King's College Summer School 2001

A QUICK TRIP THROUGH SOME GREEK GEOMETRY

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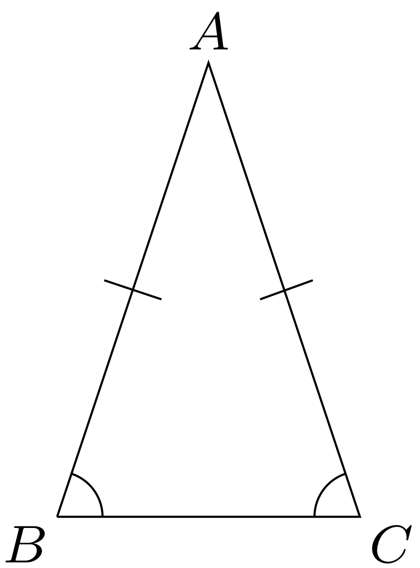
Geometry probably began in Egypt, where the River Nile regularly washed away field boundaries, and they had to have methods to recalculate the areas, for tax purposes.

In about 600BC, Thales travelled from Greece to Egypt, and brought back the methods, and it was the Greeks who first set out to support these methods with proofs.

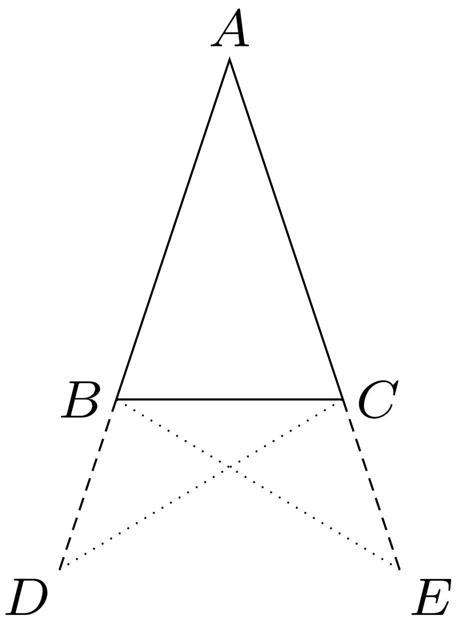
"No one before them had thought of proving such a thing as that the two base angles of an isosceles triangle are equal: the idea was an inspiration unique in the history of the world, and the fruit of it was the creation of mathematics as a science." [Heath]



Pons asinorum
(The asses' bridge)

If in $\triangle ABC$ we have $AB = AC$, then also $\angle ABC = \angle ACB$.

Euclid's proof:



Produce AB to D, and AC to E, so that $AD = AE$, and hence also $BD = CE$. Join BE and CD.

Consider $\triangle ABE$ and $\triangle ACD$. We have $AB = AC$, $AE = AD$, and $\angle BAE = \angle CAD$. So $\triangle ABE \equiv \triangle ACD$ (case side-angle-side).

In consequence, we have $BE = CD$, and $\angle ABE = \angle ACD$, and also $\angle AEB = \angle ADC$.

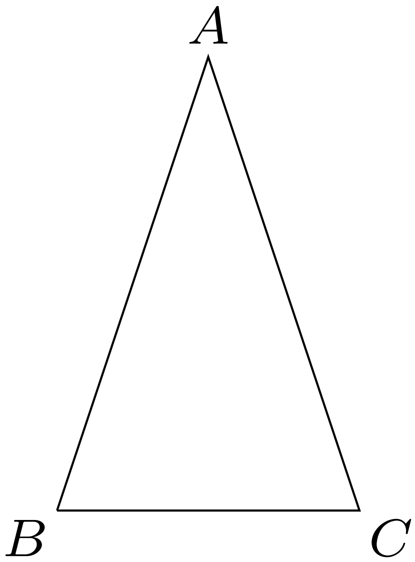
Now consider $\triangle BDC$ and $\triangle CEB$. We have $BD = CE$, $DC = EB$, and $\angle BDC = \angle CEB$. So $\triangle BDC \equiv \triangle CEB$ (case side-angle-side, again). In consequence, we have $\angle BCD = \angle CBE$.

Finally, we have

$$\angle ABC = \angle ABE - \angle CBE = \angle ACD - \angle BCD = \angle ACB,$$

as required.

(Elements, Book I, Proposition V. 300BC.)



Pappus' proof:

Consider $\triangle ABC$ and $\triangle ACB$!

We have $AB = AC$, $AC = AB$, and $\angle BAC = \angle CAB$. So

$$\triangle ABC \equiv \triangle ACB$$

(case side-angle-side).

In consequence, $\angle ABC = \angle ACB$. Done!

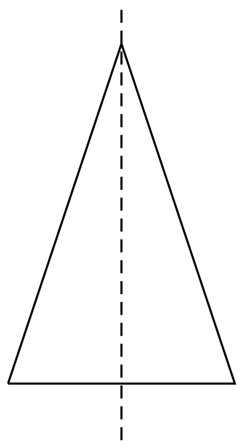
(About 300AD, or 600 years after Euclid.)

Pupil:

“Master, what shall I gain by learning these things?”

Euclid *(to his assistant)*:

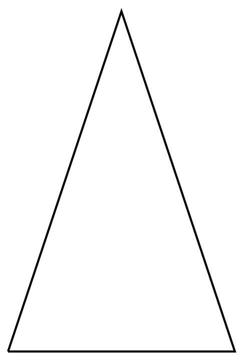
“Give him threepence, since he must *gain* by what he learns.”



Today's proof:

It's symmetrical, innit.

or more likely:

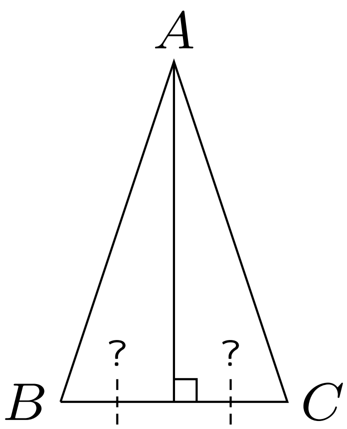


Prove it? It's obvious!

