

# Target Volatility Option Pricing <sup>\*</sup>

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## Abstract

In this paper we present two methods for the pricing of Target Volatility Options (TVOs), a recent market innovation in the field of volatility derivatives. TVOs allow investors to take a joint view on the future price of a given underlying (e.g. stocks, commodities, etc) and its realized volatility. For example, a target volatility call pays at maturity the terminal value of the underlying minus the strike, floored at zero, scaled by the ratio of a given Target Volatility (an arbitrary constant) and the realized volatility of the underlying over the life of the option. TVOs are popular with investors and hedgers because they are typically cheaper than their vanilla equivalent. We present two approaches for the pricing of TVOs: a power series expansion and a Laplace transform method. We also provide both model dependent and model independent solutions. The pricing methodologies have been tested numerically and results are provided.

## 1 Introduction

Variance and volatility swaps were the first instances of volatility derivatives. They were introduced in the late nineties to allow investors to trade pure volatility risk (see Derman et al [7] for a detailed account).

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Over the past few years, volatility products have become very liquid and widely traded instruments. Investors use volatility derivatives to hedge the volatility risk of their portfolios or to speculate on future realized volatility levels.

Variance and volatility swaps have been extensively studied in the literature. Derman et al [7] show how to price and statically hedge variance swaps in a model independent fashion by investing in a portfolio of calls and puts when the underlying exhibits continuous sample paths. In a seminal paper, Carr and Lee [4] provide several methods for pricing and hedging a large class of functions on the quadratic variation. Prices and hedges of quadratic variation claims are expressed in terms of weighted portfolios of European contracts on the terminal value of the underlying. Fritz and Gatheral [11] study some ill-posed problems connected with the replication strategy suggested by Carr and Lee [4] for certain pay-off and suggest some regularization scheme.

A new type of volatility derivatives was introduced around 2008 under the name of Target Volatility Option (TVOs). TVOs allow investors to take a joint view on the realized volatility of a given underlying and its price. For example, a target volatility call pays at maturity the terminal value of the underlying  $S_T$  minus the strike  $K$ , floored at zero, rescaled by the ratio of a given Target Volatility (an arbitrary constant, say  $\bar{\sigma}$ ) and the realized volatility  $RV_T$  of the underlying over the life of the option,

$$\phi(S_T, RV_T) = \frac{\bar{\sigma}}{RV_T} (S_T - K)^+. \quad (1.1)$$

TVOs are popular with investors and hedgers because they are typically cheaper than vanilla options. As long as realized volatilities are lower than the target volatility, the pay-off of the former is higher than the pay-off of the corresponding vanilla option.

During the 2008 financial crisis for example, implied volatilities across asset classes experienced a steep increase, with a significant impact on option (long vega) costs. The generalized increase in implied volatilities was in part a consequence of higher expected future realized volatilities, but was also connected with dealers limits/reluctance to increase their short vega positions. TVOs were introduced to allow investors to take a bullish/bearish view on the underlying asset in an option format at a relatively low cost.

In this paper we provide two methodologies for the pricing of TVOs. We shall assume that volatility is independent from the Brownian motion driving the returns underlying asset. This assumption is in general quite restrictive and its relaxation will be topic of further research. The assumption of independence however allows us to reduce the TVO pricing problem to calculating the expectation of a portfolio of quadratic variation claims.

The paper is structured as follows: in the next section we state the main assumptions and introduce some notation. In section 3 we derive the first approximation method which is based on Taylor expansions. The price of a TVO at inception is approximated by a sum of integrals of certain functions of the underlying asset variance. The subsequent section extends the result of section 3 to a generic time  $t$  to take into account the effect of the cumulated variance. A

second pricing methodology based on the log-strike Laplace transform of the option pay-off is introduced in section 5. The results of the section can be applied to a large class of stochastic volatility models to obtain TVO prices by inverting numerically the Laplace transform of the claim value. In section 6, we provide a representation of the price of the TVO in terms of a weighted portfolio of vanilla calls and puts. In particular we show that the price of a TVO is equal to the expectation of some linear combination of functions of the terminal value  $S_T$  of the underlying. We then apply the Breeden and Litzenberg [3] formula to decompose the expectation above in a weighted portfolio of calls and puts and provide a formula for the weights. Numerical Results are provided in section 7. Both the Taylor and the Laplace transform methods exhibit high levels of accuracy. Proofs of the main results are provided in section 8.

## 2 Assumptions

Our market is represented by a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  satisfying the usual conditions. Throughout the paper we will assume that there exist a money market account process  $B_t$  paying zero interest rates. We shall also assume that there exists a probability measure  $\mathbb{Q}$  under which any non-dividend-paying asset  $S_t$  satisfies the stochastic equation of the form

$$dS_t = \sigma_t S_t dW_t, \quad (2.1)$$

where  $W_t$  is a  $\mathbb{Q}$ -Brownian motion and  $\sigma_t > 0$  is a stochastic volatility process. All expectations are taken with respect to the measure  $\mathbb{Q}$ .

We will restrict our attention to  $\sigma_t$  processes satisfying a diffusion equations of the type

$$d\sigma_t = \mu_t(\sigma_t, t)dt + \nu_t(\sigma_t, t)dZ_t, \quad \sigma_0 > 0 \quad (2.2)$$

where the  $\mathbb{Q}$ -Brownian motion  $Z_t$  is independent of  $W_t$ .

