

# **Symmetry methods for quadratic Gaussian models of interest rate and FX processes**

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12 October 2010

# Outline

- Motivation
- The quadratic Gaussian distribution
- The quadratic Gaussian process
- The quadratic Gaussian model

# Part 1

# Motivation

# What is the quadratic Gaussian model?

$$\exp\left[\psi \cdot x + \frac{1}{2} x \cdot \zeta x\right]$$

$x$  : vector  
 $\psi$  : vector  
 $\zeta$  : symmetric matrix

- All discount factors and FX indexes are exponential-quadratic functions of Gaussian variables.
- Consistent, arbitrage-free, multi-factor term structure model.
- An extension of the Hull-White model of interest rates and the Black-Scholes model of FX indexes.
- Extension provides more flexibility for modelling volatility smile.

# Why the quadratic Gaussian model?

Efficient and robust implementation:

- Simple, closed-form expressions for discount factors and FX indexes.
- Simple, closed-form expressions for the measure changes between arbitrary terminal measures in any currency.
- Simple, closed-form expressions for vanilla option prices and common convexity adjustments.

# Why the quadratic Gaussian model?

## Models market features:

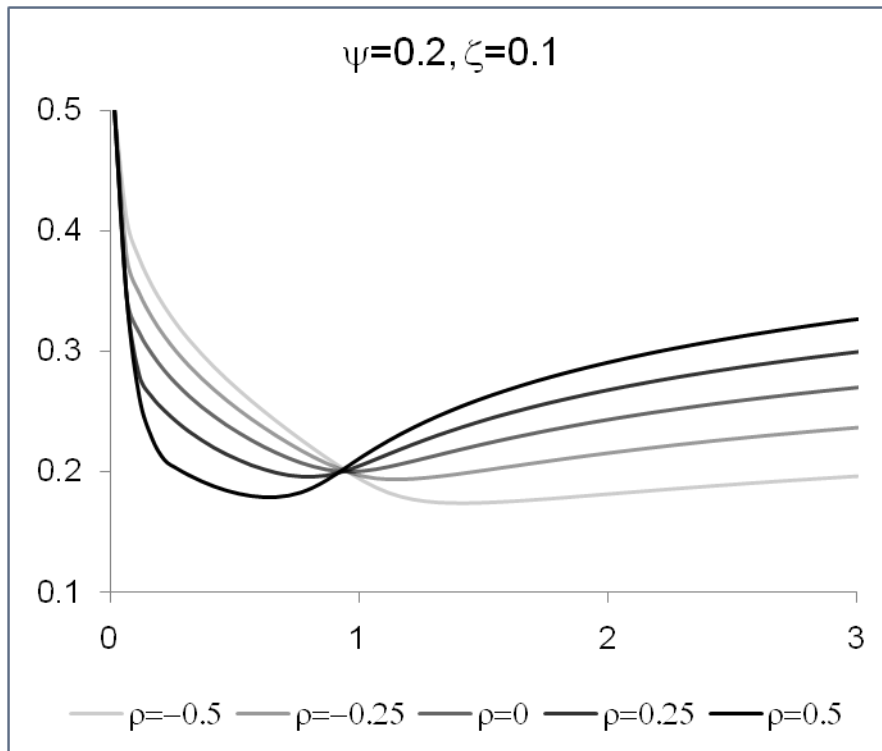
- Matches market volatility smiles for interest rate and FX index options.
- Mixture of 'stochastic volatility' and 'local volatility' characteristics enables control over the forward volatility smile dynamics.
- Multi-factor model allows mean reversion over different time horizons and decorrelation between interest rates, FX indexes and stochastic volatilities.

# Why the quadratic Gaussian model?

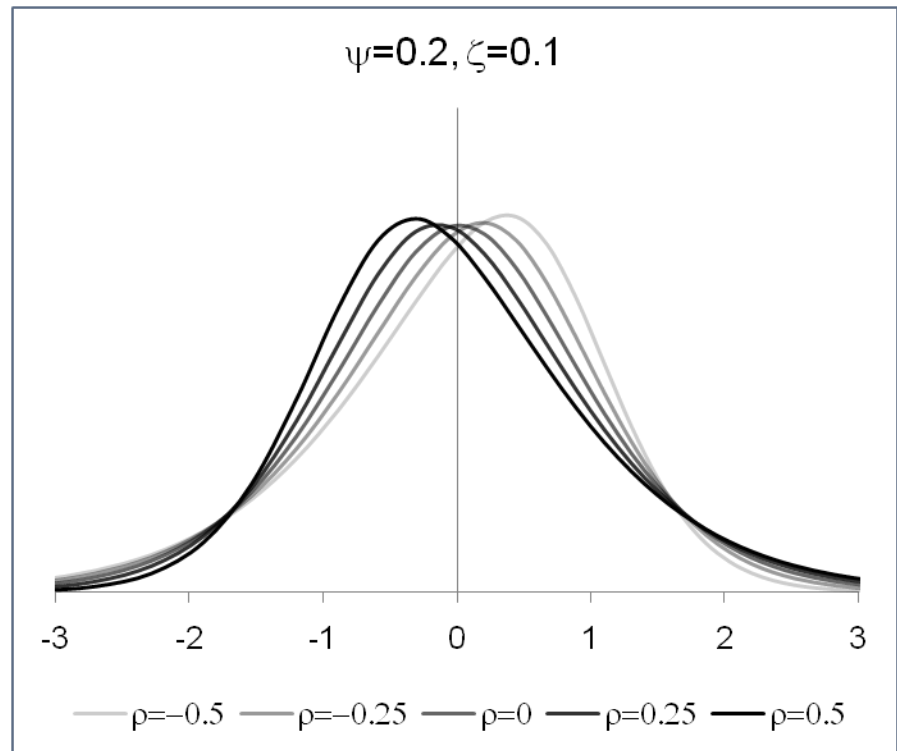
$$\exp\left[\psi x + \frac{1}{2} \zeta xy\right]$$