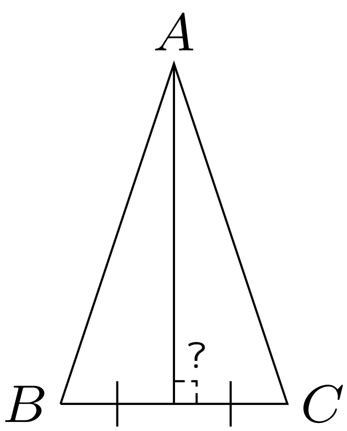
(This is called *progress*.)

COROLLARY: We are all Egyptians, now.

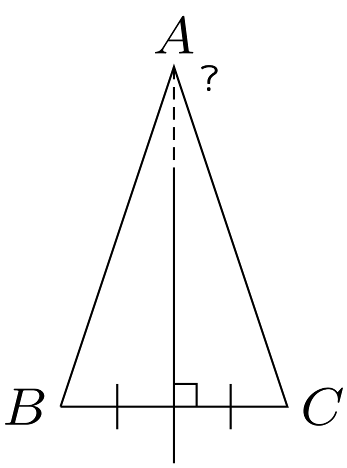
Where, *precisely*, is the mirror?



If it is the perpendicular from A to BC, why does it meet BC at its mid-point?



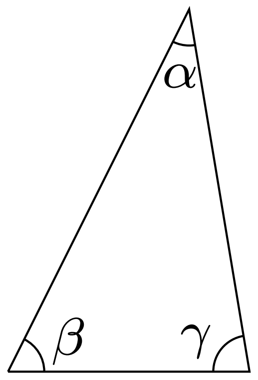
If it is the join of A to the mid-point of BC, why is it perpendicular to BC?

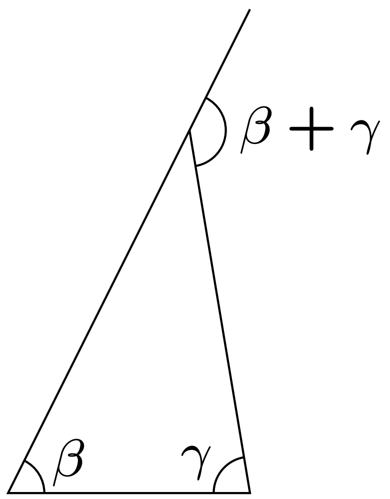


If it is the perpendicular bisector of BC, why does it go through A?

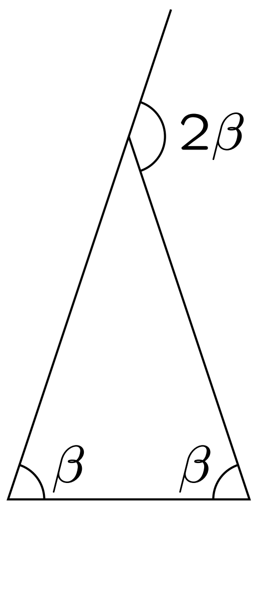
The angle-sum of a triangle

$$\alpha + \beta + \gamma = 180^\circ$$



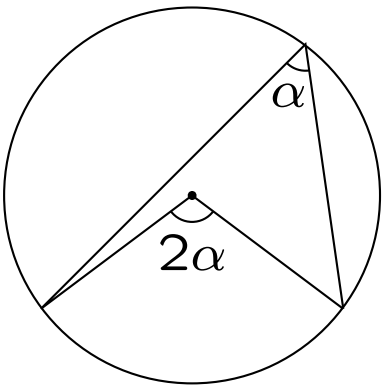


The external angle is equal to the sum of the opposite internal angles

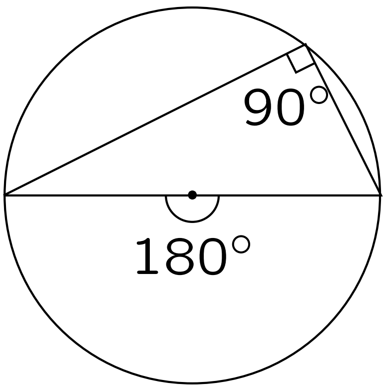


Isosceles case

Circle theorems

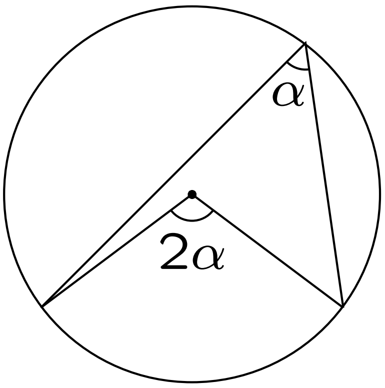


The angle at the centre is twice the angle at the circumference

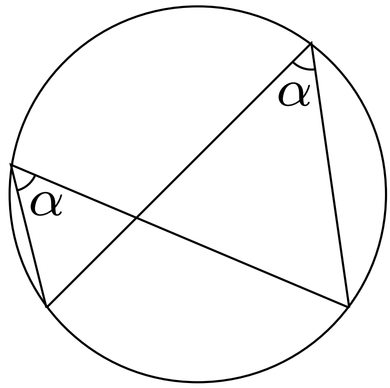


Case $\alpha = 90^\circ$: the angle in a semicircle is a right angle

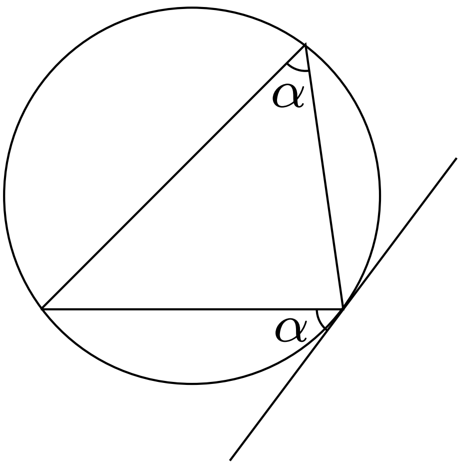
Circle theorems



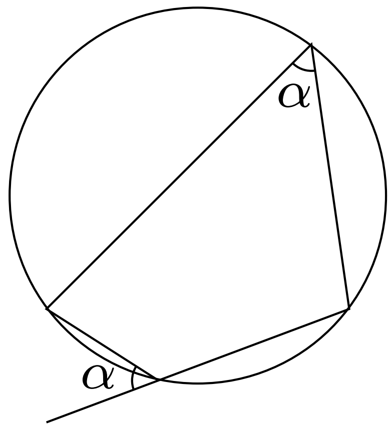
The angle at the centre is twice the angle at the circumference



