Matrices

and

Quantum Mechanics

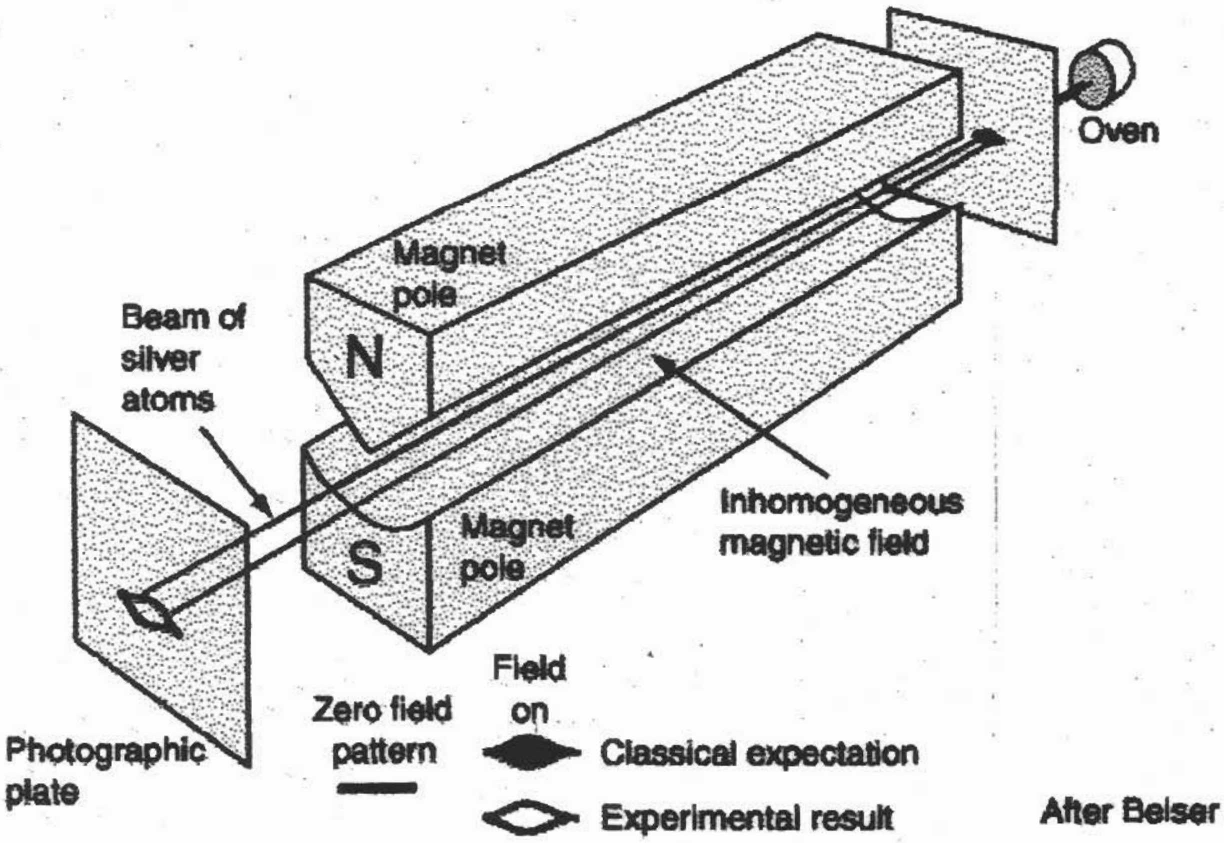
Why quantum mechanics?

