in the *a priori* probabilities. After the passage of time, new rumours and stories start circulating, and we model this by taking into account that the variance of the Brownian bridge steadily increases for the first half of its trajectory. Eventually, however, the variance falls to zero at the maturity of the bond, when the "moment of truth" arrives.

The parameter σ in this model represents the rate at which the true value of H_T is "revealed" as time progresses. Thus, if σ is low, then the value of H_T is effectively hidden until very near the maturity date of the bond; on the other hand, if σ is high, then we can think of H_T as being revealed quickly. The parameter σ has the units $\sigma \sim [\text{time}]^{-1/2}[\text{price}]^{-1}$. A rough measure for the timescale τ_D over which information is revealed is $\tau_D = 1/\sigma^2(h_1 - h_0)^2$. In particular, if $\tau_D \ll T$, then the value of H_T will be revealed rather early in the history of the bond, e.g., after the passage of a few multiples of τ_D. In this situation, if default is "destined" to occur, even though the initial value of the bond is high, then this will be signalled by a rapid decline in the bond price. On the other hand, if $\tau_D \gg T$, then H_T will only be revealed at the last minute, so to speak, and the default will come as a surprise. It is by virtue of this feature of the framework that the use of inaccessible stopping times can be avoided.

To make a closer inspection of the default timescale we proceed as follows. For simplicity, we assume that the only market information available about H_T at times earlier than T comes from observations of $\{\xi_t\}$. Specifically, if we denote by $\mathcal{F}_t^\xi$ the subalgebra of $\mathcal{F}_t$ generated by $\{\xi_s\}_{0\leq s\leq t}$, then our simplifying assumption is that $\{\mathcal{F}_t\} = \{\mathcal{F}_t^\xi\}$. That is to say, we assume that *the market filtration is generated by the information process.*

Now we are in a position to determine the price-process $\{B_{tT}\}_{0\leq t\leq T}$ for a credit-risky bond with payout H_T. In particular, the bond price is given by

$$B_{tT} = P_{tT} H_{tT}, \tag{5}$$

where H_{tT} denotes the conditional expectation of the bond payout:

$$H_{tT} = \mathbb{E}\left[H_T \middle| \mathcal{F}_t^\xi \right]. \tag{6}$$

It turns out that this conditional expectation can be worked out explicitly. The result is given by the following expression:

$$H_{tT} = \frac{p_0 h_0 \exp\left[\frac{T}{T-t}\left(\sigma h_0 \xi_t - \frac{1}{2}\sigma^2 h_0^2 t\right)\right] + p_1 h_1 \exp\left[\frac{T}{T-t}\left(\sigma h_1 \xi_t - \frac{1}{2}\sigma^2 h_1^2 t\right)\right]}{p_0 \exp\left[\frac{T}{T-t}\left(\sigma h_0 \xi_t - \frac{1}{2}\sigma^2 h_0^2 t\right)\right] + p_1 \exp\left[\frac{T}{T-t}\left(\sigma h_1 \xi_t - \frac{1}{2}\sigma^2 h_1^2 t\right)\right]}. \tag{7}$$

Thus we see that there exists a function $H(x, y)$ of two variables such that $H_{tT} = H(\xi_t, t)$, and as a consequence that the bond price can be expressed as a function of the market information.

Since $\{\xi_t\}$ is given by a combination of the random bond payout and an independent Brownian bridge, it is straightforward to simulate trajectories for $\{B_{tT}\}$. The bond price trajectories rise and fall randomly in line with the fluctuating information about the likely final payoff: in this respect, the resulting model is more satisfactory than an intensity-based model, where the only basis for shifts in credit spreads is through a random change in the default intensity.

The details of the derivation of the formula presented above will be given in the next section. First, however, let us verify that the expression in Equation (7) converges, as t approaches T, to the "actual" value of H_T. The proof is as follows. Suppose that the actual value of the payout is $H_T = h_0$. In that case $\xi_t = \sigma h_0 t + \beta_{tT}$, and by (7) we have

$$H_{tT} = \frac{p_0 h_0 \mathrm{e}^{\frac{T}{T-t}(\sigma h_0 \beta_{tT} + \frac{1}{2}\sigma^2 h_0^2 t)} + p_1 h_1 \mathrm{e}^{\frac{T}{T-t}(\sigma h_1 \beta_{tT} + \sigma h_0 h_1 t - \frac{1}{2}\sigma^2 h_1^2 t)}}{p_0 \mathrm{e}^{\frac{T}{T-t}(\sigma h_0 \beta_{tT} + \frac{1}{2}\sigma^2 h_0^2 t)} + p_1 \mathrm{e}^{\frac{T}{T-t}(\sigma h_1 \beta_{tT} + \sigma h_0 h_1 t - \frac{1}{2}\sigma^2 h_1^2 t)}}. \tag{8}$$

Dividing the numerator and the denominator by the coefficient of $p_0 h_0$ we obtain

$$H_{tT} = \frac{p_0 h_0 + p_1 h_1 \exp\left[-\frac{T}{T-t}\left(\frac{1}{2}\sigma^2 (h_1 - h_0)^2 t - \sigma (h_1 - h_0)\beta_{tT}\right)\right]}{p_0 + p_1 \exp\left[-\frac{T}{T-t}\left(\frac{1}{2}\sigma^2 (h_1 - h_0)^2 t - \sigma (h_1 - h_0)\beta_{tT}\right)\right]}. \tag{9}$$

We observe that as t approaches T the bond price converges to h_0, as required. A similar argument shows that if $H_T = h_1$, then the bond price converges to h_1. We note, in line with our heuristic arguments concerning τ_D, that the parameter $\sigma^2(h_1 - h_0)^2$ governs the speed at which H_{tT} converges to its terminal value. In particular, if the *a priori* probability of no default is high (say, $p_1 \approx 1$), and if σ is very small, and if in fact $H_T = h_0$, then it will only be when t is near T that serious decay in the bond price will set in.

3 Defaultable Discount Bond Price Processes

Let us now consider the more general situation where the discount bond pays the value $H_T = h_i$ ($i = 0, 1, \ldots, n$) with *a priori* probability $\mathbb{Q}[H_T = h_i] = p_i$. For convenience we assume $h_n > h_{n-1} > \cdots > h_1 > h_0$. The case $n = 1$ corresponds to the binary bond we have just considered. In the general situation we think of $H_T = h_n$ as the case of no default, and the other cases as various possible degrees of recovery.

Although for simplicity we work with a discrete payout spectrum for H_T, the continuous case can be formulated analogously. In that situation we assign a fixed *a priori* probability p_1 to the case of no default, and a continuous probability distribution function $p_0(x) = \mathbb{Q}[H_T < x]$ for values of x less than or equal to h, satisfying $p_1 + p_0(h) = 1$.

Now defining the information process $\{\xi_t\}$ as before, we want to find the conditional expectation (6). It follows from the Markovian property of $\{\xi_t\}$, which will be established below, that to compute (6) it suffices to take the conditional expectation of H_T with respect to the subalgebra generated by the random variable ξ_t alone. Therefore, writing $H_{tT} = \mathbb{E}[H_T|\xi_t]$ we have

$$H_{tT} = \sum_i h_i \pi_{it}, \tag{10}$$

where $\pi_{it} = \mathbb{Q}(H_T = h_i|\xi_t)$ is the conditional probability that the credit-risky bond pays out h_i. That is to say, $\pi_{it} = \mathbb{E}[\mathbf{1}_{\{H_T=h_i\}}|\xi_t]$.