$x, y$  : correlated normal variables  
 $\psi, \zeta$  : linear and quadratic coefficients  
 $\rho$  : correlation

Implied Black-Scholes volatility smile



Implied distribution of the log of the price

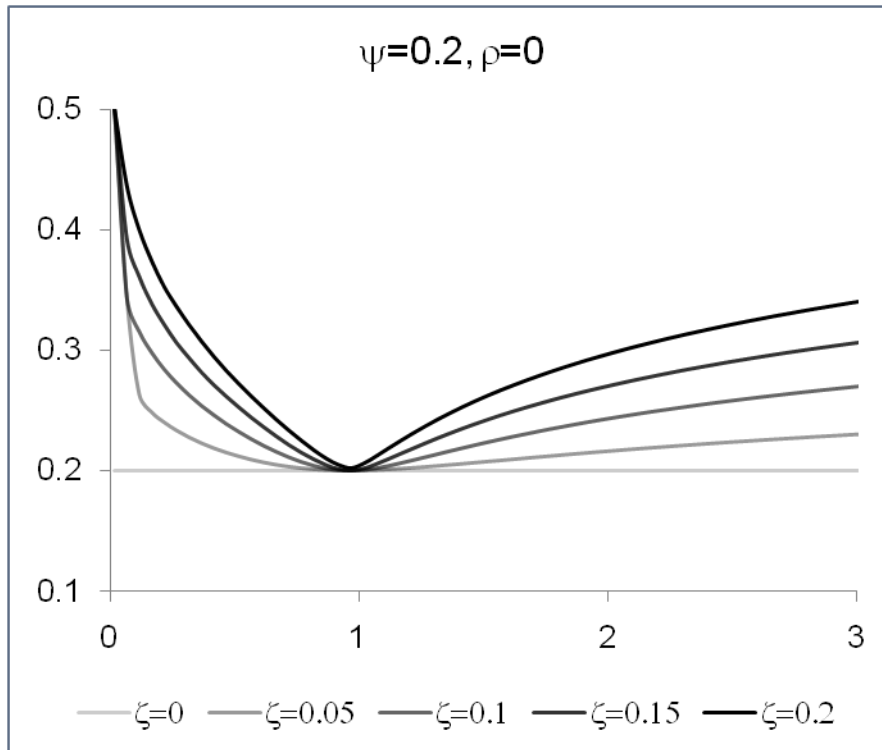


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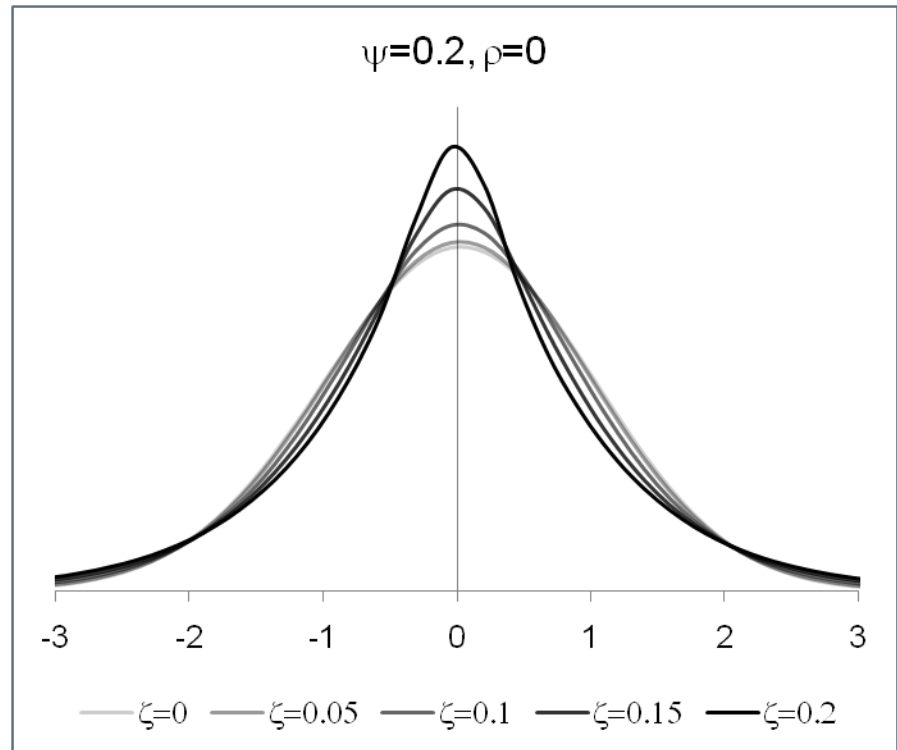
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Implied Black-Scholes volatility smile