Let  $X_t = \log(S_t/S_0)$ . The *quadratic variation* of the process  $X_t$  is given by

$$\langle X \rangle_t = \int_0^t \sigma_u^2 du. \quad (2.3)$$

We shall require that the process  $S_t$  is such that its quadratic variation  $\langle X \rangle_t$  is bounded from below for any  $t > 0$ .

Define an arbitrary constant  $\bar{\sigma} > 0$ , which we shall refer to as the *Target Volatility*. A *target volatility call option* with strike  $K$  is a contingent claim on  $S_t$  and  $\langle X \rangle_t$  with time  $t$  price given by

$$\bar{C}_t^{TV}(S_t, K, \langle X \rangle_t) = E_t \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (S_T - K)^+ \right]. \quad (2.4)$$

Similarly the time- $t$  price of a put TVO can be obtained by calculating the following expectation:

$$\bar{P}_t^{TV}(S_t, K, \langle X \rangle_t) = E_t \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (K - S_T)^+ \right]. \quad (2.5)$$

TVOs allow option buyers to take a joint bet on the price of the underlying and its volatility. The target volatility typically represents the option buyer's expectation of the future average realized volatility of  $S_t$  during the tenor of the option. In particular, if volatility realizes at or below the target level, the pay-out of the option will be greater or equal than the pay-off of the corresponding vanilla option. If implied volatilities are relatively high compared to the buyer's expectations, TVOs provide a way to gain exposure to the underlying at a reduced premium.

### 3 Taylor expansion approximation

We begin this section with a simple motivating example. Consider the pricing of an at the money (ATM) call (or put) TVO. Using the well known Bachelier approximation formula, it is straightforward to see that the price of the ATM TVO is approximately equal to the price of an equivalent vanilla with implied volatility  $\bar{\sigma}$ :

$$\begin{aligned} \bar{C}_0^{TV}(S_0, S_0, 0) &= E_0 \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} E[(S_T - S_0)^+ | \mathcal{F}_T^\sigma] \right] \\ &= E_0 \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} C^{BS}(S_0, S_0, \langle X \rangle_T) \right] \\ &\approx S_0 E_0 \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} \sqrt{\frac{\langle X \rangle_T}{2\pi}} \right] \\ &= S_0 \bar{\sigma} \sqrt{\frac{T}{2\pi}} \\ &\approx C^{BS}(S_0, S_0, \bar{\sigma}^2 T). \end{aligned} \quad (3.1)$$

Here  $(\mathcal{F}^\sigma)_{t \geq 0}$  is the filtration generated by the volatility process and  $C^{BS}(S_0, K, x)$  is the Black-Scholes price of a vanilla European call option with cumulative variance  $x$  and spot  $S_0$ , that is

$$C^{BS}(S_0, K, x) \equiv S_0 N(d^+(x)) - KN(d^-(x)), \quad (3.2)$$

and

$$d^\pm(x) = \frac{\log(S_0/K) \pm x/2}{\sqrt{x}}, \quad (3.3)$$

where  $N(\cdot)$  is the cumulative normal distribution function.

A slightly more general version of the problem above involves calculating the initial price of TVOs with a generic strike  $K$ ,

$$\bar{C}_0^{TV}(S_0, K, 0) = E_0 \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} C^{BS}(S_0, K, \langle X \rangle_T) \right]. \quad (3.4)$$

The TVO price can be calculated by approximating the Black and Scholes formula appearing on the right-hand side of equation (3.4) by its Taylor expansion around the ATM level  $S_0$ . We

shall show that each term of the expansion can be written as an integral of some exponential function of the quadratic variation  $\langle X \rangle_T$ . Expectations of such quantities can be explicitly calculated for a large class of parametric models or can be derived using a non-parametric approach *a la* Breeden and Litzenberger [3]. The following Lemma allows us to express the Black and Scholes price as a weighted sum of functions of the cumulative variance:

**Lemma 3.1.** *The Black and Scholes (call) equation admits the following Taylor expansion as a function of the strike  $K$  around the ATM point  $S$ ,*

$$C^{BS}(S, K, x) = S - (S + K)N\left(-\frac{\sqrt{x}}{2}\right) + e^{-x/8} \sum_{j=0}^{f(n)} x^{-(1/2+j)} W^{n,j}(K) + O(n+3), \quad (3.5)$$

where

$$W^{n,j}(K) \equiv \frac{1}{\sqrt{2\pi}} \sum_{k=2j}^n (-1)^k c^{f(k)-j,k} \frac{(K-S)^{k+2}}{S^{k+1}(k+2)!}, \quad (3.6)$$

and

$$f(k) = \begin{cases} \frac{k}{2}, & k \text{ even}; \\ \frac{k-1}{2}, & k \text{ odd}. \end{cases} \quad (3.7)$$

Coefficients  $c^{j,n}$  can be derived explicitly by solving a simple recursive equation (see section 8 for details). In order to calculate the TVO price we need to substitute (3.5) back into (3.4). It is convenient to simplify the functions of the quadratic variation obtained as a consequence of the previous step using the results below:

**Lemma 3.2.** *For any  $x, r > 0$  the following equalities hold:*

$$\frac{1}{\sqrt{x}} N\left(-\frac{\sqrt{x}}{2}\right) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-(z+1/8)x}}{\sqrt{z+1/8}} dz, \quad (3.8)$$

and

$$x^{-r} = \frac{1}{r\Gamma(r)} \int_0^\infty e^{-z^{1/r}x} dz. \quad (3.9)$$

Substituting equation (3.5) into (3.4) and using (3.8) and (3.9), we have the following Proposition:

**Proposition 3.3.** *The price of a call TVO can be approximated by a linear combination of integrals of some exponential function of the quadratic variation,*

$$C_0^{TV}(K) \approx \bar{\sigma}\sqrt{T} \left[ \frac{2S_0}{\sqrt{\pi}} I_0^{1/2,0} - \frac{S_0 + K}{2\sqrt{\pi}} \Phi_0^{1,1/8} + \sum_{j=0}^{f(n)} \tilde{W}^{n,j}(K) I_0^{j+1,1/8} \right], \quad (3.10)$$

where we have defined

$$I_0^{r,a} \equiv \int_0^\infty E_0 \left[ e^{\lambda^{r,a}(z)\langle X \rangle_T} \right] dz \quad (3.11)$$

$$\Phi_0^{r,a} \equiv \int_0^\infty \frac{E_0 \left[ e^{\lambda^{1,1/8}(z)\langle X \rangle_T} \right]}{\sqrt{z+1/8}} dz, \quad (3.12)$$

$$\lambda^{r,a}(z) \equiv -(z^{1/r} + a), \quad (3.13)$$

and

$$\tilde{W}^{n,j} \equiv \frac{W^{n,j}(K)}{(j+1)!}. \quad (3.14)$$

The use of Fubini's Theorem to interchange the order of integration in the formula above is justified by the fact that the process  $(\langle X \rangle)_{t>0}$  is bounded from below. Integrals  $I_0^{r,a}$  and  $\Phi_0^{r,a}$  can be calculated explicitly for a variety of parametric models for which the Laplace transform of the quadratic variation is known in closed form. For example we could model the instantaneous variance process  $\sigma_t^2$  as a CIR process. More generally we can make use of the abundant literature on affine processes (see Duffie et al [8] for a detailed treatment) to derive closed form solutions for the price of the TVO.

In paragraph 6 we shall use a model independent approach in the spirit of Carr and Lee [4], to express integrals (3.11) and (3.12) and thus the TVO price as a weighted portfolio of traded options.

## 4 Taylor expansion for $t > 0$

So far we have dealt with the pricing problem at time zero. As variance cumulates during the life of the option, the pricing problem changes slightly and formulae become slightly more involved, although the solution remains similar in nature.

Let us consider the price of a TVO at time  $t > 0$ . We need to solve an expression of the form

$$\begin{aligned} \bar{C}^{TV}(S_t, K, \langle X \rangle_t) = & E_t \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (S_T - K)^+ \right] = \\ & E_t \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\epsilon_t + \langle X \rangle_T - \langle X \rangle_t}} C^{BS}(S_t, K, \langle X \rangle_T - \langle X \rangle_t) \right], \quad (4.1) \end{aligned}$$

where we have set  $\epsilon_t \equiv \langle X \rangle_t$ . Thanks to the Markovian structure of the stock price and volatility, the  $t > 0$  pricing problem is very similar to the one encountered in the previous section. However the presence of the term  $\epsilon_t$  causes a lack of symmetry between the powers of  $\langle X \rangle_T - \langle X \rangle_t$  in the numerator and the square root in the denominator when substituting the Black and Scholes formula with its Taylor series. In particular, after expanding the time- $t$  Black and Scholes formula around the strike  $K$ , we are left with calculating expressions of the form

$$q_1(x) \equiv \frac{N(-\sqrt{x}/2)}{\sqrt{\epsilon + x}}, \quad (4.2)$$

$$q_2(x) \equiv \frac{x^{-(j+1/2)}}{\sqrt{\epsilon + x}}. \quad (4.3)$$

In principle, we could represent  $q_1(x)$  and  $q_2(x)$  as double integrals of exponential functions of  $x$  by considering the numerator and denominator separately. However, because of singularities

in some of the integrals involved, this approach does not allow us to derive a model independent prices. An alternative approach is to consider a Taylor expansion of  $N(-\sqrt{x}/2)$  and  $x^{-(j+1/2)}$  around the point  $x + \epsilon$ :

$$q_1(x) = \frac{N\left(-\frac{\sqrt{\epsilon+x}}{2}\right)}{\sqrt{\epsilon+x}} + \frac{e^{-(\epsilon+x)/8}}{\sqrt{2\pi}} \sum_{i=0}^m \omega^{i,m}(\epsilon)(\epsilon+x)^{-(i+1)} + O(m+2), \quad (4.4)$$

where

$$\omega^{i,m}(\epsilon) \equiv \sum_{k=j}^m (-1)^{k+1} \gamma^{i,k} \frac{\epsilon^{k+1}}{k+1!}, \quad (4.5)$$

and  $\gamma^{j,k}$  satisfies the following recursion <sup>1</sup>:

$$\begin{aligned} \gamma^{0,0} &= -1/4 \\ \gamma^{0,k} &= \left(-\frac{1}{8}\right) \gamma^{0,k-1}, \quad k = 1 \dots m \\ \gamma^{k,k} &= (1/2 - k) \gamma^{k-1,k-1}, \quad k = 1 \dots m \\ \gamma^{j,k} &= \left(-\frac{1}{8}\right) \gamma^{j,k-1} + (1/2 - j) \gamma^{j-1,k-1}, \quad j = 1 \dots m, k = j + 1 \dots m. \end{aligned} \quad (4.6)$$

Similarly,

$$q_2(x) = \sum_{k=0}^m \zeta^{k,j}(\epsilon)(\epsilon+x)^{-(j+k+1)} + O(m+1) \quad (4.7)$$

where we have defined  $\zeta^{0,j}(\epsilon) = 1$  and

$$\zeta^{k,j}(\epsilon) = \frac{\epsilon^k}{k!} \prod_{i=0}^{k-1} (j+i+1/2) \quad (4.8)$$

for  $k \geq 1$ .