To show that the information process satisfies the Markov property, we need to verify that

$$\mathbb{Q}(\xi_t \leq x|\mathcal{F}_s^\xi) = \mathbb{Q}(\xi_t \leq x|\xi_s) \tag{11}$$

for all $x \in \mathbb{R}$ and all s, t such that $0 \leq s \leq t \leq T$. It suffices to show that

$$\mathbb{Q}\left(\xi_t \leq x|\xi_s, \xi_{s_1}, \xi_{s_2}, \cdots, \xi_{s_k}\right) = \mathbb{Q}\left(\xi_t \leq x|\xi_s\right) \tag{12}$$

for any collection of times $t, s, s_1, s_2, \ldots, s_k$ such that $T \geq t > s > s_1 > s_2 > \cdots > s_k > 0$. First, we remark that for any times t, s, s_1 satisfying $t > s > s_1$ the Gaussian random variables β_{tT} and $\beta_{sT}/s - \beta_{s_1T}/s_1$ have vanishing covariance, and thus are independent. More generally, for $s > s_1 > s_2 > s_3$ the random variables $\beta_{sT}/s - \beta_{s_1T}/s_1$ and $\beta_{s_2T}/s_2 - \beta_{s_3T}/s_3$ are independent. Next, we note that $\xi_s/s - \xi_{s_1}/s_1 = \beta_{sT}/s - \beta_{s_1T}/s_1$. It follows that

$$\begin{aligned}
&\mathbb{Q}\left(\xi_t \leq x|\xi_s, \xi_{s_1}, \xi_{s_2}, \cdots, \xi_{s_k}\right) \\
&\quad = \mathbb{Q}\left(\xi_t \leq x \middle| \xi_s, \frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1}, \frac{\xi_{s_1}}{s_1} - \frac{\xi_{s_2}}{s_2}, \cdots, \frac{\xi_{s_{k-1}}}{s_{k-1}} - \frac{\xi_{s_k}}{s_k}\right) \\
&\quad = \mathbb{Q}\left(\xi_t \leq x \middle| \xi_s, \frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}, \frac{\beta_{s_1T}}{s_1} - \frac{\beta_{s_2T}}{s_2}, \cdots, \frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k}\right).
\end{aligned} \tag{13}$$

However, since ξ_s and ξ_t are independent of the remaining random variables $\beta_{sT}/s - \beta_{s_1T}/s_1$, $\beta_{s_1T}/s_1 - \beta_{s_2T}/s_2$, $\cdots$, $\beta_{s_{k-1}T}/s_{k-1} - \beta_{s_kT}/s_k$, the desired Markov property follows immediately.

Next we observe that the *a priori* probability p_i and the *a posteriori* probability π_{it} are related by the following version of the Bayes formula:

$$\mathbb{Q}(H_T = h_i|\xi_t) = \frac{p_i \rho(\xi_t|H_T = h_i)}{\sum_i p_i \rho(\xi_t|H_T = h_i)}. \tag{14}$$

Here the conditional density function $\rho(\xi|H_T = h_i)$, $\xi \in \mathbb{R}$, for the random variable ξ_t is defined by the relation

$$\mathbb{Q}\left(\xi_t \leq x|H_T = h_i\right) = \int_{-\infty}^{x} \rho(\xi|H_T = h_i)\, d\xi, \tag{15}$$

and is given explicitly by

$$\rho(\xi|H_T = h_i) = \frac{1}{\sqrt{2\pi t(T-t)/T}} \exp\left(-\frac{1}{2}\frac{(\xi - \sigma h_i t)^2}{t(T-t)/T}\right). \tag{16}$$

This expression can be deduced from the fact that conditional on $H_T = h_i$ the random variable ξ_t is normally distributed with mean $\sigma h_i t$ and variance $t(T-t)/T$. As a consequence of (14) and (16), we see that

$$\pi_{it} = \frac{p_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}. \tag{17}$$

It follows then, on account of (10), that

$$H_{tT} = \frac{\sum_i p_i h_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}. \tag{18}$$

This is the desired expression for the conditional expectation of the bond payoff. In particular, for the binary case $i = 0, 1$ we recover formula (7). The discount-bond price B_{tT} is then given by (5), with H_{tT} defined as in (18).

4 Defaultable Discount Bond Dynamics

In this section we analyse the dynamics of the defaultable bond price process $\{B_{tT}\}$ determined in Section 3. The key relation we need for working out the dynamics of the bond price is that the conditional probability $\{\pi_{it}\}$ satisfies a diffusion equation of the form

$$d\pi_{it} = \frac{\sigma T}{T-t}(h_i - H_{tT})\pi_{it}\, dW_t. \tag{19}$$

In particular, we can show that the process $\{W_t\}_{0\le t<T}$ arising here, defined by the expression

$$W_t = \xi_t + \int_0^t \frac{1}{T-s}\,\xi_s\, ds - \sigma T \int_0^t \frac{1}{T-s}\, H_{sT}\, ds, \tag{20}$$

is an $\{\mathcal{F}_t\}$-Brownian motion. The fact that $\{\pi_{it}\}$ satisfies (19) with $\{W_t\}$ defined as in (20) can be obtained directly from (17) by an application of Ito's lemma. We need to use the relation $(d\xi_t)^2 = dt$, which follows from the relation $(d\beta_{tT})^2 = dt$. The fact that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion can then be verified by showing that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-martingale and that $(dW_t)^2 = dt$. We proceed as follows.

To prove that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-martingale we need to show that $\mathbb{E}[W_u|\mathcal{F}_t] = W_t$ for $0 \le t \le u < T$. First we note that it follows from (20) and the Markov property of $\{\xi_t\}$ that

$$\mathbb{E}[W_u|\mathcal{F}_t] = W_t + \mathbb{E}\left[(\xi_u - \xi_t)|\xi_t\right] + \mathbb{E}\left[\int_t^u \frac{1}{T-s}\,\xi_s\,ds\bigg|\xi_t\right]$$
$$-\sigma T\,\mathbb{E}\left[\int_t^u \frac{1}{T-s}\,H_{sT}\,ds\bigg|\xi_t\right]. \qquad (21)$$

Formula (21) can be simplified if we recall that $H_{sT} = \mathbb{E}[H_T|\xi_s]$ and use the tower property in the last term on the right. Inserting the definition (4) into the second and third terms on the right in (21), we then have:

$$\mathbb{E}[W_u|\mathcal{F}_t] = W_t + \mathbb{E}[\sigma H_T u + \beta_{uT}|\xi_t] - \mathbb{E}[\sigma H_T t + \beta_{tT}|\xi_t]$$
$$+\sigma\mathbb{E}[H_T|\xi_t]\int_t^u \frac{s}{T-s}\,ds + \mathbb{E}\left[\int_t^u \frac{1}{T-s}\,\beta_{sT}\,ds\bigg|\xi_t\right]$$
$$-\sigma\mathbb{E}[H_T|\xi_t]\int_t^u \frac{T}{T-s}\,ds. \qquad (22)$$

It follows immediately that all of the terms involving the random variable H_T cancel each other in (22). This leads us to the following relation:

$$\mathbb{E}[W_u|\mathcal{F}_t] = W_t + \mathbb{E}[\beta_{uT}|\xi_t] - \mathbb{E}[\beta_{tT}|\xi_t] + \int_t^u \frac{1}{T-s}\,\mathbb{E}[\beta_{sT}|\xi_t]\,ds. \qquad (23)$$

Next we use the tower property and the independence of $\{\beta_{tT}\}$ and H_T to deduce that

$$\mathbb{E}[\beta_{uT}|\xi_t] = \mathbb{E}[\mathbb{E}[\beta_{uT}|H_T,\beta_{tT}]|\xi_t] = \mathbb{E}[\mathbb{E}\left[\beta_{uT}|\beta_{tT}\right]|\xi_t]\,. \qquad (24)$$

To calculate the inner expectation $\mathbb{E}[\beta_{uT}|\beta_{tT}]$ we use the fact that the random variable $\beta_{uT}/(T-u) - \beta_{tT}/(T-t)$ is independent of the random variable β_{tT}. This can be established by calculating their covariance, and using the relation $\mathbb{E}[\beta_{uT}\beta_{tT}] = t(T-u)/T$. We conclude after a short calculation that

$$\mathbb{E}[\beta_{uT}|\beta_{tT}] = \frac{T-u}{T-t}\,\beta_{tT}. \qquad (25)$$

Inserting this result into (24) we obtain the following formula:

$$\mathbb{E}[\beta_{uT}|\xi_t] = \frac{T-u}{T-t}\,\mathbb{E}[\beta_{tT}|\xi_t]. \qquad (26)$$

Applying this formula to the second and fourth terms on the right side of (23), we immediately deduce that $\mathbb{E}[W_u|\mathcal{F}_t] = W_t$. That establishes that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-martingale. Now we need to show that $(dW_t)^2 = dt$. This follows if we insert (4) into the definition of $\{W_t\}$ above and use again the fact that $(d\beta_{tT})^2 = dt$. Hence, by virtue of Lévy's criterion (see, e.g., Karatzas and Shreve [26]), we conclude that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion.