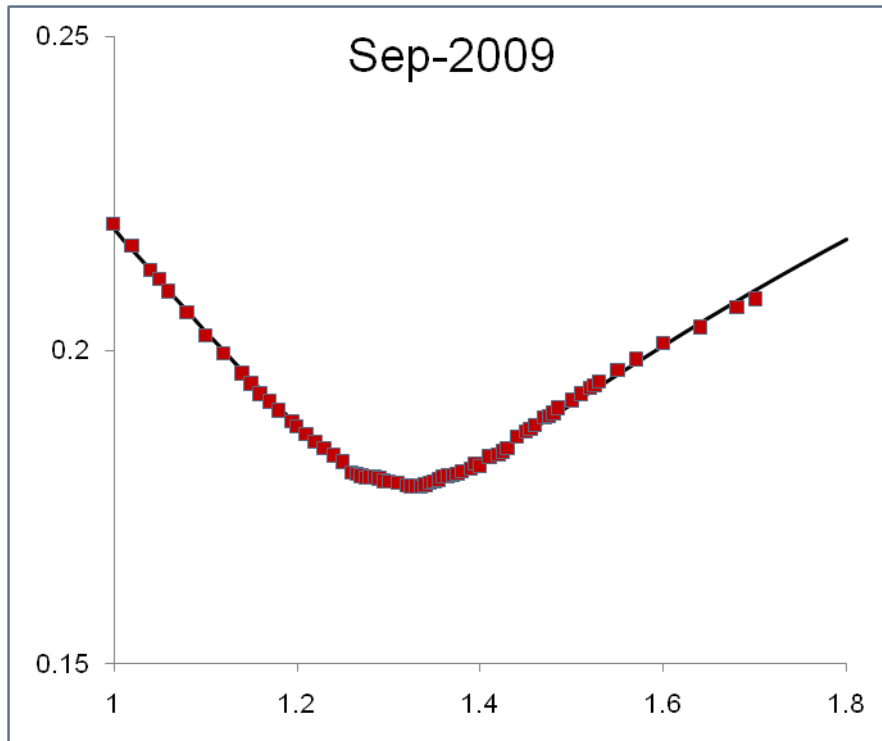
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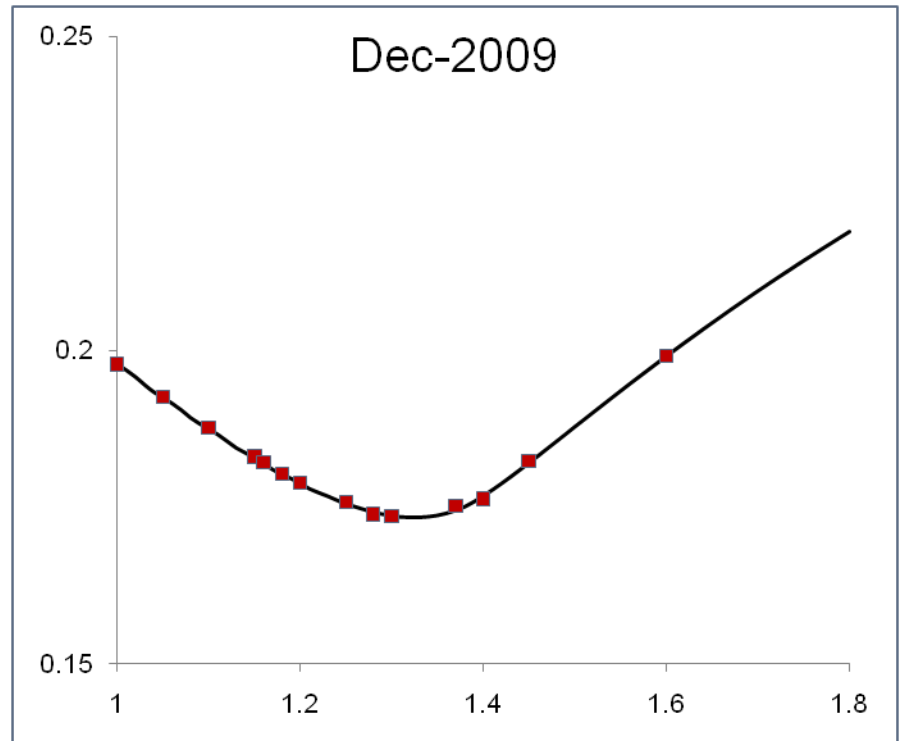
# Why the quadratic Gaussian model?