Angles in the same segment are equal

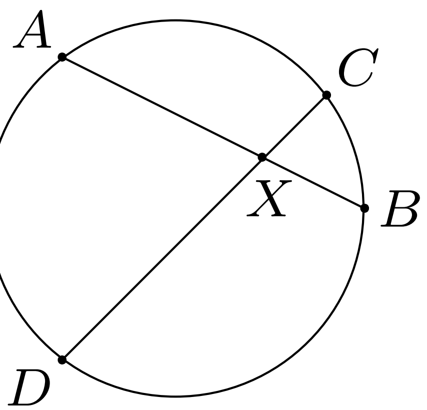


Angle between tangent and chord equals angle in alternate segment

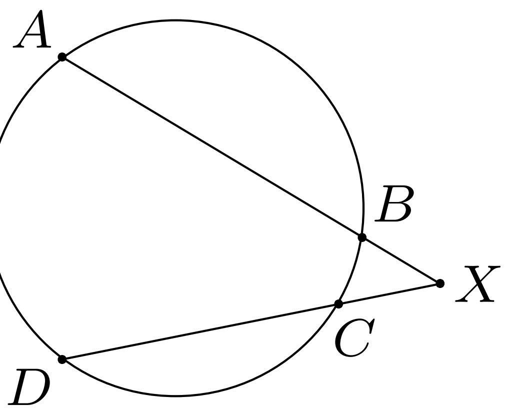


Exterior angle of cyclic quadrilateral equals opposite interior angle

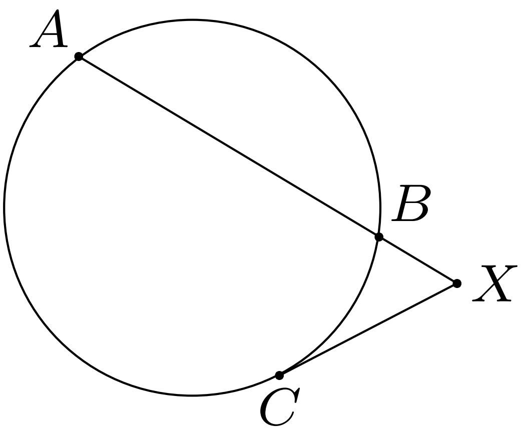
Rectangular properties



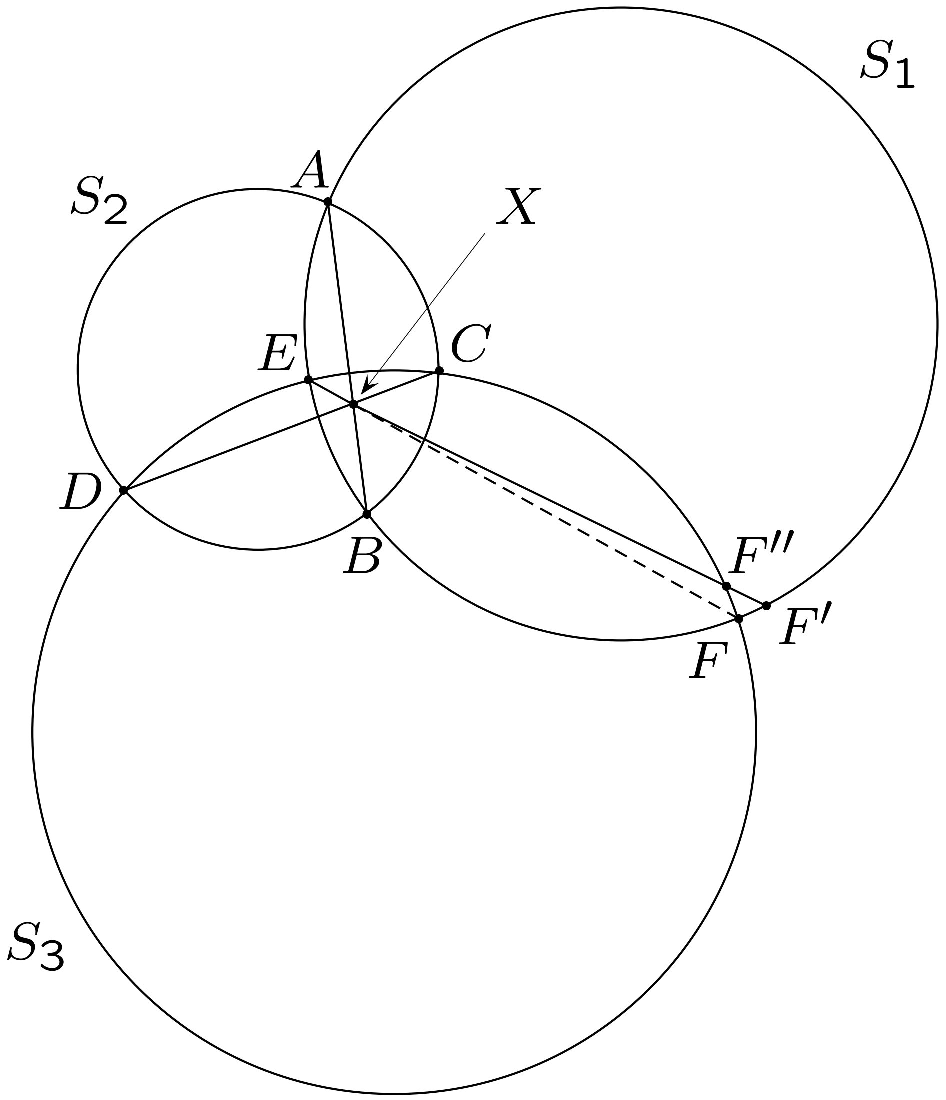
$$XA.XB = XC.XD$$



$$XA.XB = XC.XD$$



$$XA.XB = XC^2$$

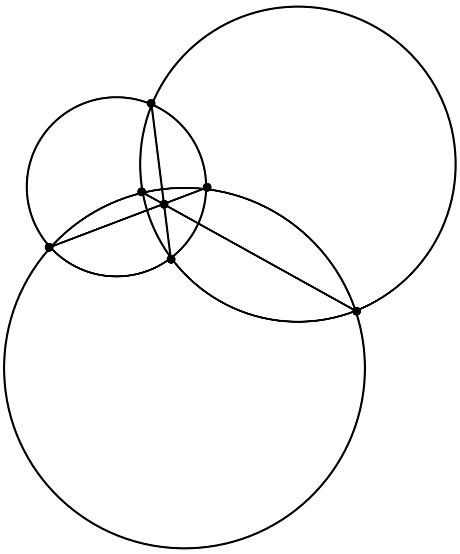


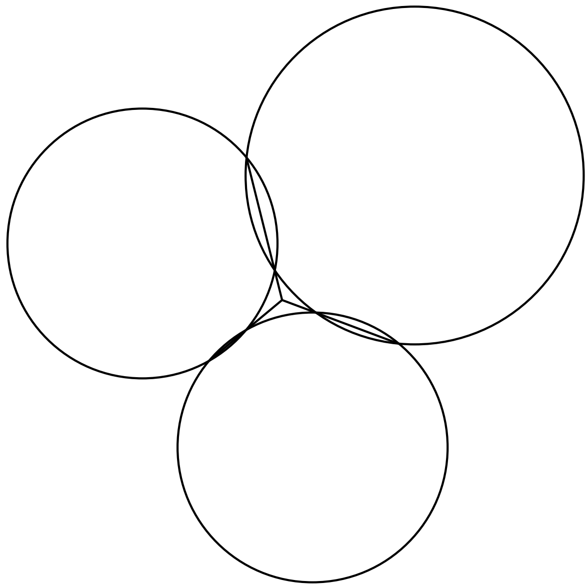
S_1, S_2 meet at A, B; S_2, S_3 meet at C, D, and S_3, S_1 meet at E, F. AB, CD meet at X, and EX meets S_1 again at F', and S_2 again at F''. Then

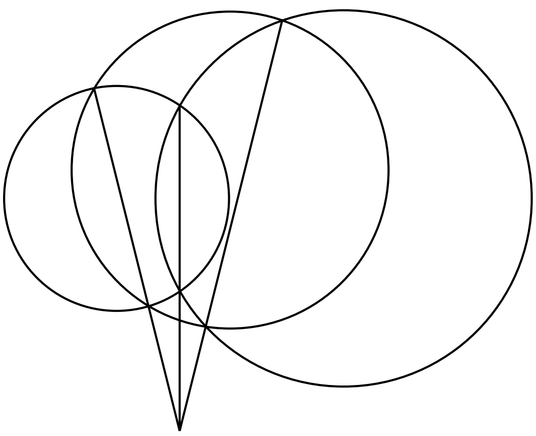
$$XE.XF' \underset{S_1}{=} XA.XB \underset{S_2}{=} XC.XD \underset{S_3}{=} XE.XF'',$$

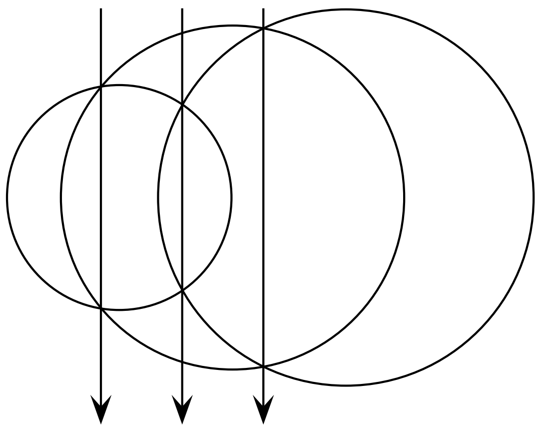