These approximations can be substituted in the Taylor expansion of the time- $t$  Black and Scholes price. By putting all terms  $\langle X \rangle_T - \langle X \rangle_t$  as common factor, after some rearrangement we obtain

**Proposition 4.1.**

$$C_t^{TV}(K) \approx \bar{\sigma} \sqrt{T} \left[ \frac{2S_t}{\sqrt{\pi}} I_t^{1/2,0,0} - \frac{S_t + K}{2\sqrt{\pi}} \Phi_t^{1,1/8} + \sum_{j=0}^{m+f(n)} \hat{W}_t^{n,m,j}(K, \langle X \rangle_t) I_t^{j+1,1/8,0} \right], \quad (4.9)$$

where

$$I_t^{r,a,b} = \int_0^\infty e^{-(z^{1/r}+b)\langle X \rangle_t} E_t \left[ e^{\lambda^{r,a}(\langle X \rangle_T - \langle X \rangle_t)} \right] dz, \quad (4.10)$$

$$\Phi_t^{1,a} = \int_0^\infty \frac{e^{-(z+a)\langle X \rangle_t}}{\sqrt{z+a}} E_t \left[ e^{\lambda^{1,a}(\langle X \rangle_T - \langle X \rangle_t)} \right] dz, \quad (4.11)$$

$$\lambda^{r,a}(z) \equiv -(z^{1/r} + a), \quad (4.12)$$

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<sup>1</sup>The closed form solution of the recursion is  $\gamma^{j,k} = -\frac{1}{4} \prod_{i=1}^j \frac{1-2i}{2} \left(-\frac{1}{8}\right)^{k-j} \binom{k}{j}$  for  $j \geq 1$  and  $k \geq j$ .

and the weights of the linear combination are given by

$$\begin{aligned} \hat{W}_t^{n,m,j}(K, \epsilon) &\equiv \frac{1}{(j+1)!} \left\{ -\frac{S_t + K}{\sqrt{2\pi}} e^{-\epsilon/8} \omega^{j,m}(\epsilon) \right. \\ &+ \sum_{k=0}^{\min(j,f(n))} W^{n,k}(K) \zeta^{j-k,k}(\epsilon) I_{j \leq m} \\ &\left. + \sum_{k=0}^{\min(m,f(n)-j+m)} W^{n,j-m+k}(K) \zeta^{m-k,j-m+k}(\epsilon) I_{j > m} \right\}. \end{aligned} \quad (4.13)$$

Note that to simplify the notation, we have imposed that the summation in the Taylor expansion of  $q_1(x)$  and  $q_2(x)$  is up to  $m$  for both functions.

## 5 TVO pricing using Laplace Transforms

An alternative approach to the use of Taylor series to derive the price of a TVO is based on Laplace transform techniques. In particular, we shall consider the Laplace transform in the log-strike of the pay-off. As we will show later in this section, this approach leads to very simple semi-analytical solutions which are also efficient from a computational point of view. The main drawback of the methodology is that our model independent approach cannot be applied.

Let us consider the pricing problem of a put TVO. It is convenient to express the pay-off of the option in terms of the log-strike  $k \equiv \log K$  and the log-terminal value  $s_T \equiv \log S_T$

$$\bar{P}_t(S_t, e^k, \langle X \rangle_t) = E_t \left[ \frac{\bar{\sigma} \sqrt{T}}{\sqrt{\langle X \rangle_T}} (e^k - S_T)^+ \right] \equiv P(k). \quad (5.1)$$

By Laplace transforming the option price in the log-strike, we can eliminate the max function appearing in the pay-off of the TVO. This will allow us to reduce the problem to the pricing of a quadratic variation claim rather than a joint claim on the terminal value of the stock and the quadratic variation.

For any complex  $\alpha$  such that  $\text{Re}(\alpha) > 1$ , the Laplace transform of  $P(k)$  is equal to

$$\begin{aligned} \hat{P}_t(\alpha) &\equiv \int_0^\infty e^{-\alpha k} P_t(k) dk \\ &= \bar{\sigma} \sqrt{T} S_t^{1-\alpha} E_t \left[ \frac{1}{\sqrt{\epsilon_t + \langle X \rangle_T - \langle X \rangle_t}} \frac{e^{(1-\alpha)(X_T - X_t)}}{\alpha(\alpha-1)} \right]. \end{aligned} \quad (5.2)$$

Using formula (3.9) we can represent the denominator of (5.2) in integral form:

$$\frac{1}{\sqrt{\epsilon + x}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2(\epsilon+x)} dz. \quad (5.3)$$