The Brownian motion $\{W_t\}$, the existence of which we have just established, can be regarded as part of the information accessible to market participants. Unlike β_{tT}, the value of W_t contains "real" information relevant to the bond payoff. It follows from (10) and (19) that for the discount bond dynamics we have

$$dB_{tT} = r_t B_{tT}\, dt + \Sigma_{tT}\, dW_t. \tag{27}$$

Here $r_t = -\partial \ln P_{0t}/\partial t$ is the short rate, and the absolute bond volatility Σ_{tT} is given by

$$\Sigma_{tT} = \frac{\sigma T}{T-t} P_{tT} V_{tT}, \tag{28}$$

where V_{tT} is the conditional variance of the terminal payoff H_T, defined by:

$$V_{tT} = \sum_i (h_i - H_{tT})^2 \pi_{it}. \tag{29}$$

We observe that as the maturity date is approached the absolute discount bond volatility will be high unless the conditional probability has most of its mass concentrated around the "true" outcome; this ensures the correct level can be eventually reached.

It follows as a consequence of (20) that $\{\xi_t\}$ satisfies the following stochastic differential equation:

$$d\xi_t = \frac{1}{T-t}\big(\sigma T H(\xi_t, t) - \xi_t\big)dt + dW_t. \tag{30}$$

We see that $\{\xi_t\}$ is a diffusion process; and since $H(\xi_t, t)$ is monotonic in its dependence on ξ_t, we deduce that $\{B_{tT}\}$ is also a diffusion process. To establish that B_{tT} is increasing in ξ_t we note that $P_{tT} H'(\xi_t, t) = \Sigma_{tT}$, where $H'(\xi, t) = \partial H(\xi, t)/\partial \xi$. It is interesting to observe that, in principle, instead of "deducing" the dynamics of $\{B_{tT}\}$ from the information-based arguments of the previous sections, we might have simply "proposed" on an *ad hoc* basis the one-factor diffusion process (30), noting that it leads to the correct default dynamics. This reasoning shows that our framework can be viewed, if desired, as leading to purely "classical" financial models, based on observable price processes.

5 Simulation of Bond Price Processes

The present framework allows for a highly efficient simulation methodology for the dynamics of defaultable bonds. In the case of a defaultable discount bond all we need to do is to simulate the dynamics of $\{\xi_t\}$. For each run of the simulation we choose at random a value for H_T (in accordance with the *a priori* probabilities), and a sample path for the Brownian bridge. That is to say, each simulation corresponds to a choice of $\omega \in \Omega$, and for each such choice we simulate the path $\xi_t(\omega) = \sigma t H_T(\omega) + \beta_{tT}(\omega)$ for $t \in [0, T]$. One way to simulate a Brownian bridge is to write $\beta_{tT} = B_t - (t/T)B_T$, where $\{B_t\}$ is a standard Brownian motion. It is straightforward to verify that if $\{\beta_{tT}\}$ is defined in this way then it has the correct mean and covariance. Since the bond price at t is expressed directly as a function of the random variable ξ_t, this means that pathwise simulation of the bond price trajectory is feasible for any number of recovery levels.

In Figure 1 we present some sample trajectories of the defaultable bond price process for various values of σ. These illustrations are fascinating inasmuch as they show that for small values of σ the default comes almost as a "surprise" near the maturity date of the bond; whereas for large values of σ the default, if it occurs, effectively takes place early in the life of bond.

6 Digital Bonds and Binary Bonds With Partial Recovery

It is interesting to ask whether in the case of a binary bond with partial recovery, with possible payoffs $\{h_0, h_1\}$, the bond-price process admits the representation

$$B_{tT} = P_{tT} h_0 + D_{tT}(h_1 - h_0). \tag{31}$$

Here D_{tT} denotes the value of a digital credit-risky bond that at maturity pays a unit of currency with probability p_1 and zero with probability $p_0 = 1 - p_1$. Thus h_0 is the amount that is guaranteed, whereas $h_1 - h_0$ is the part that is "at risk". It is well known that such a relation can be deduced in intensity-based models. The problem now is to find a process $\{D_{tT}\}$ consistent with our scheme such that (31) holds. It turns out that this can be achieved.

Suppose we consider a digital payoff structure $D_T \in \{0, 1\}$ for which the information-flow parameter σ is replaced by $\bar{\sigma} = \sigma(h_1 - h_0)$. In other words, in establishing the appropriate dynamics for $\{D_{tT}\}$ we "renormalise" σ by replacing it with $\bar{\sigma}$. The information available to market participants in the case of the digital bond is represented by the process $\{\bar{\xi}_t\}$ defined by

$$\bar{\xi}_t = \bar{\sigma} D_T t + \beta_{tT}. \tag{32}$$

It follows from (18) that the corresponding digital bond price is given by

$$D_{tT} = P_{tT} \frac{p_1 \exp\left[\frac{T}{T-t}\left(\bar{\sigma}\bar{\xi}_t - \frac{1}{2}\bar{\sigma}^2 t\right)\right]}{p_0 + p_1 \exp\left[\frac{T}{T-t}\left(\bar{\sigma}\bar{\xi}_t - \frac{1}{2}\bar{\sigma}^2 t\right)\right]}. \tag{33}$$

A calculation making use of (7) then allows us to confirm that (31) holds, where D_{tT} is given by (33). Thus even though *prima facie* the general binary bond process (7) does not appear to admit a decomposition of the form

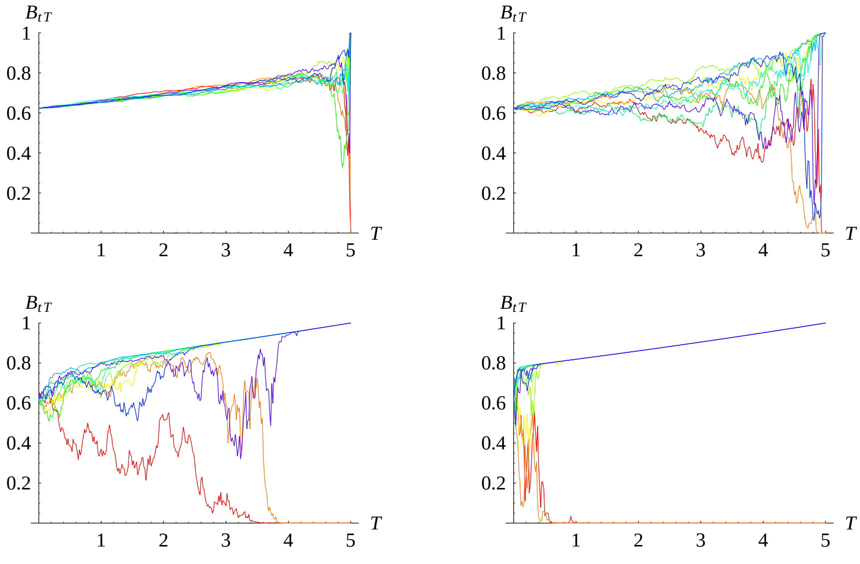

Fig. 1. *Bond price processes for various information-flow rates.* The parameter σ governs the rate at which information is released to market participants concerning the payout of a defaultable discount bond. Four values of σ are illustrated, given by .04, .2, 1, and 5. The bond has a maturity of five years, and the default-free interest-rate system has a constant short rate given by $r = 5\%$. The *a priori* probability of default is taken to be 20%. For low values of σ, collapse of the bond price, if it occurs, takes place only "at the last minute".