so $XF' = XF''$, and therefore $F' = F'' = F$. Thus EF also passes through X, that is, AB, CD, EF are concurrent.

Theorem. The common chords of three circles, taken in pairs, meet in a point.*
[This point is called the *radical centre* of the circles.]









*I lied: in *this* case the radical centre is at ∞ !

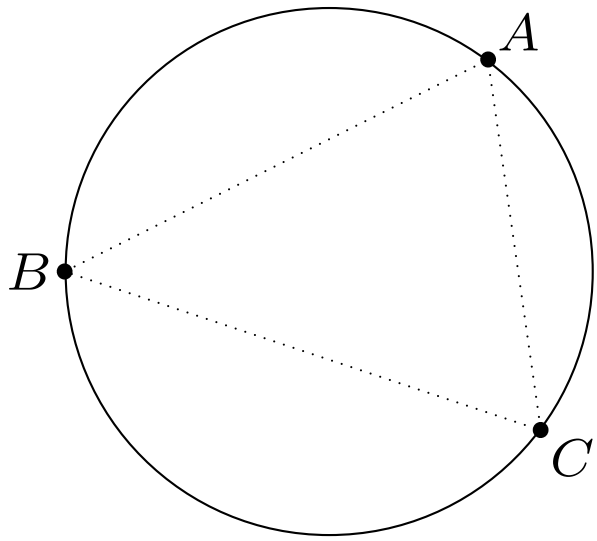
APOLLONIUS of Perga (? 262–200BC) was known as ‘the great geometer’.

His 8 volumes on **Conics** was the authoritative treatise on the subject, and he gave them their names, *parabola, ellipse, hyperbola*.

Among his many other works was a treatise on *Contacts* or *Tangencies*, in two volumes. It solves the following problem: **given** three things, each of which may be a point, a line, or a circle, **construct** a circle which passes through the point or points (if such are given) and touches the lines or circles, as the case may be.