- Snooker ball picture breaks down

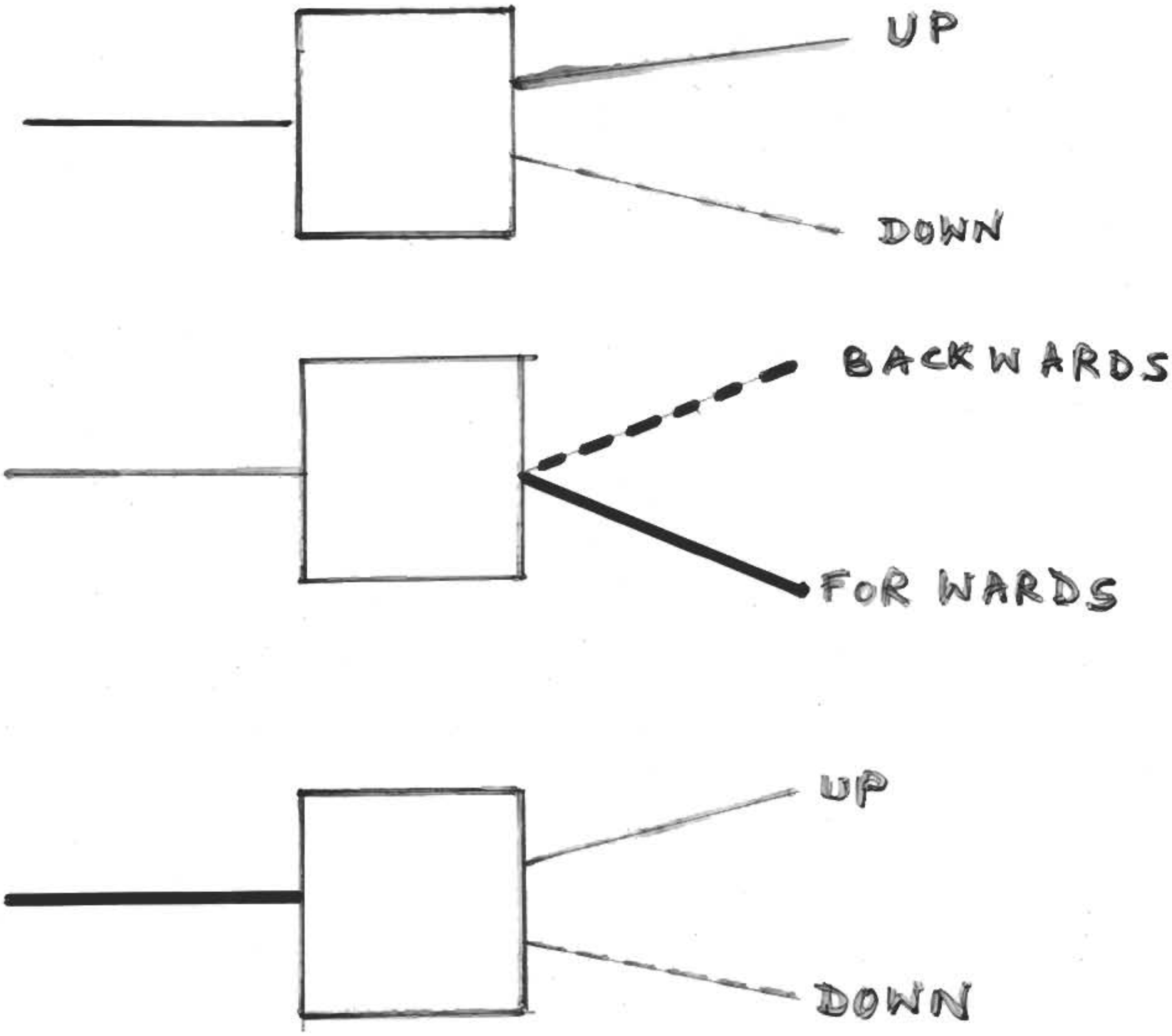
- Inexplicable observations

- Experiment of Stern and Gerlach (1921)

Stern Gerlach Apparatus



Chain of Stern Gerlach Apparatus



Two by Two Matrices

Matrices provide a quick and systematic way of encoding operations one often does on a pair of numbers to get a new pair of numbers,eg