Under the assumption of independence of  $\sigma_t$  and  $W_t$ , after applying Fubini's Theorem, we can write  $\hat{P}(\alpha)$  in terms of  $S_t$  and the quadratic variation

$$\begin{aligned}
\hat{P}_t(\alpha) &= 2\bar{\sigma}\sqrt{\frac{T}{\pi}}S_t^{1-\alpha}\int_0^\infty e^{-z^2\langle X\rangle_t}E_t\left[\frac{e^{-z^2(\langle X\rangle_T-\langle X\rangle_t)+(1-\alpha)(X_T-X_t)}}{\alpha(\alpha-1)}\right]dz \\
&= 2\bar{\sigma}\sqrt{\frac{T}{\pi}}S_t^{1-\alpha}\int_0^\infty \frac{e^{-z^2\langle X\rangle_t}E_t[e^{\lambda_{z,\alpha}(\langle X\rangle_T-\langle X\rangle_t)}]}{\alpha(\alpha-1)}dz,
\end{aligned} \tag{5.4}$$

where we have defined the function

$$\lambda_{z,\alpha} = -(z^2 + \alpha(1 - \alpha)/2). \tag{5.5}$$

The Laplace transform (5.4) can be calculated explicitly for a variety stochastic volatility models (e.g., affine models). The price of the TVO options can be obtained by inverting (5.4) numerically. In particular, pricing the TVO option amounts to calculating the following integral

$$P_t(k) = \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}}\int_0^\infty\int_0^\infty e^{-z^2\langle X\rangle_t}\text{Re}\left(\frac{S_t^{1-a-iu}E_t[e^{\lambda_{z,a+iu}(\langle X\rangle_T-\langle X\rangle_t)}]}{(a+iu)(a+iu-1)}\right)\cos(uk)dzdu. \tag{5.6}$$

Numerical integration can be achieved by using, for example, the Abate-Whitt method [1] which is based on a smart application of the Trapezium rule combined with the Euler summation. The Laplace method is fast, easy to implement and produces accurate and stable results.

## 6 Robust pricing

Many authors in recent years (see for example Schoutens et al [15]) have highlighted the problems related to model dependence in the context of exotic option pricing and hedging. For example, local volatility models are known to lead to significantly different results from stochastic volatility models when pricing forward starting and cliquet options. Even within the stochastic volatility class, different models lead to different prices for path dependent options when calibrated to the same volatility surface.

In a pioneering paper, Breeden and Litzenberger [3] showed how to obtain model independent prices for European options with sufficiently smooth second derivatives by forming a portfolio of traded call and put options (see also Carr and Madan [5]). Volatility derivatives are in general path dependent and the results of [3] are not immediately applicable.

Carr and Lee [4] however proved that under the assumption of independence, expectations of exponential functions of the quadratic variation are equal to expectations of some function of the terminal value of the underlying. The results of [4] allow us to calculate the price of a special class of highly path dependent claims as if they were European options. For any complex  $\lambda$  we have

$$E_t[e^{\lambda(\langle X\rangle_T-\langle X\rangle_t)}] = E_t[e^{(X_T-X_t)p(\lambda)}] = E_t\left(\frac{S_T}{S_t}\right)^{p(\lambda)}, \tag{6.1}$$

where

$$p(\lambda) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}. \quad (6.2)$$

In the previous sections, we showed how to approximate the TVOs by a linear combination of integrals involving expressions of the form (6.1). We shall now show how to apply the Breeden-Litzenberger formula (6.3) to represent the TVO price in terms of traded option prices. If  $f(S)$  is a twice-differentiable pay-off, for some arbitrary constant  $\eta$  we have,

$$f(S) = f(\eta) + f'(\eta)[S - \eta] + \int_{\eta}^{\infty} f''(x)(S - x)^+ dx + \int_0^{\eta} f''(x)(x - S)^+ dx. \quad (6.3)$$

By taking conditional expectations on both sides of (6.3), we obtain a representation of the price of the claim  $f(S)$  in terms of vanilla call and put prices

$$E_t[f(S_T)] = f(\eta) + f'(\eta)[S_t - \eta] + \int_{\eta}^{\infty} f''(x)C_t^M(S_t, x)dx + \int_0^{\eta} f''(x)P_t^M(S_t, x)dx, \quad (6.4)$$

where  $C_t^M(S_t, x)$  and  $P_t^M(S_t, x)$  are prices of call and puts respectively with strike  $x$ .

## 6.1 Robust pricing via Taylor

In order to apply formula (6.4) to price TVOs, we need to find a function  $f(S_T)$  such that  $E_t[f(S_T)]$  equals the TVO price. As shown in formula (4.9), the price of a call TVO can be approximated by a linear combination of terms of the form  $I_t^{r,a,b}$  and  $\Phi_t^{1,a}$  for a set of integer valued  $r$  and real constants  $a$  and  $b$ . In order to derive model independent prices for TVOs thus, it is sufficient to apply Breeden-Litzenberger formula to  $I_t^{r,a,b}$  and  $\Phi_t^{1,a}$  and derive the portfolio of forwards, calls and puts yielding the model independent price.

Carr-Lee formula (6.1) can then be used to express  $I_t^{r,a,b}$  and  $\Phi_t^{1,a}$  as an expectation of some integral of the terminal value of the underlying  $S_T$ . In particular, it can be shown that

$$\begin{aligned} I_t^{r,a,b} &= \int_0^{\infty} e^{-(z^{1/r}+b)\langle X \rangle_t} E_t \left[ e^{\lambda^{r,a}(\langle X \rangle_T - \langle X \rangle_t)} \right] \\ &= E_t \int_0^{\infty} e^{-(z^{1/r}+b)\langle X \rangle_t} \operatorname{Re} \left( \frac{S_T}{S_t} \right)^{p^{r,a}(z)} dz, \end{aligned} \quad (6.5)$$

where we have applied Fubini's Theorem to exchange the order of integration and defined

$$p^{r,a} \equiv 1/2 \pm \sqrt{1/4 - 2z^{1/r} - 2a}. \quad (6.6)$$

Similarly,

$$\begin{aligned} \Phi_t^{1,a} &= \int_0^{\infty} \frac{e^{-(z+a)\langle X \rangle_t}}{\sqrt{z+a}} E_t \left[ e^{\lambda^{1,a}(\langle X \rangle_T - \langle X \rangle_t)} \right] dz \\ &= E_t \int_0^{\infty} \frac{e^{-(z+a)\langle X \rangle_t}}{\sqrt{z+a}} \operatorname{Re} \left( \frac{S_T}{S_t} \right)^{p^{1,a}(z)} dz. \end{aligned} \quad (6.7)$$