Euro/Dollar options on 27-Mar-2009 for Sep-2009 and Dec-2009 expiry

Implied Black-Scholes volatility smile



Implied Black-Scholes volatility smile



## Part 2

# The quadratic Gaussian distribution

# The measure and expectation groups

$$\exp\left[\psi \cdot x + \frac{1}{2} x \cdot \zeta x\right]$$

$x$  : vector  
 $\psi$  : vector  
 $\zeta$  : symmetric matrix

- The measure group represents the measure changes generated by exponential-quadratic functions of Gaussian variables.
- The expectation group represents the conditional expectations of exponential-quadratic functions of Gaussian variables.
- Measure change and conditional expectation are reduced to purely algebraic operations.

# The measure and expectation groups

$x$  : vector  
 $\mu$  : vector  
 $\sigma^2$  : symmetric matrix  
 $\psi$  : vector  
 $\zeta$  : symmetric matrix

- Define the generators of the quadratic Gaussian model:

$$u[(\psi, \zeta), x] = \psi \cdot x + \frac{1}{2} x \cdot \zeta x$$

$$\begin{aligned} n[(\psi, \zeta), (\mu, \sigma^2)] &= (1 - \zeta \sigma^2)^{-1} \psi \cdot \mu + \frac{1}{2} \mu \cdot (1 - \zeta \sigma^2)^{-1} \zeta \mu \\ &\quad + \frac{1}{2} \psi \cdot \sigma^2 (1 - \zeta \sigma^2)^{-1} \psi - \frac{1}{2} \log \det[1 - \zeta \sigma^2] \end{aligned}$$

# The measure and expectation groups

- The definition is motivated by the relationship:

$$\mathbf{E} \exp[u[(\psi, \zeta), x]] = \exp[n[(\psi, \zeta), (\mu, \sigma^2)]]$$

This holds when  $x$  has Gaussian distribution  $x \sim N[(\mu, \sigma^2)]$ .

- The generators simplify when the quadratic coefficient vanishes:

$$u[(\psi, 0), x] = \psi \cdot x$$
$$n[(\psi, 0), (\mu, \sigma^2)] = \psi \cdot \mu + \frac{1}{2} \psi \cdot \sigma^2 \psi$$

# The measure group

$x$  : vector  
 $\mu$  : vector  
 $\sigma^2$  : symmetric matrix  
 $\alpha$  : vector  
 $\psi$  : vector  
 $\zeta$  : symmetric matrix

- The generators satisfy the algebraic relationship:

$$n[(\alpha + \psi, \zeta), (\mu, \sigma^2)] - n[(\psi, \zeta), (\mu, \sigma^2)] = u[M[(\psi, \zeta)] \circ (\mu, \sigma^2), \alpha]$$

- The measure group action is defined by:

$$M\left[\begin{pmatrix} \psi \\ \zeta \end{pmatrix}\right] \circ \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} (1 - \sigma^2 \zeta)^{-1} (\mu + \sigma^2 \psi) \\ (1 - \sigma^2 \zeta)^{-1} \sigma^2 \end{pmatrix}$$

- The action simplifies when the quadratic coefficient vanishes:

$$M\left[\begin{pmatrix} \psi \\ 0 \end{pmatrix}\right] \circ \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} \mu + \sigma^2 \psi \\ \sigma^2 \end{pmatrix}$$

# The measure group

- Suppose that  $x$  has distribution  $x \sim N[(\mu, \sigma^2)]$  in the measure  $\mathbf{E}$ . Define the equivalent measure  $\mathbf{E}^*$  via the Radon-Nikodym derivative:

$$\frac{d\mathbf{E}^*}{d\mathbf{E}} = \frac{\exp[u[(\psi, \zeta), x]]}{\mathbf{E} \exp[u[(\psi, \zeta), x]]}$$

Then  $x$  has distribution  $x \sim N[M[(\psi, \zeta)] \circ (\mu, \sigma^2)]$  in the measure  $\mathbf{E}^*$ . This can be seen from the characteristic functions:

$$\mathbf{E} \exp[\alpha \cdot x] = \exp[u[(\mu, \sigma^2), \alpha]]$$

$$\mathbf{E}^* \exp[\alpha \cdot x] = \exp[u[M[(\psi, \zeta)] \circ (\mu, \sigma^2), \alpha]]$$

- The transformation of the coefficients is given by the measure group.