(31), we see that it does, once a suitably renormalised value for the market information flow-rate parameter has been introduced.

More generally, if the bond has a number of recovery levels, so the random variable H_T can take the values $\{h_0, h_1, \ldots, h_n\}$, then h_0 can be regarded as the "risk-free" component of the bond. The bond price process admits an additive decomposition into two parts, namely, a default-free discount bond that pays h_0, and a credit-risky discount bond that pays $\bar{h}_i = h_i - h_0$ with *a priori* probability p_i. This decomposition is given by

$$B_{tT} = P_{tT}h_0 + P_{tT}\frac{\sum_{i=1}^{n} p_i \bar{h}_i \exp\left[\frac{T}{T-t}\left(\sigma \bar{h}_i \bar{\xi}_t - \frac{1}{2}\sigma^2 \bar{h}_i^2 t\right)\right]}{p_0 + \sum_{i=1}^{n} p_i \exp\left[\frac{T}{T-t}\left(\sigma \bar{h}_i \bar{\xi}_t - \frac{1}{2}\sigma^2 \bar{h}_i^2 t\right)\right]}, \tag{34}$$

where $\bar{\xi}_t = \sigma(H_T - h_0)t + \beta_{tT}$ is the resulting price-shifted information process. Note that $\bar{h}_0 = 0$, which makes the summation in the numerator begin from $i = 1$; thus the second term in the right-hand side of (34) is the multiple recovery-level analogue of formula (33) above.

7 Dynamic Consistency and Model Calibration

The technique of renormalising the information flow rate has another useful application. It turns out that the information-based framework exhibits a property that might appropriately be called "dynamic consistency". Loosely speaking, the question is: if the information process is given as described, but then we re-initialise the model at some specified intermediate time, is it still the case that the dynamics of the model moving forward from that intermediate time can be consistently represented by an information process?

To answer this question we proceed as follows. First we define a standard Brownian bridge over the interval $[t, T]$ to be a Gaussian process $\{\gamma_{uT}\}_{t\leq u\leq T}$ satisfying $\gamma_{tT}=0$, $\gamma_{TT}=0$, $\mathbb{E}[\gamma_{uT}]=0$ for all $u\in[t,T]$, and $\mathbb{E}[\gamma_{uT}\gamma_{vT}]=(u-t)(T-v)/(T-t)$ for all u, v such that $t\leq u\leq v\leq T$. We make note of the following observation: *Let* $\{\beta_{tT}\}_{0\leq t\leq T}$ *be a standard Brownian bridge over the interval* $[0,T]$*, and define the process* $\{\gamma_{uT}\}_{t\leq u\leq T}$ *by*

$$\gamma_{uT}=\beta_{uT}-\frac{T-u}{T-t}\,\beta_{tT}. \tag{35}$$

Then $\{\gamma_{uT}\}_{t\leq u\leq T}$ *is a standard Brownian bridge over the interval* $[t,T]$*, and is independent of* $\{\beta_{sT}\}_{0\leq s\leq t}$. The result is easily proved by use of the covariance relation $\mathbb{E}[\beta_{tT}\beta_{uT}]=t(T-u)/T$. We need to recall also that a necessary and sufficient condition for a pair of Gaussian random variables to be independent is that their covariance should vanish. Now let the information process $\{\xi_s\}_{0\leq s\leq T}$ be given, and fix an intermediate time $t\in(0,T)$. Then for all $u\in[t,T]$ let us define a process $\{\eta_u\}_{0\leq u\leq T}$ by

$$\eta_u=\xi_u-\frac{T-u}{T-t}\,\xi_t. \tag{36}$$

We claim that $\{\eta_u\}$ is an information process over the time interval $[t,T]$. In fact, a short calculation establishes that

$$\eta_u=\tilde{\sigma}H_T(u-t)+\gamma_{uT}, \tag{37}$$

where $\{\gamma_{uT}\}_{t\leq u\leq T}$ is a standard Brownian bridge over the interval $[t,T]$, independent of H_T, and the new information flow rate is given by $\tilde{\sigma}=\sigma T/(T-t)$. The interpretation of these results is as follows. The "original" information process proceeds from time 0 up to time t. At that time we can re-calibrate the model by taking note of the value of the random variable ξ_t, and introducing the re-initialised information process $\{\eta_u\}$. The new information process depends on H_T; but since the value of ξ_t is supplied, the *a priori* probability distribution for H_T now changes to the appropriate *a posteriori* distribution consistent with the information gained from the knowledge of ξ_t at time t.

These interpretive remarks can be put into a more precise form as follows. Let $0\leq t\leq u<T$. What we want is a formula for the conditional probability π_{iu} expressed in terms of the information η_u and the "new" *a priori* probability π_{it}. Such a formula exists, and is given as follows:

$$\pi_{iu} = \frac{\pi_{it} \exp\left[\frac{T-t}{T-u}\left(\tilde{\sigma} h_i \eta_u - \frac{1}{2}\tilde{\sigma}^2 h_i^2 (u-t)\right)\right]}{\sum_i \pi_{it} \exp\left[\frac{T-t}{T-u}\left(\tilde{\sigma} h_i \eta_u - \frac{1}{2}\tilde{\sigma}^2 h_i^2 (u-t)\right)\right]}. \tag{38}$$

This remarkable relation can be verified by substituting the given expressions for π_{it}, η_u, and $\tilde{\sigma}$ into the right-hand side of (38). But (38) has the same structure as the original formula (17) for π_{it}, and thus we see that the model exhibits *manifest dynamic consistency.*

8 Options on Credit-Risky Bonds

We now turn to consider the pricing of options on credit-risky bonds. As we shall demonstrate shortly, in the case of a binary bond there is an exact solution for the valuation of European-style vanilla options. The resulting expression for the option price exhibits a structure that is strikingly analogous to that of the Black-Scholes option pricing formula.

We consider the value at time 0 of an option that is exercisable at a fixed time $t > 0$ on a credit-risky discount bond that matures at time $T > t$. The value C_0 of a call option is

$$C_0 = P_{0t}\mathbb{E}\left[(B_{tT} - K)^+\right], \tag{39}$$

where K is the strike price. Inserting formula (5) for B_{tT} into the valuation formula (39) for the option, we obtain

$$\begin{aligned} C_0 &= P_{0t}\,\mathbb{E}\left[(P_{tT}H_{tT} - K)^+\right] \\ &= P_{0t}\,\mathbb{E}\left[\left(\sum_{i=0}^{n} P_{tT}\pi_{it}h_i - K\right)^+\right] \\ &= P_{0t}\,\mathbb{E}\left[\left(\frac{1}{\Phi_t}\sum_{i=0}^{n} P_{tT}p_{it}h_i - K\right)^+\right] \\ &= P_{0t}\,\mathbb{E}\left[\frac{1}{\Phi_t}\left(\sum_{i=0}^{n}\Big(P_{tT}h_i - K\Big)p_{it}\right)^+\right]. \end{aligned} \tag{40}$$

Here the random variables p_{it}, $i = 0, 1, \ldots, n$, are the "unnormalised" conditional probabilities, defined by

$$p_{it} = p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \tfrac{1}{2}\sigma^2 h_i^2 t\right)\right]. \tag{41}$$