$$\begin{aligned} 3x + 5y &= 13 \\ 4x + 6y &= 16 \end{aligned}$$

is written

$$\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 16 \end{pmatrix}$$

Some more examples of matrix multiplication

$$\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

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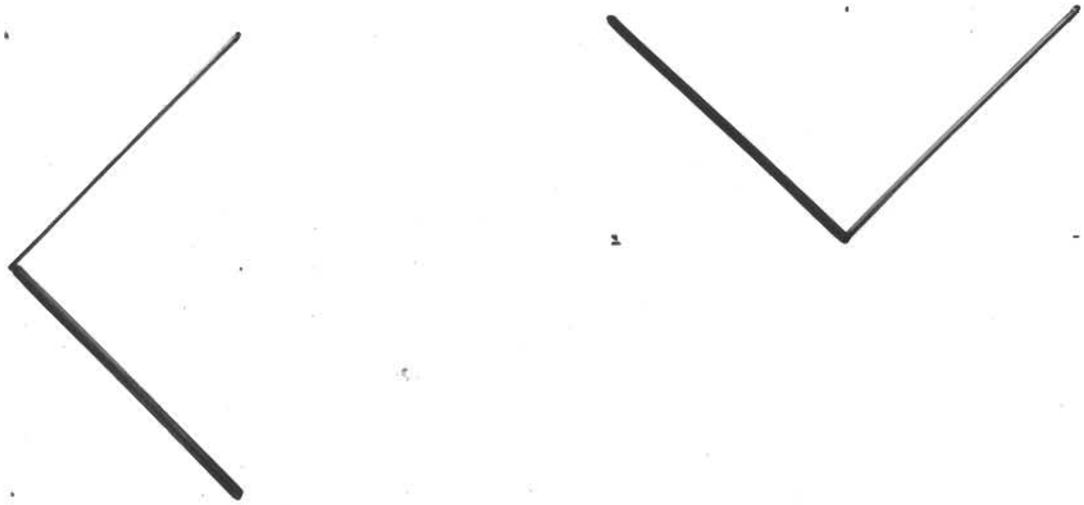

Quantum Mechanics

State	Column vector
Observable	Matrix
Result	Eigenvalue, Eigenvector

Example: an electron

State $\begin{pmatrix} a \\ b \end{pmatrix}$

Observable Up/down spin

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Result $+1$ or -1, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The sequel

Top beam through

forward/backward Stern Gerlach

State $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Observable Forward/backward spin

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Result $+1$ or -1

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

...and more

Forward beam through up/down Stern Gerlach

State $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Observable Up/down spin

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Result $+1$ or -1, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Mathematical Aside

Can we find a function $y(x)$ such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \quad ?$$

Try

$$y = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \ldots$$

Writing this as

$$\begin{aligned} y \;=\; & 1 + x + \tfrac{1}{1\times 2}x^2 + \tfrac{1}{1\times 2\times 3}x^3 \\ & + \tfrac{1}{1\times 2\times 3\times 4}x^4 + \ldots \end{aligned}$$

suggests how we can keep building this up

We need to generalise this a bit, and find a function $\psi(t)$ such that

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 3t$$

This time try

$$\psi = 1+3t+\frac{9}{2}t^2+\frac{27}{6}t^3+\frac{81}{24}t^4+\ldots$$

Again, writing this as

$$\begin{aligned}\psi \;=\; & 1 + 3t + \frac{1}{1\times 2}(3t)^2 \\ & + \frac{1}{1\times 2\times 3}(3t)^3 \\ & + \frac{1}{1\times 2\times 3\times 4}(3t)^4 + \ldots\end{aligned}$$

suggests how we can keep building this up

Towards waves

State ψ varying with time,
$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Equation of motion

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = A\psi \text{ with } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Solution

$$\psi = \exp(At) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

What is $\exp(At)$?

$$\exp(At) = 1 + At + \frac{1}{2}(At)^2 + \ldots$$

$$(At)^2 = \begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix},$$

$$(At)^3 = \begin{pmatrix} 0 & t^3 \\ -t^3 & 0 \end{pmatrix}, \ldots$$

$$\exp(At) =$$

$$\begin{pmatrix} 1 - \frac{1}{2}t^2\text{etc} & -t + \frac{1}{6}t^3\text{etc} \\ t - \frac{1}{6}t^3\text{etc} & 1 - \frac{1}{2}t^2\text{etc} \end{pmatrix}$$

$$\exp(At) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Finally we get

$$\psi(t) =$$

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Check

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

and

$$A\psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

as required.

Also, when $t = 0$, $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.