# The measure group

- Vanilla payoffs are modelled as linear combinations of exponential-quadratic functions of a Gaussian variable  $x \sim N[(\mu, \sigma^2)]$  in the pricing measure  $\mathbf{E}$ .
- The option price generalises the Black-Scholes formula:

$$\mathbf{E}\left(\sum_i \lambda_i \frac{\exp[u[(\psi_i, \zeta_i), x]]}{\mathbf{E} \exp[u[(\psi_i, \zeta_i), x]]}\right)^+ = \sum_i \lambda_i \mathbf{E}^i(x \in D)$$

where  $x \sim N[M[(\psi_i, \zeta_i)]^\circ(\mu, \sigma^2)]$  in the measure  $\mathbf{E}^i$  and  $D$  is the domain:

$$(x \in D) = \left(\sum_i \lambda_i \frac{\exp[u[(\psi_i, \zeta_i), x]]}{\mathbf{E} \exp[u[(\psi_i, \zeta_i), x]]} > 0\right)$$

# The measure group

- The composition rule for the measure group is:

$$M\left[\begin{pmatrix} \psi_1 \\ \zeta_1 \end{pmatrix}\right] \circ M\left[\begin{pmatrix} \psi_2 \\ \zeta_2 \end{pmatrix}\right] = M\left[\begin{pmatrix} \psi_1 + \psi_2 \\ \zeta_1 + \zeta_2 \end{pmatrix}\right]$$

- The inverse for the measure group is:

$$M\left[\begin{pmatrix} \psi \\ \zeta \end{pmatrix}\right]^{-1} = M\left[\begin{pmatrix} -\psi \\ -\zeta \end{pmatrix}\right]$$

# The expectation group

$x$  : vector  
 $A$  : vector  
 $B$  : matrix  
 $\Sigma^2$  : symmetric matrix  
 $\psi$  : vector  
 $\zeta$  : symmetric matrix

- The generators satisfy the algebraic relationship:

$$n[(\psi, \zeta), (A + Bx, \Sigma^2)] - n[(\psi, \zeta), (A, \Sigma^2)] = u[E[(A, B, \Sigma^2)] \circ (\psi, \zeta), x]$$

- The expectation group action is defined by:

$$E\left[\begin{pmatrix} A \\ B \\ \Sigma^2 \end{pmatrix}\right] \circ \begin{pmatrix} \psi \\ \zeta \end{pmatrix} = \begin{pmatrix} B^* (1 - \zeta \Sigma^2)^{-1} (\zeta A + \psi) \\ B^* (1 - \zeta \Sigma^2)^{-1} \zeta B \end{pmatrix}$$

- The action simplifies when the quadratic coefficient vanishes:

$$E\left[\begin{pmatrix} A \\ B \\ \Sigma^2 \end{pmatrix}\right] \circ \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} B^* \psi \\ 0 \end{pmatrix}$$

# The expectation group

- The generator is closed under conditional expectation:

$$\frac{\mathbf{E} \exp[u[(\psi, \zeta), X]]}{\exp[n[(\psi, \zeta), (A, \Sigma^2)]]} = \exp[u[E[(A, B, \Sigma^2)] \circ (\psi, \zeta), x]]$$

This holds when  $X$  has Gaussian distribution  $X \sim N[(A + Bx, \Sigma^2)]$ .

- The expectation of the exponential-quadratic function of the variable  $X$  is an exponential-quadratic function of the variable  $x$ .
- The transformation of the coefficients is given by the expectation group.

# The expectation group

- The composition rule for the expectation group is:

$$E\left[\begin{pmatrix} A_1 \\ B_1 \\ \Sigma_1^2 \end{pmatrix}\right] \circ E\left[\begin{pmatrix} A_2 \\ B_2 \\ \Sigma_2^2 \end{pmatrix}\right] = E\left[\begin{pmatrix} B_2 A_1 + A_2 \\ B_2 B_1 \\ B_2 \Sigma_1^2 B_2^* + \Sigma_2^2 \end{pmatrix}\right]$$

- The inverse for the expectation group is:

$$E\left[\begin{pmatrix} A \\ B \\ \Sigma^2 \end{pmatrix}\right]^{-1} = \begin{pmatrix} -B^{-1}A \\ B^{-1} \\ -B^{-1}\Sigma^2 B^{-*} \end{pmatrix}$$

## Part 3

# The quadratic Gaussian process

# Gaussian processes

$$(x_T | x_t) \sim N[(A_{tT} + B_{tT}x_t, \Sigma_{tT}^2)]$$

- The coefficients of the Gaussian process satisfy:

$$A_{\tau T} = B_{tT} A_{\tau t} + A_{tT}$$

$$B_{\tau T} = B_{tT} B_{\tau t}$$

$$\Sigma_{\tau T}^2 = B_{tT} \Sigma_{\tau t}^2 B_{tT}^* + \Sigma_{tT}^2$$

- The coefficients can be generated from their initial values:

$$A_{tT} = A_{0T} - B_{0T} B_{0t}^{-1} A_{0t}$$

$$B_{tT} = B_{0T} B_{0t}^{-1}$$

$$\Sigma_{tT}^2 = \Sigma_{0T}^2 - B_{0T} B_{0t}^{-1} \Sigma_{0t}^2 B_{0t}^{-*} B_{0T}^*$$