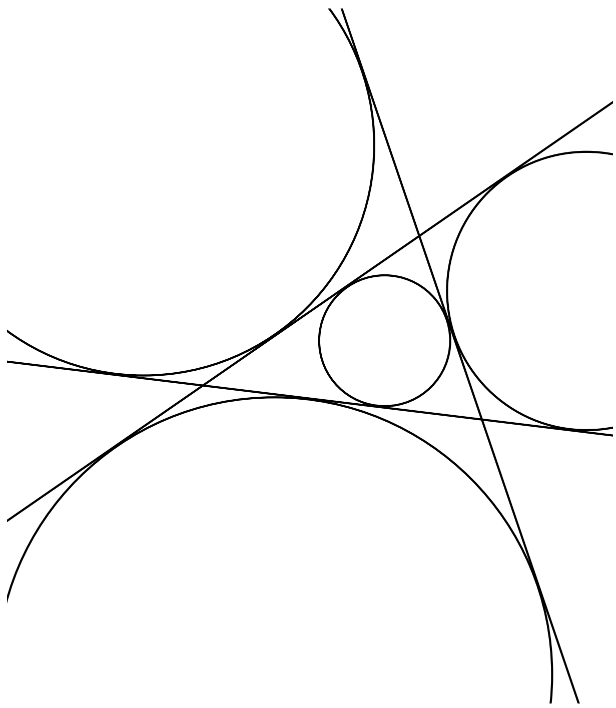
There are ten cases: PPP, PPL, PPC, PLL, PCC, PLC, LLL, LLC, LCC, CCC. The first eight of these came in Book I, and the last two in Book II.

Problem PPP is the familiar one of finding a circle through three given points A, B, C, that is, finding the *circumcircle* of $\triangle ABC$.



Here the answer is unique, except in the case where A, B, C are collinear (when there is *no* answer).

Problem LLL is that of finding a circle to touch three lines. If no two of the lines are parallel, then there are four answers: the incircle and excircles of the triangle they form.



(If two of the lines are parallel, there are only *two* answers—check this!—and if they are all parallel there is *no* answer.)

Let n be the number of answers to a particular case of Apollonius' problem.

We have just seen that in case PPP, $n = 1$ or 0, and in case LLL, $n = 4$ or 2 or 0.

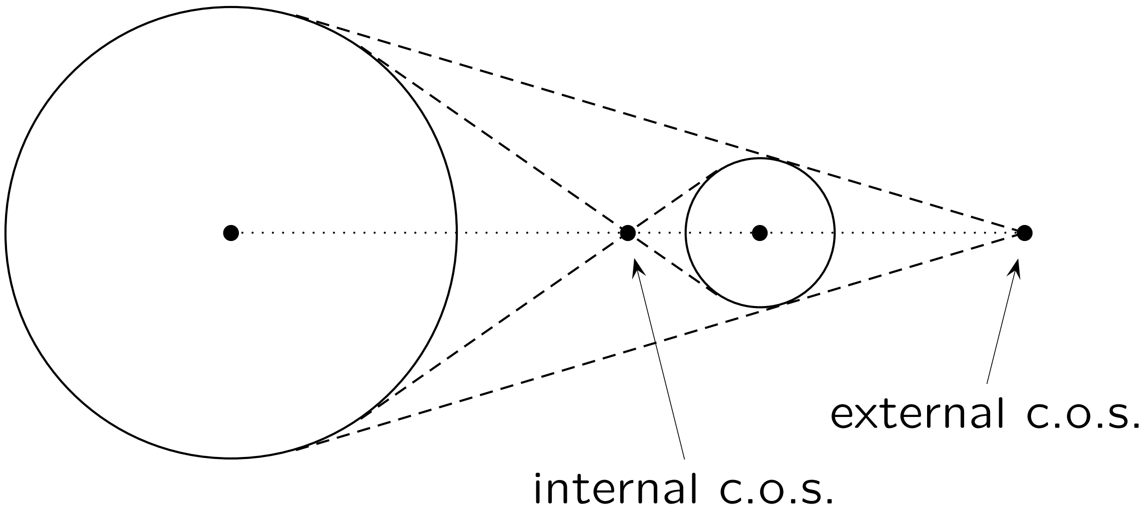
Exercise 1: decide what values of n are possible in each of the other cases.

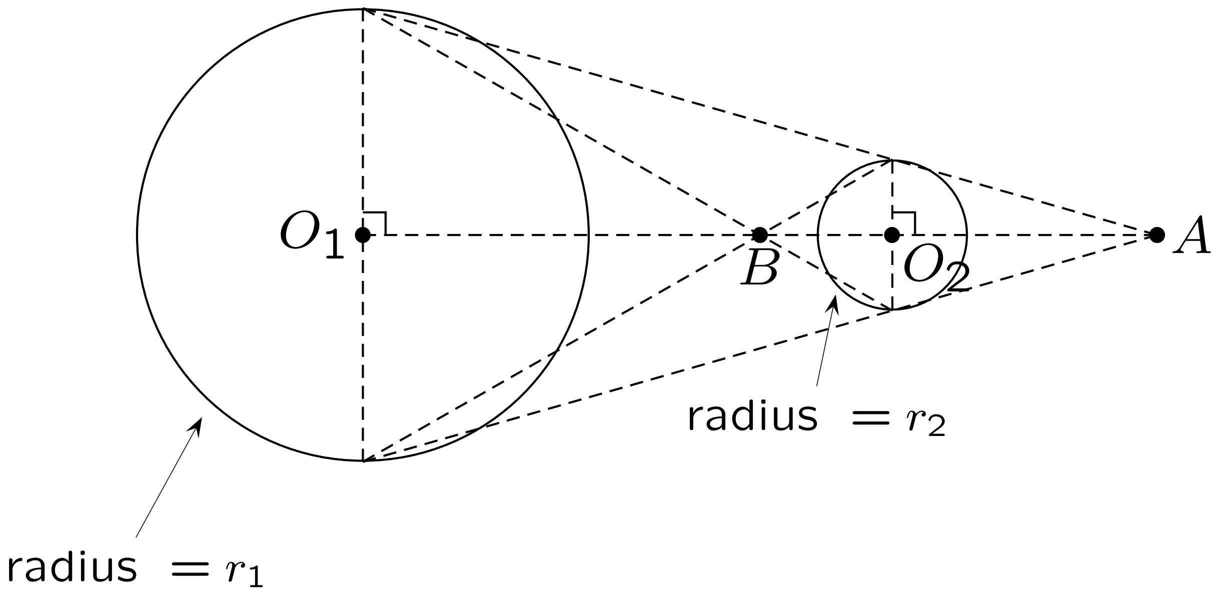
Exercise 2: solve Apollonius' problem in as many cases as you can, that is, give a construction for the required circle or circles.