Then $\pi_{it} = p_{it}/\Phi_t$ where $\Phi_t = \sum_i p_{it}$, or, more explicitly,

$$\Phi_t = \sum_{i=0}^{n} p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \tfrac{1}{2}\sigma^2 h_i^2 t\right)\right]. \qquad (42)$$

Our plan now is to use the factor $1/\Phi_t$ appearing in (40) to make a change of probability measure on $(\Omega, \mathcal{F}_t)$. To this end, we fix a time horizon u at or beyond the option expiration but before the bond maturity, so $t \leq u < T$. We define a process $\{\Phi_t\}_{0\leq t\leq u}$ by use of the expression (42), where now we let t vary in the range $[0, u]$. It is a straightforward exercise in Ito calculus, making use of (30), to verify that

$$d\Phi_t = \sigma^2 \left(\frac{T}{T-t}\right)^2 H_{tT}^2 \Phi_t \, dt + \sigma \frac{T}{T-t} H_{tT} \Phi_t \, dW_t. \qquad (43)$$

It follows then that

$$d\Phi_t^{-1} = -\sigma \frac{T}{T-t} H_{tT} \Phi_t^{-1} dW_t, \qquad (44)$$

and hence that

$$\Phi_t^{-1} = \exp\left(-\sigma \int_0^t \frac{T}{T-s} H_{sT} \, dW_s - \tfrac{1}{2}\sigma^2 \int_0^t \frac{T^2}{(T-s)^2} H_{sT}^2 \, ds\right). \qquad (45)$$

Since $\{H_{sT}\}$ is bounded, and $s \leq u < T$, we see that the process $\{\Phi_s^{-1}\}_{0\leq s\leq u}$ is a martingale. In particular, since $\Phi_0 = 1$, we deduce that $\mathbb{E}^{\mathbb{Q}}[\Phi_t^{-1}] = 1$, where t is the option maturity date, and hence that the factor $1/\Phi_t$ in (40) can be used to effect a change of measure on $(\Omega, \mathcal{F}_t)$. Writing $\mathbb{B}_T$ for the new probability measure thus defined, we have

$$C_0 = P_{0t}\, \mathbb{E}^{\mathbb{B}_T}\left[\left(\sum_{i=0}^{n} \Big(P_{tT} h_i - K\Big) p_{it}\right)^+\right]. \qquad (46)$$

We call $\mathbb{B}_T$ the "bridge" measure because it has the special property that it makes $\{\xi_s\}_{0\leq s\leq t}$ a $\mathbb{B}_T$-Gaussian process with mean zero and covariance $\mathbb{E}^{\mathbb{B}_T}[\xi_r \xi_s] = r(T-s)/T$ for $0 \leq r \leq s \leq t$. In other words, with respect to the measure $\mathbb{B}_T$, and over the interval $[0, t]$, the information process has the law of a standard Brownian bridge over the interval $[0, T]$. Armed with this fact, we proceed to calculate the expectation in (46).

The proof that $\{\xi_s\}_{0\leq s\leq t}$ has the claimed properties under the measure $\mathbb{B}_T$ is as follows. For convenience we introduce a process $\{W_t^*\}_{0\leq t\leq u}$ which we define as the following Brownian motion with drift in the $\mathbb{Q}$-measure:

$$W_t^* = W_t + \sigma \int_0^t \frac{T}{T-s} H_{sT} \, ds. \qquad (47)$$

It is straightforward to check that on $(\Omega, \mathcal{F}_t)$ the process $\{W_t^*\}_{0\leq t\leq u}$ is a Brownian motion with respect to the measure defined by use of the density martingale $\{\Phi_t^{-1}\}_{0\leq t\leq u}$ given by (45). It then follows from the definition of $\{W_t\}$, given in (20), that

$$W_t^* = \xi_t + \int_0^t \frac{1}{T-s}\,\xi_s\,ds. \tag{48}$$

Taking the stochastic differential of each side of this relation, we deduce that

$$d\xi_t = -\frac{1}{T-t}\,\xi_t\,dt + dW_t^*. \tag{49}$$

We note, however, that (49) is the stochastic differential equation satisfied by a Brownian bridge (see, e.g., Karatzas & Shreve [26], Yor [37], Protter [34]) over the interval $[0,T]$. Thus we see that in the measure $\mathbb{B}_T$ defined on $(\Omega,\mathcal{F}_t)$ the process $\{\xi_s\}_{0\leq s\leq t}$ has the properties of a standard Brownian bridge over $[0,T]$, albeit restricted to the interval $[0,t]$.

For the transformation back from $\mathbb{B}_T$ to $\mathbb{Q}$ on $(\Omega,\mathcal{F}_t)$, the appropriate density martingale $\{\Phi_t\}_{0\leq t\leq u}$ with respect to $\mathbb{B}_T$ is given by:

$$\Phi_t = \exp\left(\sigma\int_0^t \frac{T}{T-s}\,H_{sT}\,dW_s^* - \tfrac{1}{2}\sigma^2\int_0^t \frac{T^2}{(T-s)^2}\,H_{sT}^2\,ds\right). \tag{50}$$

The crucial point that follows from this analysis is that the random variable ξ_t is $\mathbb{B}_T$-Gaussian. In the case of a binary discount bond, therefore, the relevant expectation for determining the option price can be carried out by standard techniques, and we are led to a formula of the Black-Scholes type. In particular, for a binary bond, Equation (46) reads

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\Big((P_{tT}h_1-K)p_{1t} + (P_{tT}h_0-K)p_{0t}\Big)^+\right], \tag{51}$$

where p_{0t} and p_{1t} are given by

$$\begin{aligned} p_{0t} &= p_0\exp\left[\tfrac{T}{T-t}\left(\sigma h_0\xi_t - \tfrac{1}{2}\sigma^2 h_0^2 t\right)\right], \\ p_{1t} &= p_1\exp\left[\tfrac{T}{T-t}\left(\sigma h_1\xi_t - \tfrac{1}{2}\sigma^2 h_1^2 t\right)\right]. \end{aligned} \tag{52}$$

To compute the value of (51) there are essentially three different cases that have to be considered: (i) $P_{tT}h_1 > P_{tT}h_0 > K$, (ii) $K > P_{tT}h_1 > P_{tT}h_0$, and (iii) $P_{tT}h_1 > K > P_{tT}h_0$. In case (i) the option is certain to expire in the money. Thus, making use of the fact that ξ_t is $\mathbb{B}_T$-Gaussian with mean zero and variance $t(T-t)/T$, we see that $\mathbb{E}^{\mathbb{B}_T}[p_{it}] = p_i$; hence in case (i) we have $C_0 = B_{0T} - P_{0t}K$. In case (ii) the option expires out of the money, and thus $C_0 = 0$. In case (iii) the option can expire in or out of the money, and there is a "critical" value of ξ_t above which the argument of (51) is positive. This is obtained by setting the argument of (51) to zero and solving for ξ_t. Writing $\bar{\xi}_t$ for the critical value, we find that $\bar{\xi}_t$ is determined by the relation

$$\frac{T}{T-t}\sigma(h_1-h_0)\bar{\xi}_t = \ln\left[\frac{p_0(P_{tT}h_0-K)}{p_1(K-P_{tT}h_1)}\right] + \tfrac{1}{2}\sigma^2(h_1^2-h_0^2)\tau, \tag{53}$$

where $\tau = tT/(T-t)$. Next we note that since ξ_t is $\mathbb{B}_T$-Gaussian with mean zero and variance $t(T-t)/T$, for the purpose of computing the expectation in (51) we can set $\xi_t = Z\sqrt{t(T-t)/T}$, where Z is $\mathbb{B}_T$-Gaussian with zero mean and unit variance. Then writing $\bar{Z}$ for the corresponding critical value of Z, we obtain

$$\bar{Z} = \frac{\ln\left[\frac{p_0(P_{tT}h_0-K)}{p_1(K-P_{tT}h_1)}\right] + \frac{1}{2}\sigma^2(h_1^2-h_0^2)\tau}{\sigma\sqrt{\tau}(h_1-h_0)}. \tag{54}$$

With this expression at hand, we can work out the expectation in (51). We are thus led to the following option pricing formula:

$$C_0 = P_{0t}\Big[p_1(P_{tT}h_1-K)N(d^+) - p_0(K-P_{tT}h_0)N(d^-)\Big]. \tag{55}$$

Here d^+ and d^- are defined by

$$d^{\pm} = \frac{\ln\left[\frac{p_1(P_{tT}h_1-K)}{p_0(K-P_{tT}h_0)}\right] \pm \frac{1}{2}\sigma^2(h_1-h_0)^2\tau}{\sigma\sqrt{\tau}(h_1-h_0)}. \tag{56}$$

A short calculation shows that the corresponding option delta, defined by $\Delta = \partial C_0/\partial B_{0T}$, is given by

$$\Delta = \frac{(P_{tT}h_1-K)N(d^+) + (K-P_{tT}h_0)N(d^-)}{P_{tT}(h_1-h_0)}. \tag{57}$$

This can be verified by using (3) to determine the dependency of the option price C_0 on the initial bond price B_{0T}.