$x_t$ : Gaussian process

$\tau < t < T$ : times

$A_{tT}$ : vector

$B_{tT}$ : matrix

$\Sigma_{tT}^2$ : symmetric matrix

# Gaussian processes

- The Gaussian process satisfies the SDE:

$$dx_t = (\bar{\mu}_t + \hat{\mu}_t x_t) dt + \sigma_t dz_t$$

$$0 = \frac{dA_{tT}}{dt} + B_{tT} \bar{\mu}_t$$

$$0 = \frac{dB_{tT}}{dt} + B_{tT} \hat{\mu}_t$$

$$0 = \frac{d\Sigma_{tT}^2}{dt} + B_{tT} \sigma_t^2 B_{tT}^*$$

- The Gaussian process  $x_t$  is related to a time-changed Brownian process  $w_t$  with clock  $B_{0t}^{-1} \Sigma_{0t}^2 B_{0t}^{-*}$ :

$$x_t = A_{0t} + B_{0t} w_t$$

$x_t$ : Gaussian process  
 $t < T$ : times  
 $\bar{\mu}_t$ : vector  
 $\hat{\mu}_t$ : matrix  
 $\sigma_t$ : matrix  
 $A_{tT}$ : vector  
 $B_{tT}$ : matrix  
 $\Sigma_{tT}^2$ : symmetric matrix

# Quadratic Gaussian processes

- Exponential-quadratic functions of Gaussian processes are closed under conditional expectation:

$$\mathbf{E}_t \frac{\exp[u[(\phi, \xi), x_T]]}{\mathbf{E} \exp[u[(\phi, \xi), x_T]]} = \frac{\exp[u[E[(A_{tT}, B_{tT}, \Sigma_{tT}^2)] \circ (\phi, \xi), x_t]]}{\mathbf{E} \exp[u[E[(A_{tT}, B_{tT}, \Sigma_{tT}^2)] \circ (\phi, \xi), x_t]]}$$

- The process:

$$m_t = \frac{\exp[u[(\psi_t, \zeta_t), x_t]]}{\mathbf{E} \exp[u[(\psi_t, \zeta_t), x_t]]}$$

is a martingale when the coefficients satisfy:

$$(\psi_t, \zeta_t) = E[(A_{0t}, B_{0t}, \Sigma_{0t}^2)]^{-1} \circ (\psi_0, \zeta_0)$$

# Quadratic Gaussian processes

- Define the equivalent measure  $\mathbf{E}^*$  via the Radon-Nikodym derivative:

$$\frac{d\mathbf{E}_t^*}{d\mathbf{E}_t} = \frac{\exp[u[(\psi_t, \zeta_t), x_t]]}{\mathbf{E} \exp[u[(\psi_t, \zeta_t), x_t]]}$$

- The process  $x_t$  is a Gaussian process in both measures:

$$(x_T | x_t) \sim N[(A_{tT} + B_{tT}x_t, \Sigma_{tT}^2)]$$

$$(x_T | x_t) \sim N[M[(\psi_T, \zeta_T)] \circ (A_{tT} + B_{tT}x_t, \Sigma_{tT}^2)]$$

in the measures  $\mathbf{E}$  and  $\mathbf{E}^*$  respectively.

## Part 4

# The quadratic Gaussian model

# Single currency model

- Single currency pricing model  $(\mathbf{E}, b_t)$ :  $\mathbf{E}$  is the pricing measure,  $b_t$  is the discount process.
- By definition, the discounted price process  $b_t a_t$  is an  $\mathbf{E}$ -martingale:

$$\mathbf{E}_t b_T a_T = b_t a_t$$

- Zero-coupon bond process  $p_{tT}$  and forward rate process  $r_{tT}$ :

$$p_{tT} = \frac{\mathbf{E}_t b_T}{b_t}$$

$$r_{tT} = -\frac{d}{dT} \log[\mathbf{E}_t b_T]$$

# Multiple currency model

- Multiple currency pricing model ( $\mathbf{E}^c, b_t^c, m_t^c$ ): for each currency  $c$ ,  $\mathbf{E}^c$  is the pricing measure,  $b_t^c$  is the discount process, and  $m_t^c = d\mathbf{E}_t^c / d\mathbf{E}_t$  is the measure process that maps to the universal pricing measure  $\mathbf{E}$ .
- Zero-coupon bond process  $p_{tT}^c$ , forward rate process  $r_{tT}^c$  and FX index process  $s_t^{cd}$ :

$$p_{tT}^c = \frac{\mathbf{E}_t^c b_T^c}{b_t^c}$$

$$r_{tT}^c = -\frac{d}{dT} \log[\mathbf{E}_t^c b_T^c]$$

$$s_t^{cd} = \frac{m_t^d b_t^d}{m_t^c b_t^c}$$

# Quadratic Gaussian model

- The measure process is an exponential-quadratic function of the Gaussian process  $x_t$ :

$$m_t^c = s_0^c \frac{\exp[u[(\psi_t^c, \zeta_t^c), x_t]]}{\mathbf{E} \exp[u[(\psi_t^c, \zeta_t^c), x_t]]}$$

$$(\psi_t^c, \zeta_t^c) = E[(A_{0t}, B_{0t}, \Sigma_{0t}^2)]^{-1} \circ (\psi_0^c, \zeta_0^c)$$