We shall now do the hardest case, CCC, that is, finding a circle to touch three given circles.

First, some preliminary results.

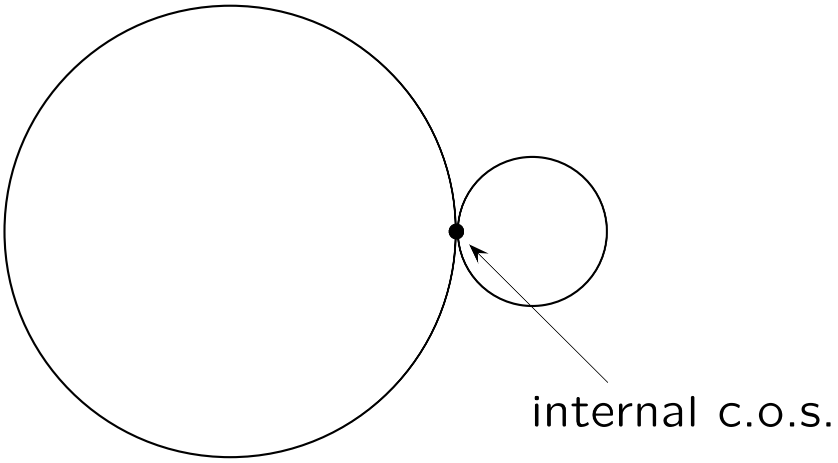
Centres of Similitude

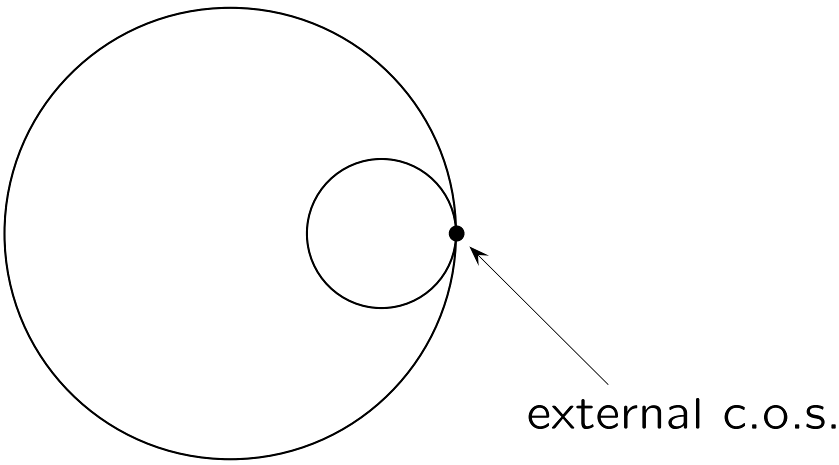




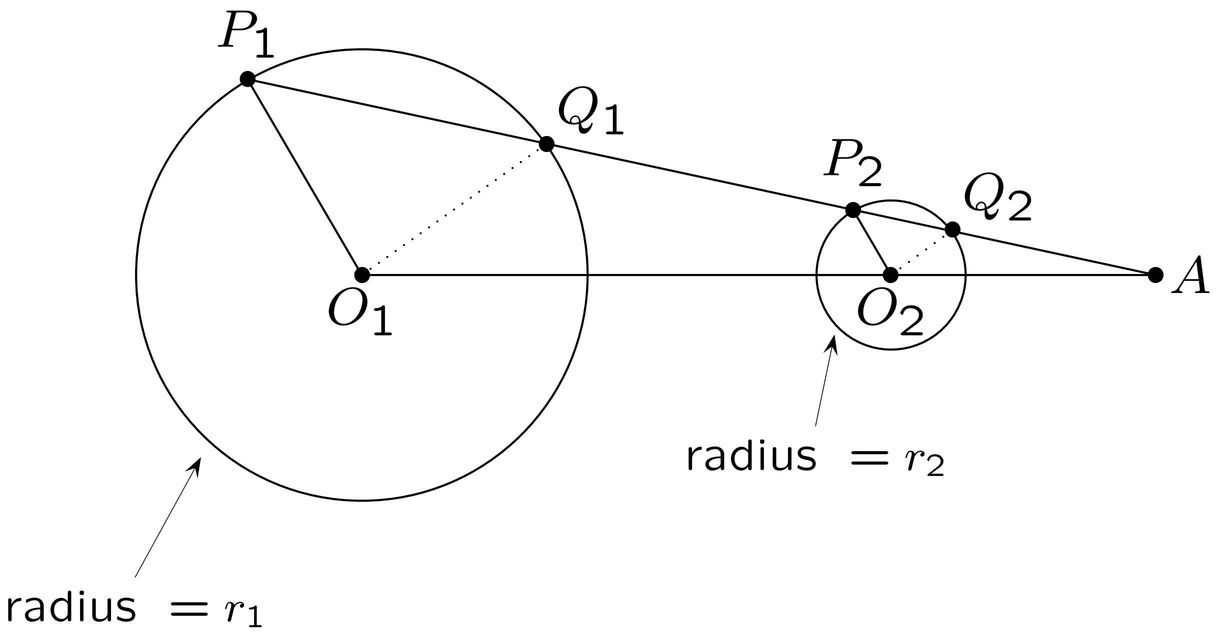
$$O_1A : O_2A = r_1 : r_2 = O_1B : BO_2$$