It is interesting to note that the parameter σ plays a role like that of the volatility parameter in the Black-Scholes model. The more rapidly information is "leaked out" about the true value of the bond repayment, the higher the volatility. It is straightforward to verify that the option price has a positive vega, i.e. that C_0 is an increasing function of σ. This means that we can use bond option prices (or, equivalently, caps and floors) to back out an implied value for σ, and hence to calibrate the model. Writing $\mathcal{V} = \partial C_0/\partial\sigma$ for the corresponding option vega, we obtain the following positive expression:

$$\mathcal{V} = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-rt-\frac{1}{2}A}(h_1-h_0)\sqrt{\tau p_0 p_1(P_{tT}h_1-K)(K-P_{tT}h_0)}, \tag{58}$$

where

$$A = \frac{1}{\sigma^2\tau(h_1-h_0)^2}\left(\ln\left[\frac{p_1(P_{tT}h_1-K)}{p_0(K-P_{tT}h_0)}\right]\right)^2 + \tfrac{1}{4}\sigma^2\tau(h_1-h_0)^2. \tag{59}$$

We remark that in the more general case of a stochastic recovery, a semi-analytic option pricing formula can be obtained that, for practical purposes, can be regarded as fully tractable. In particular, starting from (46) we consider the case where the strike price K lies in the range $P_{tT}h_{k+1} > K > P_{tT}h_k$ for some value of $k \in \{0, 1, \ldots, n\}$. It is an exercise to verify that there exists a unique critical value of ξ_t such that the summation appearing in the argument of the $\max(x, 0)$ function in (46) vanishes. Writing $\bar{\xi}_t$ for the critical value, which can be obtained by numerical methods, we define the scaled critical value $\bar{Z}$ as before, by setting $\bar{\xi}_t = \bar{Z}\sqrt{t(T-t)/T}$. A calculation then shows that the option price is given by the following expression:

$$C_0 = P_{0t} \sum_{i=0}^{n} p_i \left(P_{tT}h_i - K\right) N(\sigma h_i \sqrt{\tau} - \bar{Z}). \tag{60}$$

9 Bond Option Price Processes

In the previous section we obtained the initial value C_0 of an option on a binary credit-risky bond. In the present section we determine the price process $\{C_s\}_{0 \le s \le t}$ of such an option. We fix the bond maturity T and the option maturity t. Then the price C_s of a call option at time $s \le t$ is given by

$$\begin{aligned} C_s &= P_{st}\, \mathbb{E}\left[(B_{tT} - K)^+ | \mathcal{F}_s\right] \\ &= \frac{P_{st}}{\Phi_s}\, \mathbb{E}^{\mathbb{B}_T}\left[\Phi_t (B_{tT} - K)^+ | \mathcal{F}_s\right] \\ &= \frac{P_{st}}{\Phi_s}\, \mathbb{E}^{\mathbb{B}_T}\left[\left(\sum_{i=0}^{n} \Big(P_{tT}h_i - K\Big) p_{it}\right)^+ \Bigg| \mathcal{F}_s\right]. \end{aligned} \tag{61}$$

We recall that p_{it}, defined in (41), is a function of ξ_t. The calculation can thus be simplified by use of the fact that $\{\xi_t\}$ is a $\mathbb{B}_T$-Brownian bridge. To determine the conditional expectation (61) we note that the $\mathbb{B}_T$-Gaussian random variable Z_{st} defined by

$$Z_{st} = \frac{\xi_t}{T-t} - \frac{\xi_s}{T-s} \tag{62}$$

is independent of $\{\xi_u\}_{0 \le u \le s}$. We can then express $\{p_{it}\}$ in terms of ξ_s and Z_{st} by writing

$$p_{it} = p_i \exp\left[\frac{T}{T-s}\sigma h_i T \xi_s - \tfrac{1}{2}\frac{T}{T-t}\sigma^2 h_i^2 t + \sigma h_i Z_{st} T\right]. \tag{63}$$

Substituting (63) into (61), we find that C_s can be calculated by taking an expectation involving the random variable Z_{st}, which has mean zero and variance v_{st}^2, given by

$$v_{st}^2 = \frac{t-s}{(T-t)(T-s)}. \tag{64}$$

In the case of a call option on a binary discount bond that pays h_0 or h_1, we can obtain a closed-form expression for (61). In that case the option price is given as follows:

$$C_s = \frac{P_{st}}{\Phi_s}\,\mathbb{E}^{\mathbb{B}_T}\left[\left((P_{tT}h_0 - K)p_{0t} + (P_{tT}h_1 - K)p_{1t}\right)^+ \middle| \mathcal{F}_t\right]. \tag{65}$$

Substituting (63) in (65) we find that the expression in the expectation is positive only if the inequality $Z_{st} > \bar{Z}$ is satisfied, where

$$\bar{Z} = \frac{\ln\left[\frac{\pi_{0s}(K-P_{tT}h_0)}{\pi_{1s}(P_{tT}h_1-K)}\right] + \frac{1}{2}\sigma^2(h_1^2-h_0^2)v_{st}^2 T}{\sigma v_{st}T(h_1-h_0)}. \tag{66}$$

It will be convenient to set $Z_{st} = v_{st}Z$, where Z is a $\mathbb{B}_T$-Gaussian random variable with zero mean and unit variance. The computation of the expectation in (65) then reduces to a pair of Gaussian integrals, and we obtain

$$C_s = P_{st}\Big[\pi_{1s}\,(P_{tT}h_1 - K)\,N(d_s^+) - \pi_{0s}\,(K - P_{tT}h_0)\,N(d_s^-)\Big], \tag{67}$$

where the conditional probabilities $\{\pi_{is}\}$ are as defined in (17), and

$$d_s^{\pm} = \frac{\ln\left[\frac{\pi_{1s}(P_{tT}h_1-K)}{\pi_{0s}(K-P_{tT}h_0)}\right] \pm \frac{1}{2}\sigma^2 v_{st}^2 T^2(h_1-h_0)^2}{\sigma v_{st}T(h_1-h_0)}. \tag{68}$$

We note that $d_s^+ = d_s^- + \sigma v_{st}T(h_1-h_0)$, and that $d_0^{\pm} = d^{\pm}$.