The last step is to define the following functions of the terminal value of the underlying  $S$ ,

$$\tilde{I}_t^{r,a,b}(S) \equiv \int_0^\infty \frac{e^{-(z+a)\langle X \rangle_t}}{\sqrt{z+a}} \operatorname{Re} \left( \frac{S}{S_t} \right)^{p^{1,a}(z)} dz \quad (6.8)$$

$$\tilde{\Phi}_t^{r,a,b}(S) \equiv \int_0^\infty e^{-(z^{1/r}+b)\langle X \rangle_t} \operatorname{Re} \left( \frac{S}{S_t} \right)^{p^{r,a}(z)} dz. \quad (6.9)$$

For  $t > 0$  the second derivative in  $S$  of functions  $I_t^{r,a,b}(S)$  and  $\Phi_t^{r,a,b}(S)$  is well defined as  $\langle X \rangle_t$  is strictly positive. The left-hand side of equalities (6.5) and (6.7) it is equal to the conditional expectation of functions  $\tilde{I}_t^{r,a,b}(S)$  and  $\tilde{\Phi}_t^{r,a,b}(S)$  respectively. We can now apply formula (6.4) to the functions above with  $\eta = S_t$  and substitute the result in (4.9) to obtain a representation of the TVO price in terms of traded call and put options.

Note that for  $t = 0$  integrals in expressions (6.8) and (6.9) do not converge and we cannot use Fubini's Theorem to interchange integrals in equations (6.5) and (6.7). It is therefore not always possible to calculate the TVO price using formula (6.4).

However, if the TVO contract is redefined as

$$\bar{C}_0^{TV}(S_0, K, 0) = E_0 \left[ \frac{\bar{\sigma}\sqrt{T}}{\sqrt{c + \langle X \rangle_T}} (S_T - K)^+ \right] \quad (6.10)$$

for some small arbitrary constant  $c$ , a "robust price", in the sense defined in this paper, does exist.

## 6.2 Robust pricing via Laplace transforms

Can the robust pricing approach be applied to Laplace transform method introduced in section 5? Using the Carr-Lee formula (6.1) we can express the Laplace transform of the TVO as a conditional expectation of some function of the terminal value  $S_T$ ,

$$P_t(k) = \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}} E_t \left[ \int_0^\infty e^{-z^2\langle X \rangle_t} \int_0^\infty \operatorname{Re} \left( \frac{S_t^{1-a-iu}(S_T/S_t)^{p_{z,\alpha}^\pm}}{(a+iu)(a+iu-1)} \right) \cos(uk) dudz \right], \quad (6.11)$$

where we have set  $\alpha = a + iu$  and defined

$$p_{z,\alpha}^\pm \equiv 1/2 \pm \sqrt{1/4 - 2z^2 - \alpha(1-\alpha)} \quad (6.12)$$

In principle, we could define the function  $f(S)$  as

$$f(S) \equiv \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}} \int_0^\infty e^{-z^2\langle X \rangle_t} \int_0^\infty \operatorname{Re} \left( \frac{S_t^{1-a-iu}(S/S_t)^{p_{z,\alpha}^\pm}}{(a+iu)(a+iu-1)} \right) \cos(uk) dudz. \quad (6.13)$$

However the second derivative of  $f(S)$  does not exist, because the  $dz$  the integral we obtain after differentiating (6.13) twice does not converge. It is therefore not possible to apply the Breeden-Litzenberger decomposition to derive a model independent price for the TVO using the Laplace transform method introduced in section 5.

## 7 Numerical results

We have implemented and tested the pricing formulae presented in the previous sections using MATLAB. In particular, we have assumed Heston dynamics for the underlying asset process,

$$dS_t = v_t^{1/2} S_t dW_t, \quad (7.1)$$

with CIR instantaneous variance process for the variance given by the SDE

$$dv_t = \kappa(\theta - v_t)dt + \eta v_t^{1/2} dZ_t. \quad (7.2)$$

As for the rest of the paper,  $W_t$  and  $Z_t$  are independent Brownian motions.

Unless otherwise specified, we have used the following parameters:

$$S_0 = 100, \quad v_0 = 0.2, \quad \bar{\sigma} = 0.1, \quad \kappa = 0.5, \quad \theta = 0.2, \quad \eta = 0.3. \quad (7.3)$$

In order to test the accuracy of our results we compared formulae (3.10), (4.9) and (5.6) with a Monte Carlo simulation of the pay-off across various strikes, maturities, and realized volatility values. Both the Taylor approximation and the Laplace transform formula proved to be very accurate.