Special cases





Lemma



A is the external centre of similitude of the two circles, and a line through A meets the circles as shown. Then $\triangle AO_1P_1$ is similar to $\triangle AO_2P_2$, so

$$AP_1 : AP_2 = r_1 : r_2.$$

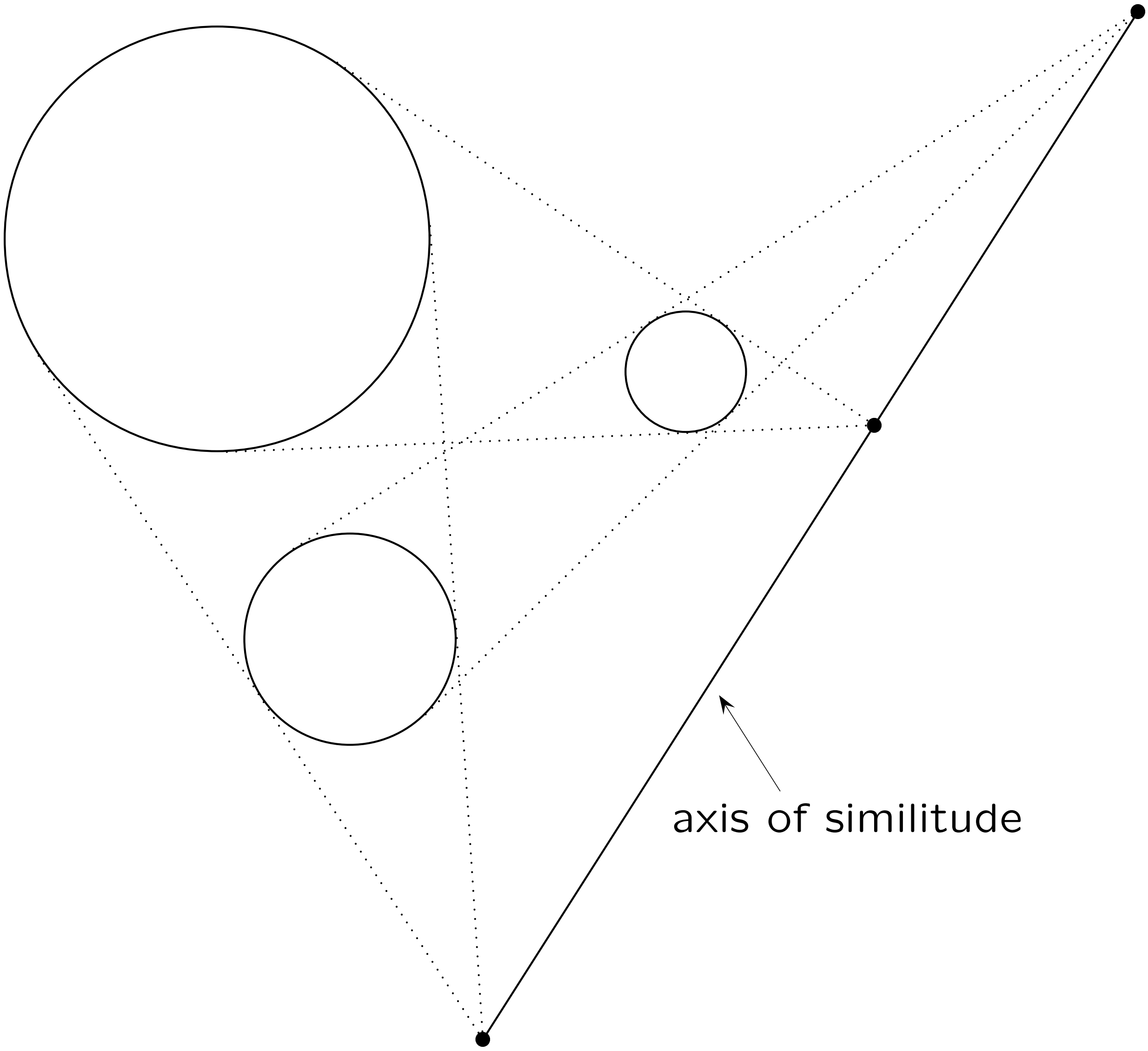
Likewise, $\triangle AO_1Q_1$ is similar to $\triangle AO_2Q_2$, so

$$AQ_1 : AQ_2 = r_1 : r_2,$$

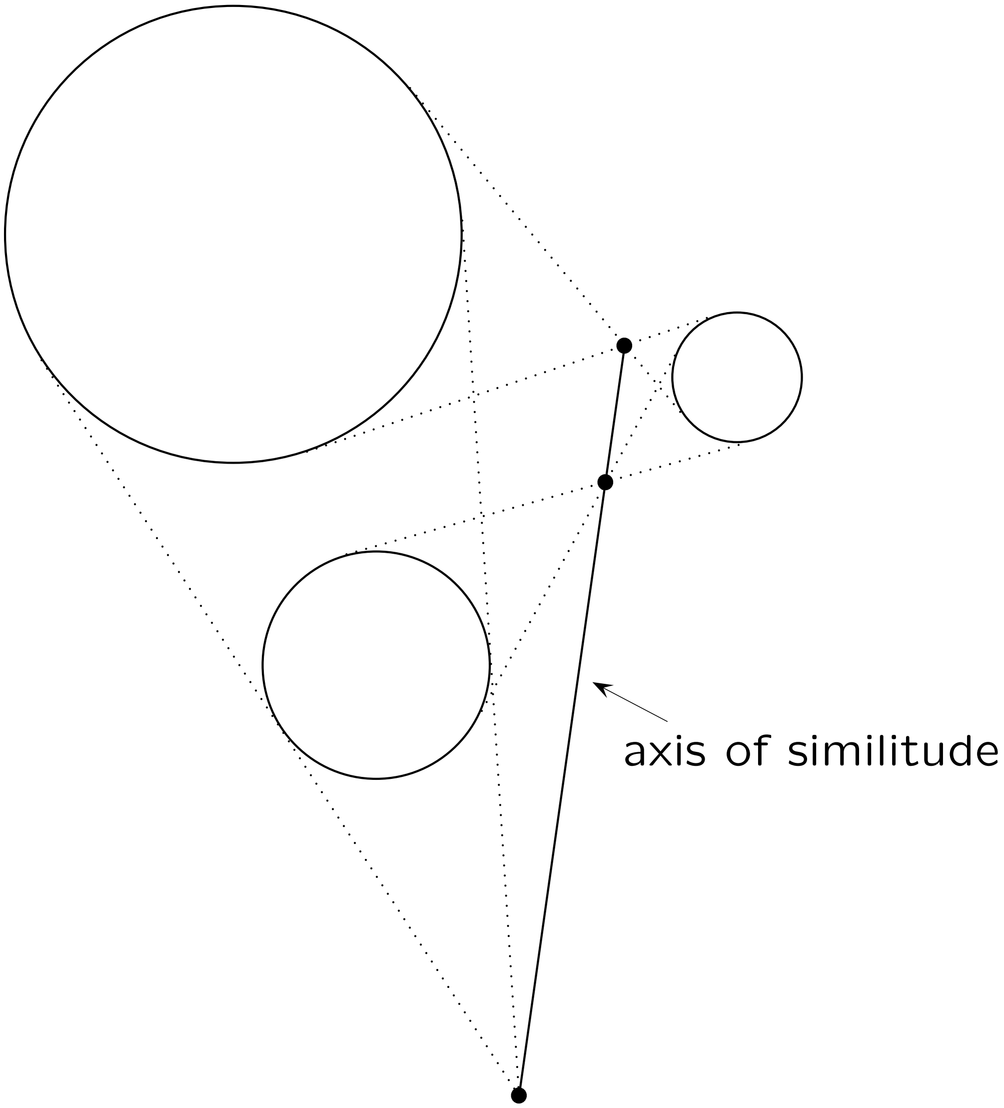
also.