The process  $x_t$  is then a Gaussian process in all measures:

$$(A_{tT}^c + B_{tT}^c x_t, \Sigma_{tT}^{c2}) = M[(\psi_T^c, \zeta_T^c)] \circ (A_{tT} + B_{tT} x_t, \Sigma_{tT}^2)$$

- The discount process is an exponential-quadratic function of the Gaussian process  $x_t$ :

$$b_t^c = p_{0t}^c \frac{\exp[u[(\phi_t^c, \xi_t^c), x_t]]}{\mathbf{E}^c \exp[u[(\phi_t^c, \xi_t^c), x_t]]}$$

# Financial processes

- Zero-coupon bond process  $p_{tT}^c$ :

$$p_{tT}^c = \frac{p_{0T}^c}{p_{0t}^c} \exp[u[E[(A_{tT}^c, B_{tT}^c, \Sigma_{tT}^{c2})] \circ (\phi_T^c, \xi_T^c) - (\phi_t^c, \xi_t^c), x_t] \\ - n[E[(A_{tT}^c, B_{tT}^c, \Sigma_{tT}^{c2})] \circ (\phi_T^c, \xi_T^c), (A_{0t}^c, \Sigma_{0t}^{c2})] \\ + n[(\phi_t^c, \xi_t^c), (A_{0t}^c, \Sigma_{0t}^{c2})]]$$

- FX index process  $s_t^{cd}$ :

$$s_t^{cd} = s_0^{cd} \frac{p_{0t}^d}{p_{0t}^c} \exp[u[(\psi_t^d, \zeta_t^d) - (\psi_t^c, \zeta_t^c) + (\phi_t^d, \xi_t^d) - (\phi_t^c, \xi_t^c), x_t] \\ - n[(\psi_t^d, \zeta_t^d), (A_{0t}^d, \Sigma_{0t}^{d2})] + n[(\psi_t^c, \zeta_t^c), (A_{0t}^c, \Sigma_{0t}^{c2})] \\ - n[(\phi_t^d, \xi_t^d), (A_{0t}^d, \Sigma_{0t}^{d2})] + n[(\phi_t^c, \xi_t^c), (A_{0t}^c, \Sigma_{0t}^{c2})]]$$

# Financial processes

$$s_t = f_t \frac{\exp[u[(\bar{\psi}_t, \bar{\zeta}_t), x_t]]}{\mathbf{E} \exp[u[(\bar{\psi}_t, \bar{\zeta}_t), x_t]]}$$

$$\frac{ds_t}{s_t} = \dots + (\bar{\psi}_t + \bar{\zeta}_t x_t) \cdot dx_t$$

- Zero-coupon bond process  $p_{tT}^c$ :

$$\begin{pmatrix} \bar{\psi}_t \\ \bar{\zeta}_t \end{pmatrix} = E \left[ \begin{pmatrix} A_{tT}^c \\ B_{tT}^c \\ \Sigma_{tT}^{c2} \end{pmatrix} \right] \circ \begin{pmatrix} \phi_T^c \\ \xi_T^c \\ \zeta_T^c \end{pmatrix} - \begin{pmatrix} \phi_t^c \\ \xi_t^c \\ \zeta_t^c \end{pmatrix}$$

- FX index process  $s_t^{cd}$ :

$$\begin{pmatrix} \bar{\psi}_t \\ \bar{\zeta}_t \end{pmatrix} = \begin{pmatrix} \psi_t^d \\ \zeta_t^d \end{pmatrix} - \begin{pmatrix} \psi_t^c \\ \zeta_t^c \end{pmatrix} + \begin{pmatrix} \phi_t^d \\ \xi_t^d \\ \zeta_t^d \end{pmatrix} - \begin{pmatrix} \phi_t^c \\ \xi_t^c \\ \zeta_t^c \end{pmatrix}$$

# Pricing vanilla options

- Model parameters in the  $S$ -terminal measure of currency  $c$ :

$$\begin{aligned} (A_{tT}^{cS} + B_{tT}^{cS} x_t, \Sigma_{tT}^{cS2}) &= M[E[(A_{TS}^c, B_{TS}^c, \Sigma_{TS}^{c2})] \circ (\phi_S^c, \xi_S^c)] \circ (A_{tT}^c + B_{tT}^c x_t, \Sigma_{tT}^{c2}) \\ (\psi_t^{cS}, \zeta_t^{cS}) &= (\psi_t^c, \zeta_t^c) + E[(A_{tS}^c, B_{tS}^c, \Sigma_{tS}^{c2})] \circ (\phi_S^c, \xi_S^c) \\ (\phi_t^{cS}, \xi_t^{cS}) &= (\phi_t^c, \xi_t^c) - E[(A_{tS}^c, B_{tS}^c, \Sigma_{tS}^{c2})] \circ (\phi_S^c, \xi_S^c) \end{aligned}$$