**Table 1: Maturity  $T = 3$ ,  $t = 0$ . TVOs prices for different strikes, using different pricing methods**

Strike	Taylor polynomial of order n					Laplace Transform	Monte Carlo simulation
	n=1	n=2	n=3	n=4	n=5		
60	10.1534	11.1768	11.3814	11.4006	11.3958	11.3919	11.3897
80	8.4475	8.7033	8.7289	8.7301	8.7300	8.7301	8.7281
100	6.7416	6.7416	6.7416	6.7416	6.7416	6.7416	6.7415
120	5.0357	5.2915	5.2659	5.2671	5.2673	5.2672	5.2618
140	3.3298	4.3532	4.1485	4.1677	4.1725	4.1699	4.1643

**Table 2: Maturity  $T = 0.25$ ,  $t = 0$ . TVOs prices for different strikes, using different pricing methods**

Strike	Taylor polynomial of order n					Laplace Transform	Monte Carlo simulation
	n=1	n=2	n=3	n=4	n=5		
60	6.1232	9.4084	10.0654	9.9672	9.8556	9.0774	9.1058
80	4.0569	4.8782	4.9603	4.9542	4.9507	4.8989	4.9051
100	1.9906	1.9906	1.9906	1.9906	1.9906	1.9906	1.9906
120	-0.0757	0.7456	0.6634	0.6573	0.6608	0.6310	0.6291
140	-2.1420	1.1431	0.4861	0.3879	0.4994	0.1706	0.1703

**Table 3: Maturity  $T = 4$ ,  $t = 3$ ,  $\langle X \rangle_t = 0.3$ . TVOs prices for different strikes, using different pricing methods.**

Strike	Taylor polynomial of order n					Laplace Transform	Monte Carlo simulation
	n=1	n=2	n=3	n=4	n=5		
60	9.7467	11.4962	11.8461	11.8810	11.8740	11.9816	11.9827
80	7.4080	7.8454	7.8892	7.8913	7.8911	7.8390	7.8400
100	5.0694	5.0694	5.0694	5.0694	5.0694	4.9175	4.9067
120	2.7307	3.1681	3.1243	3.1265	3.1267	3.0313	3.0338
140	0.3920	2.1415	1.7916	1.8266	1.8336	1.8690	1.8680

**Table 4: Maturity  $T = 5$   $T = 2$ ,  $K = 100$ ,  $S_t = 120$ . TVOs prices for various realized volatility levels.**

$\langle X \rangle_t$	Taylor polynomial of order n					Laplace Transform	Monte Carlo simulation
	n=1	n=2	n=3	n=4	n=5		
0.2	10.7404	10.9632	10.9817	10.9828	10.9828	10.9780	10.9772
0.4	9.5530	9.7438	9.7597	9.7607	9.7607	9.7416	9.7391
0.6	8.7105	8.8795	8.8935	8.8945	8.8945	8.8601	8.8577
0.8	8.0694	8.2222	8.2349	8.2358	8.2358	8.1872	8.1894
1	7.5591	7.6993	7.7110	7.7118	7.7118	7.6503	7.6416

## 8 Appendix

### Proposition 3.3.

*Proof.* Consider first the Taylor expansion of the Black and Scholes formula  $C^{BS}(S, K, x) \equiv C(K)$  with respect to the strike  $K$  around the ATM point  $S$

$$C(K) = C(S) + C^{(1)}(K - S_0) + \sum_{k=0}^{\infty} C^{(k+2)}(S) \frac{(K - S)^{k+2}}{(k+2)!}, \quad (8.1)$$

where  $C^i(S)$  represents the  $i^{th}$  derivative with respect to the strike  $K$  evaluated at the ATM point  $S$ . Following Estrella [10], the Taylor series converges for  $0 < K < 2S$ , and it is possible to derive the generic expression for  $C^{k+2}(S)$  for all  $k \geq 0$ :

$$C^{k+2}(S) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\hat{\sigma}^2}{8}\right) \frac{P_k(d^+)}{S^{k+1} \hat{\sigma}^{k+1}} (-1)^k, \quad (8.2)$$

where we have defined the time scaled volatility  $\hat{\sigma} \equiv \sigma\sqrt{t} = x$ . It can be shown that  $P_n(d^+)$  satisfies the following recursive equation:  $P_0(d^+) = 1$  and

$$P_k(d^+) = (d^+ + k\hat{\sigma})P_{k-1}(d^+) - P'_{k-1}(d^+) \quad (8.3)$$

where  $d^+$  is defined in (3.3). Noting that for  $K = S$  it is  $d^+ = \hat{\sigma}/2$ , we can write the generic term  $P_n$  as a sum of powers of the volatility term, namely

$$P_k = \sum_{j=0}^{f(k)} c^{j,k} \hat{\sigma}^{\gamma(j,k)}, \quad (8.4)$$

where  $c^{j,k}$  is the  $j^{\text{th}}$  term of the polynomial  $P_k$ ,  $f(k)$  is defined in (3.7) and

$$\gamma(j,k) = \begin{cases} 2j, & k \text{ even;} \\ 2j + 1, & k \text{ odd.} \end{cases} \quad (8.5)$$

Polynomials  $P_k$  consist of a sum of even (odd) powers of  $\hat{\sigma}$  for  $k$  even (odd) and are of degree  $k$ . The scaled polynomials

$$\bar{P}_k \equiv \frac{P_k}{\hat{\sigma}^{k+1}} \quad (8.6)$$

consist only of odd powers of the volatility term  $\hat{\sigma}$  and are equal to

$$\bar{P}_k = \sum_{j=0}^{f(k)} c^{f(k)-j,k} \hat{\sigma}^{-(1+2j)}. \quad (8.7)$$