One important feature of the model worth pointing out in the present context is that a position in a bond option can be hedged with a position in the underlying bond. This is because the option price process and the underlying bond price process are one-dimensional diffusions driven by the same Brownian motion. Since C_t and B_{tT} are both monotonic in their dependence on ξ_t, it follows that C_t can be expressed as a function of B_{tT}; the delta of the option can then be obtained in the conventional way as the derivative of the option price with respect to the underlying. In the case of a binary bond, the resulting hedge ratio process $\{\Delta_s\}_{0\le s\le t}$ is given by

$$\Delta_s = \frac{(P_{tT}h_1 - K)N(d_s^+) + (K - P_{tT}h_0)N(d_s^-)}{P_{tT}(h_1-h_0)}. \tag{69}$$

This brings us to another interesting point. For certain types of instruments it may be desirable to model the occurrence of credit events (e.g., credit-rating downgrades) taking place at some time preceding a cash-flow date. In particular, we may wish to consider contingent claims based on such events. In the present framework we can regard such contingent claims as derivative structures for which the payoff is triggered by the level of $\{\xi_t\}$. For example, it may be that a credit event is established if B_{tT} drops below some specific level, or if the credit spread widens beyond some threshold. As a consequence, a number of different types of contingent claims can be valued by use of barrier option methods in this framework (cf. Albanese et al. [1], Chen & Filipović [7], Albanese & Chen [2]).

10 Coupon Bonds: the *X*-Factor Approach

The discussion so far has focused on simple structures, such as discount bonds and options on discount bonds. One of the advantages of the present approach, however, is that its tractability extends to situations of a more complex nature. In this section we consider the case of a credit-risky coupon bond. One should regard a coupon bond as being a rather complicated instrument from the point of view of credit risk management, since default can occur at any of the coupon dates. The market will in general possess partial information concerning all of the future coupon payments, as well as the principal payment.

As an illustration, we consider a bond with two payments remaining—a coupon H_{T_1} at time T_1, and a coupon plus the principal totalling H_{T_2} at time T_2. We assume that if default occurs at T_1, then no further payment is made at T_2. On the other hand, if the T_1-coupon is paid, default may still occur at T_2. We model this by setting

$$H_{T_1} = \mathbf{c} X_{T_1}, \qquad H_{T_2} = (\mathbf{c}+\mathbf{p}) X_{T_1} X_{T_2}, \tag{70}$$

where X_{T_1} and X_{T_2} are independent random variables each taking the values $\{0,1\}$, and the constants $\mathbf{c}$ and $\mathbf{p}$ denote the coupon and principal. Let us write $\{p_0^{(1)}, p_1^{(1)}\}$ for the *a priori* probabilities that $X_{T_1} = \{0,1\}$, and $\{p_0^{(2)}, p_1^{(2)}\}$ for the *a priori* probabilities that $X_{T_2} = \{0,1\}$. We introduce a pair of information processes

$$\xi_t^{(1)} = \sigma_1 X_{T_1} t + \beta_{tT_1}^{(1)} \quad \text{and} \quad \xi_t^{(2)} = \sigma_2 X_{T_2} t + \beta_{tT_2}^{(2)}, \tag{71}$$

where $\{\beta_{tT_1}^{(1)}\}$ and $\{\beta_{tT_2}^{(2)}\}$ are independent Brownian bridges, and σ_1 and σ_2 are parameters. Then for the credit-risky coupon-bond price process we have

$$S_t = \mathbf{c} P_{tT_1} \mathbb{E}\left[X_{T_1} \middle| \xi_t^{(1)}\right] + (\mathbf{c}+\mathbf{p}) P_{tT_2} \mathbb{E}\left[X_{T_1} \middle| \xi_t^{(1)}\right] \mathbb{E}\left[X_{T_2} \middle| \xi_t^{(2)}\right]. \tag{72}$$

The two conditional expectations appearing in this formula can be worked out explicitly using the techniques already described. The result is:

$$\mathbb{E}\left[X_{T_i} \middle| \xi_t^{(i)}\right] = \frac{p_1^{(i)} \exp\left[\frac{T_i}{T_i - t}\left(\sigma_i \xi_t^{(i)} - \frac{1}{2}\sigma_i^2 t\right)\right]}{p_0^{(i)} + p_1^{(i)} \exp\left[\frac{T_i}{T_i - t}\left(\sigma_i \xi_t^{(i)} - \frac{1}{2}\sigma_i^2 t\right)\right]}, \tag{73}$$

for $i = 1, 2$. It should be evident that in the case of a bond with two payments remaining we obtain a natural "two-factor" model—the factors being the two independent Brownian motions arising in connection with the information processes $\{\xi_t^{(i)}\}_{i=1,2}$. Similarly, if there are n outstanding coupons, we model the payments by $H_{T_k} = \mathbf{c}X_{T_1} \cdots X_{T_k}$ for $k \leq n-1$ and $H_{T_n} = (\mathbf{c}+\mathbf{p})X_{T_1} \cdots X_{T_n}$, and introduce the information processes

$$\xi_t^{(i)} = \sigma_i X_{T_i} t + \beta_{tT_i}^{(i)} \qquad (i = 1, 2, \ldots, n). \tag{74}$$

The case of n outstanding payments leads to an n-factor model. The independence of the random variables $\{X_{T_i}\}_{i=1,2,\ldots,n}$ implies that the price of a credit-risky coupon bond admits a closed-form expression analogous to (72).

With a slight modification of these expressions we can consider the case when there is partial recovery in the event of default. In the two-coupon example discussed above, for instance, we can extend the model by saying that in the event of default on the first coupon the effective recovery rate (as a percentage of coupon plus principal) is R_1; whereas in the case of default on the final payment the recovery rate is R_2. Then $H_{T_1} = \mathbf{c}X_{T_1} + R_1(\mathbf{c}+\mathbf{p})(1 - X_{T_1})$ and $H_{T_2} = \mathbf{c} + \mathbf{p}X_{T_1}X_{T_2} + R_2(\mathbf{c}+\mathbf{p})X_{T_1}(1 - X_{T_2})$. A further extension of this reasoning allows for the introduction of a stochastic recovery rates.

11 Credit Default Swaps

Swap-like structures can also be readily treated. For example, in the case of a basic credit default swap we have a series of premium payments, each of the amount $\mathbf{g}$, made to the seller of protection. The payments continue until the failure of a coupon payment in the reference bond, at which point a payment $\mathbf{n}$ is made to the buyer of protection.

As an illustration, we consider two reference coupons, letting X_{T_1} and X_{T_2} be the associated independent market factors, following the pattern of the previous example. We assume for simplicity that the default-swap premium payments are made immediately after the coupon dates. Then the value of the default swap, from the point of view of the seller of protection, is:

$$\begin{aligned} V_t = {} & \mathbf{g}P_{tT_1}\mathbb{E}\left[X_{T_1}\middle|\xi_t^{(1)}\right] - \mathbf{n}P_{tT_1}\mathbb{E}\left[1 - X_{T_1}\middle|\xi_t^{(1)}\right] \\ & + \mathbf{g}P_{tT_2}\mathbb{E}\left[X_{T_1}\middle|\xi_t^{(1)}\right]\mathbb{E}\left[X_{T_2}\middle|\xi_t^{(2)}\right] \\ & - \mathbf{n}P_{tT_2}\mathbb{E}\left[X_{T_1}\middle|\xi_t^{(1)}\right]\mathbb{E}\left[1 - X_{T_2}\middle|\xi_t^{(2)}\right]. \end{aligned} \tag{75}$$

After some rearrangement of terms, this can be expressed more compactly as follows:

$$\begin{aligned} V_t = {} & -\mathbf{n}P_{tT_1} + \left[(\mathbf{g}+\mathbf{n})P_{tT_1} - \mathbf{n}P_{tT_2}\right]\mathbb{E}\left[X_{T_1}\middle|\xi_t^{(1)}\right] \\ & + (\mathbf{g}+\mathbf{n})P_{tT_2}\mathbb{E}\left[X_{T_1}\middle|\xi_t^{(1)}\right]\mathbb{E}\left[X_{T_2}\middle|\xi_t^{(2)}\right], \end{aligned} \tag{76}$$

which can then be written explicitly in terms of the expressions given in (73). A similar approach can be adapted in the multi-name credit situation. The point that we would like to emphasise is that there is a good deal of flexibility available in the manner in which the various cash-flows can be modelled to depend on one another, and in many situations tractable expressions emerge that can be used as the basis for the modelling of complex credit instruments.