- Vanilla option price:

$$\mathbf{E}^c b_t^c \left( \sum_i \lambda_i s_t^{cc_i} p_{tT_i}^{c_i} \right)^+ = \sum_i \lambda_i s_0^{cc_i} p_{0T_i}^{c_i} \mathbf{E}^{c_i T_i} (x_t \in D)$$

where  $D$  is the domain:

$$(x_t \in D) = \left( \sum_i \lambda_i p_{0T_i}^{c_i} m_t^{c_i T_i} > 0 \right)$$

# Example: Forward volatility

$$s_t = f_t \frac{\exp[u[(\bar{\psi}_t, \bar{\zeta}_t), x_t]]}{\mathbf{E} \exp[u[(\bar{\psi}_t, \bar{\zeta}_t), x_t]]}$$

$$(s_T | x_t) = \mathbf{E}_t[s_T] \frac{\exp[u[(\Psi, \Omega), z]]}{\exp[n[(\Psi, \Omega), (0,1)]]}$$

$$\Psi = \sum_{tT}^* (\bar{\psi}_T + \bar{\zeta}_T (A_{tT} + B_{tT} x_t))$$

$$\Omega = \sum_{tT}^* \bar{\zeta}_T \Sigma_{tT}$$

- Closed-form expressions for the financial variables provide transparency in the analysis of forward volatility smile dynamics.
- Dynamics blends local volatility and stochastic volatility features.
- Sticky strike/delta properties of the dynamics can be imposed as constraints on the parameter curves.

# Example: Cliquet convexity

$$s_t = f_t \frac{\exp[u[(\bar{\psi}_t, \bar{\zeta}_t), x_t]]}{\mathbf{E} \exp[u[(\bar{\psi}_t, \bar{\zeta}_t), x_t]]}$$

$$\mathbf{E}\left(\frac{s_T}{s_t} - 1\right) = \frac{f_T}{f_t} \exp[\lambda] - 1$$

$$\begin{aligned} \lambda = & n[E[A_{tT}, B_{tT}, \Sigma_{tT}^2] \circ (\bar{\psi}_T, \bar{\zeta}_T) - (\bar{\psi}_t, \bar{\zeta}_t), (A_{0t}, \Sigma_{0t}^2)] \\ & - (n[E[A_{tT}, B_{tT}, \Sigma_{tT}^2] \circ (\bar{\psi}_T, \bar{\zeta}_T), (A_{0t}, \Sigma_{0t}^2)] - n[(\bar{\psi}_t, \bar{\zeta}_t), (A_{0t}, \Sigma_{0t}^2)]) \end{aligned}$$

- Closed-form expression for the convexity adjustment to the cliquet forward incorporates the effects of smile.
- Expression useful for volatility bonds, equity cliquets and inflation year-on-year options.
- Similar expressions exist for other convexity adjustments.

# Possible problems

- Model is over-parametrised. Successful parameter reductions can be achieved by imposing stationarity constraints on the forward volatility smile.
- Martingale constraint on the measure process means there is no additional control on the term structure of FX smile beyond that implied from the IR smiles. This is alleviated by adding more factors, and could be considered as a benefit for long-dated FX models.
- Pay delay and quanto convexity adjustments can diverge. In practice quadratic coefficients are small enough for this not to be an issue. The problem is no worse than occurs with other stochastic volatility models.

# Practical benefits

- Core analytics for the model based entirely on the definitions for the generators, the measure and expectation groups, and the Gaussian process.
- Simple, exact expressions for vanilla option prices enable robust and efficient calibration without resort to analytic proxies.
- Closed-form expressions for the basic financial processes in terms of a state process that has Gaussian increments in all pricing measures facilitates numerical integration methods, such as lattice or Monte-Carlo schemes.
- The QGM stands as an acceptable model in its own right. It can also be used as an approximation for evaluating more complex models, for example by restricting to the quadratic terms of the SDE, or by projecting to the QGM using Markovian projection techniques.

# References

- McCloud P, 2008 *Exponential-quadratic semimartingale models of interest rate and asset volatility*  
Available at <http://ssrn.com/abstract=1269577>
- McCloud P, 2010 *Putting the smile back on the face of derivatives*  
*Risk* January, pages 117-123
- Piterbarg V, 2009 *Rates squared*  
*Risk* January, pages 100-105
- Rogers L C G, 1997 *The potential approach to the term structure of interest rates and foreign exchange rates*  
*Mathematical Finance* 7, pages 157-176