Substituting (8.7) into (8.2) and using the result in the Taylor expansion of the Black and Scholes formula, we obtain

$$C(K) = C(S) + C^{(1)}(K - S) + e^{-\hat{\sigma}^2/8} \sum_{k=0}^n (-1)^k \frac{(K - S)^{k+2}}{(k + 2)!} \sum_{j=0}^{f(k)} c^{f(k)-j,n} \hat{\sigma}^{-(1+2j)} + O(n + 3). \quad (8.8)$$

Inverting the order of summation, taking  $\hat{\sigma}$  as a common factor and using the definition of  $C^0(S)$  and  $C^1(S)$

$$\begin{aligned} C(K) &= S \left\{ N\left(\frac{\hat{\sigma}}{2}\right) - N\left(-\frac{\hat{\sigma}}{2}\right) \right\} - N\left(-\frac{\hat{\sigma}}{2}\right) (K - S) \\ &+ \frac{e^{-\hat{\sigma}^2/8}}{\sqrt{2\pi}} \sum_{j=0}^{f(n)} \hat{\sigma}^{-(1+2j)} \sum_{k=2j}^n (-1)^k c^{f(k)-j,k} \frac{(K - S)^{k+2}}{S^{k+1}(k + 2)!} + O(n + 3). \end{aligned} \quad (8.9)$$

Formula (3.5) follows. □

**Lemma 3.2.**

*Proof.* For any  $a > 0$ , the Laplace transform of the function  $g(z) \equiv 1/\sqrt{\pi(z+a)}$  is equal to

$$\begin{aligned}\hat{g}(x, a) &\equiv \int_0^\infty \frac{e^{-xz}}{\sqrt{\pi(z+a)}} dz \\ &= \frac{e^{xa}}{\sqrt{x}} \operatorname{erfc}(\sqrt{ax}) \\ &= \frac{2e^{ax}}{\sqrt{x}} N\left(-\frac{\sqrt{x}}{2}\right).\end{aligned}\tag{8.10}$$

Setting  $a = 1/8$  and rearranging we obtain formula (3.8).

The proof of the second equality can be found in Schrueger [16].  $\square$

**Proposition 4.1 .**

*Proof.* Let us first calculate the quantity

$$\tilde{C}^{BS}(S, K, x, \epsilon) \equiv \frac{C^{BS}(S, K, x)}{\sqrt{\epsilon+x}}.\tag{8.11}$$

Substituting (3.5) in the right-hand side of (8.11) and using (4.4) and (4.7), we can approximate the rescaled Black and Scholes price (8.11) as follows:

$$\begin{aligned}\tilde{C}^{BS}(S, K, x, \epsilon) &\approx \frac{S}{\sqrt{\epsilon+x}} - \frac{(S+K)}{\sqrt{\epsilon+x}} N\left(-\frac{\sqrt{\epsilon+x}}{2}\right) \\ &\quad - (S+K)e^{-(x+\epsilon)/8} \sum_{j=0}^m \omega^{j,m} (\epsilon+x)^{-(1+j)} + e^{(-x/8)} \sum_{j=0}^{f(n)} W^{n,j}(K) \sum_{k=0}^m \zeta^{k,j} (\epsilon+x)^{-(1+k+j)}.\end{aligned}\tag{8.12}$$

Expanding the sum in the equation above and putting all terms in  $(\epsilon+x)^{-(1+j)}$  as common factor, after a bit of algebra we obtain

$$\begin{aligned}\tilde{C}^{BS}(S, K, x) &\approx \frac{S}{\sqrt{\epsilon+x}} - \frac{(S+K)}{\sqrt{\epsilon+x}} N\left(-\frac{\sqrt{\epsilon+x}}{2}\right) \\ &\quad - (S+K)e^{-x/8} \sum_{j=0}^m (j+1)! \hat{W}(n, m, j)(K, \epsilon) (\epsilon+x)^{-(1+j)}.\end{aligned}\tag{8.13}$$

The time- $t$  TVO price is equal to the  $t$ -conditional expectation

$$C_t^{TV}(K) = E_t \left[ \tilde{C}^{BS}(S_t, K, \langle X \rangle_T - \langle X \rangle_t, \langle X \rangle_t) \right];\tag{8.14}$$

substituting (8.13) on the right-hand side of (8.14), making use of integral representations (3.9) and (3.2) and applying Fubini's Theorem to exchange the expectation with the integral in  $dz$ , we obtain the desired result.  $\square$

**Applicability of Fubini's Theorem to formulae (6.5) and (6.7).**

*Proof.* In order to justify the interchange of the order of integration in equation (6.5), it is sufficient to prove that that

$$\int_0^\infty E_t \left[ \left| e^{-(z^{1/r}+b)\langle X \rangle_t} \operatorname{Re} \left( \frac{S_T}{S_t} \right)^{1/2 \pm \sqrt{1/4 - 2z^{1/r} - 4a}} \right| \right] dz < \infty. \quad (8.15)$$

Without loss of generality, set  $b = 0$ . For  $z \rightarrow \infty$  the function

$$\begin{aligned} I(z) &\equiv E_t \left[ \left| e^{-z^{1/r}\langle X \rangle_t} \operatorname{Re} \left( \frac{S_T}{S_t} \right)^{1/2 \pm \sqrt{1/4 - 2z^{1/r} - 4a}} \right| \right] \\ &\sim e^{-z^{1/r}\langle X \rangle_t} E_t \left( \frac{S_T}{S_t} \right)^{1/2} \rightarrow 0 \end{aligned} \quad (8.16)$$

decays exponentially since  $E_t (S_T/S_t)^{1/2} < \infty$ .

□

The justification of equality (6.7) is similar.

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