12 Baskets of Credit-Risky Bonds

We consider now the valuation problem for a basket of bonds where there are correlations in the payoffs. We shall obtain a closed-form expression for the value of a basket of defaultable bonds with various maturities.

For definiteness we consider a set of digital bonds. It will be convenient to label the bonds in chronological order with respect to maturity. We let H_{T_1} denote the payoff of the bond that expires first; we let H_{T_2} ($T_2 \geq T_1$) denote the payoff of the first bond that matures after T_1; and so on. In general the various bond payouts will not be independent. We propose to model this set of dependent random variables in terms of an underlying set of independent market factors. To achieve this we let X denote the random variable associated with the payoff of the first bond: $H_{T_1} = X$. The random variable X takes on the values $\{1, 0\}$ with *a priori* probabilities $\{p, 1-p\}$. The payoff of the second bond H_{T_2} can then be represented in terms of three independent random variables: $H_{T_2} = XX_1 + (1-X)X_0$. Here X_0 takes the values $\{1, 0\}$ with the probabilities $\{p_0, 1-p_0\}$, and X_1 takes the values $\{1, 0\}$ with the probabilities $\{p_1, 1-p_1\}$. Clearly, the payoff of the second bond is unity if and only if the random variables (X, X_0, X_1) take the values $(0,1,0)$, $(0,1,1)$, $(1,0,1)$, or $(1,1,1)$. Since these random variables are independent, the *a priori* probability that the second bond does not default is $p_0 + p(p_1 - p_0)$, where p is the *a priori* probability that the first bond does not default. To represent the payoff of the third bond we introduce four additional independent random variables:

$$\begin{aligned} H_{T_3} = {} & XX_1X_{11} + X(1-X_1)X_{10} + (1-X)X_0X_{01} \\ & +(1-X)(1-X_0)X_{00}. \end{aligned} \tag{77}$$

The market factors $\{X_{ij}\}_{i,j=0,1}$ here take the values $\{1, 0\}$ with probabilities $\{p_{ij}, 1-p_{ij}\}$. It is a matter of elementary combinatorics to determine the *a priori* probability that $H_{T_3} = 1$ in terms of p, $\{p_i\}$, and $\{p_{ij}\}$.

The scheme above can be extended to represent the payoff of a generic bond in the basket, with an expression of the following form:

$$H_{T_{n+1}} = \sum_{\{k_j\}=1,0} X^{\omega(k_1)} X_{k_1}^{\omega(k_2)} X_{k_1k_2}^{\omega(k_3)} \cdots X_{k_1k_2\cdots k_{n-1}}^{\omega(k_n)} X_{k_1k_2\cdots k_{n-1}k_n}. \tag{78}$$

Here, for any random variable X we define $X^{\omega(0)} = 1-X$ and $X^{\omega(1)} = X$. The point to observe is that if we have a basket of N digital bonds with arbitrary *a priori* default probabilities and correlations, then we can introduce $2^N - 1$ independent digital random variables to represent the N correlated random variables associated with the bond payoffs.

One advantage of the decomposition into independent market factors is that we retain analytical tractability for the pricing of the basket. In particular, since the random variables $\{X_{k_1 k_2 \cdots k_n}\}$ are independent, it is natural to introduce $2^N - 1$ independent Brownian bridges to represent the noise that veils the values of the independent market factors:

$$\xi_t^{k_1 k_2 \cdots k_n} = \sigma_{k_1 k_2 \cdots k_n} X_{k_1 k_2 \cdots k_n} t + \beta_{tT_{n+1}}^{k_1 k_2 \cdots k_n}. \tag{79}$$

The number of independent factors grows rapidly with the number of bonds in the portfolio. As a consequence, a market that consists of correlated bonds is in general highly incomplete. This fact provides an economic justification for the creation of products such as CDSs and CDOs that enhance the "hedgeability" of such portfolios.

13 Homogeneous Baskets

In the case of a "homogeneous" basket the number of independent factors determining the payoff of the portfolio can be reduced. We assume now that the basket contains n defaultable discount bonds, each maturing at time T, and each paying 0 or 1, with the same *a priori* probability of default. This situation is of interest as a first step in the analysis of the more general setup. Our goal is to model default correlations in the portfolio, and in particular to model the flow of market information concerning default correlation. Let us write H_T for the payoff at time T of the homogeneous portfolio, and set

$$H_T = n - X_1 - X_1X_2 - X_1X_2X_3 - \cdots - X_1X_2 \ldots X_n, \tag{80}$$

where the random variables $\{X_j\}_{j=1,2,\ldots,n}$, each taking the values $\{0, 1\}$, are assumed to be independent. Thus if $X_1 = 0$, then $H_T = n$; if $X_1 = 1$ and $X_2 = 0$, then $H_T = n-1$; if $X_1 = 1$, $X_2 = 1$, and $X_3 = 0$, then $H_T = n-2$; and so on. Now suppose we write $p_j = \mathbb{Q}(X_j = 1)$ and $q_j = \mathbb{Q}(X_j = 0)$ for $j = 1, 2, \ldots, n$. Then $\mathbb{Q}(H_T = n) = q_1$, $\mathbb{Q}(H_T = n-1) = p_1 q_2$, $\mathbb{Q}(H_T = n-2) = p_1 p_2 q_3$, and so on. More generally, we have $\mathbb{Q}(H_T = n-k) = p_1 p_2 \ldots p_k q_{k+1}$. Thus, for example, if $p_1 \ll 1$ but $p_2, p_3, \ldots, p_k$ are large, then we are in a situation of low default probability and high default correlation; that is to say, the probability of a default occurring in the portfolio is small, but conditional on at least one default occurring, the probability of several defaults is high.

The market will take a view on the likelihood of various numbers of defaults occurring in the portfolio. We model this by introducing a set of independent information processes $\{\eta_t^j\}$ defined by

$$\eta_t^j = \sigma_j X_j t + \beta_{tT}^j, \tag{81}$$

where $\{\sigma_j\}_{j=1,2,\ldots,n}$ are parameters, and $\{\beta^j_{tT}\}_{j=1,2,\ldots,n}$ are independent Brownian bridges. The market filtration $\{\mathcal{F}_t\}$ is generated collectively by $\{\eta^j_t\}_{j=1,2,\ldots,n}$, and for the portfolio value $H_t = P_{tT}\mathbb{E}\left[H_T|\mathcal{F}_t\right]$ we have

$$H_t = P_{tT}\Big[n - \mathbb{E}_t[X_1] - \mathbb{E}_t[X_1]\mathbb{E}_t[X_2] - \cdots - \mathbb{E}_t[X_1]\mathbb{E}_t[X_2]\ldots\mathbb{E}_t[X_n]\Big]. \quad (82)$$

The conditional expectations appearing here can be calculated by means of formulae established earlier in the paper. The resulting dynamics for $\{H_t\}$ can then be used to describe the evolution of correlations in the portfolio. For example, if $\mathbb{E}_t[X_1]$ is low and $\mathbb{E}_t[X_2]$ is high, then the conditional probability at time t of a default at time T is small; whereas if $\mathbb{E}_t[X_1]$ were to increase suddenly, then the conditional probability of two or more defaults at T would rise as a consequence. Thus, the model is sufficiently rich to admit a detailed account of the correlation dynamics of the portfolio. The losses associated with individual tranches can be identified, and derivative structures associated with such tranches can be defined. For example, a digital option that pays out in the event that there are three or more defaults has the payoff structure $H^{(3)}_T = X_1X_2X_3$. The homogeneous portfolio model has the property that the dynamics of equity-level and mezzanine-level tranches involve a relatively small number of factors. The market prices of tranches can be used to determine the *a priori* probabilities, and the market prices of options on tranches can be used to fix the information-flow parameters.

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