NB Care is required in the last lemma. It is no good saying "$AO_1 : AO_2 = O_1P_1 : O_2P_2$ and $\angle O_1AP_1 = \angle O_2AP_2$, so $\triangle AO_1P_1$ is similar to $\triangle AO_2P_2$", because exactly the same argument would give "$AO_1 : AO_2 = O_1P_1 : O_2Q_2$ and $\angle O_1AP_1 = \angle O_2AQ_2$, so $\triangle AO_1P_1$ is similar to $\triangle AO_2Q_2$", which is patently false. For similarity, or congruence, we need two sides and the *included* angle, which is not what we have here.

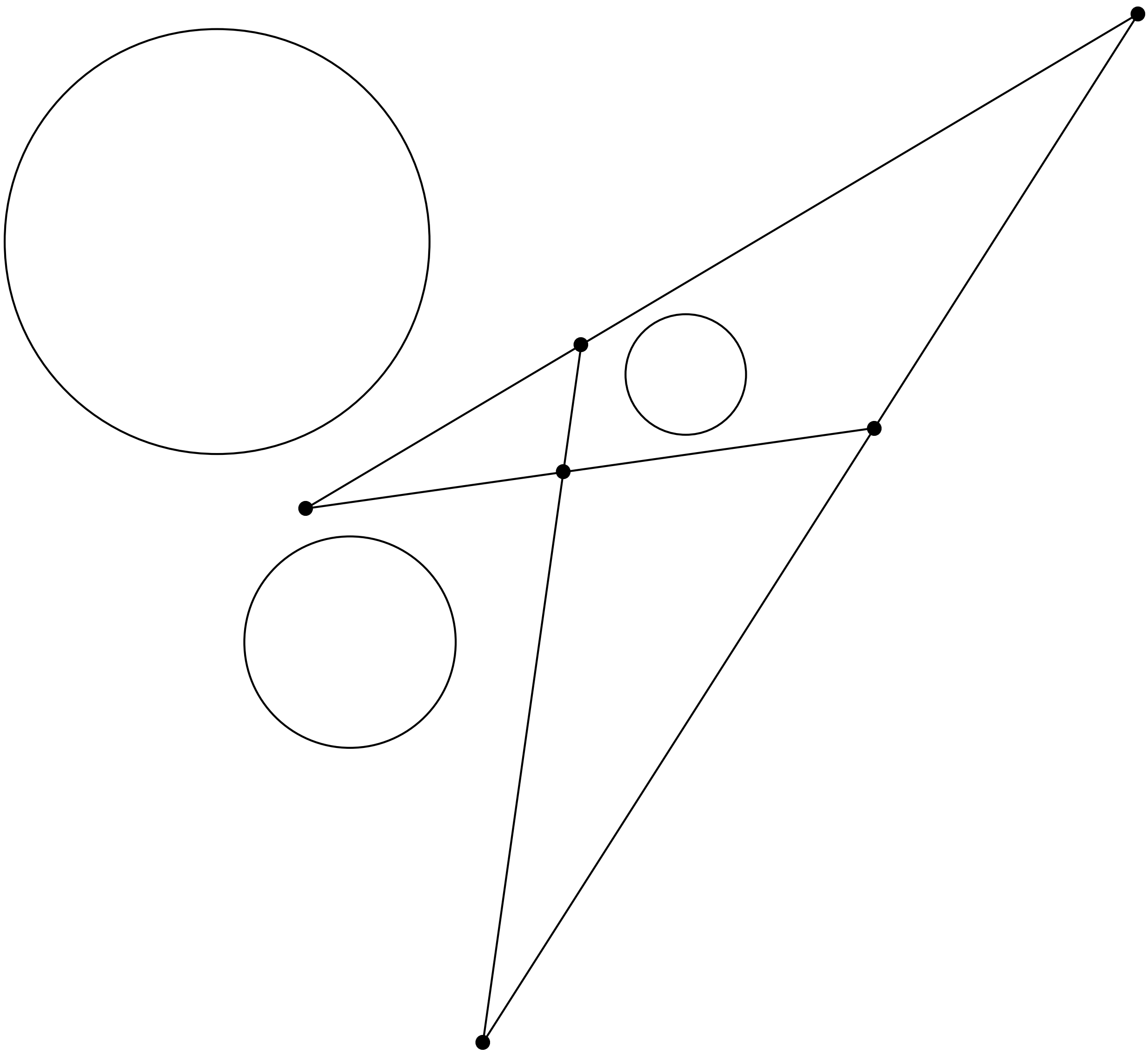
A correct method would be to choose P_1 on the one circle, and then put P_2 on the other circle so that O_1P_1 and O_2P_2 are parallel. (So we don't—yet—know that P_2 is on AP_1.) We have $AO_1 : AO_2 = O_1P_1 : O_2P_2$ and $\angle AO_1P_1 = \angle AO_2P_2$, so $\triangle AO_1P_1$ is similar to $\triangle AO_2P_2$. (This time the angle *is* the included angle.) It follows that $\angle O_1AP_1 = \angle O_2AP_2$, so that P_2 is on AP_1.



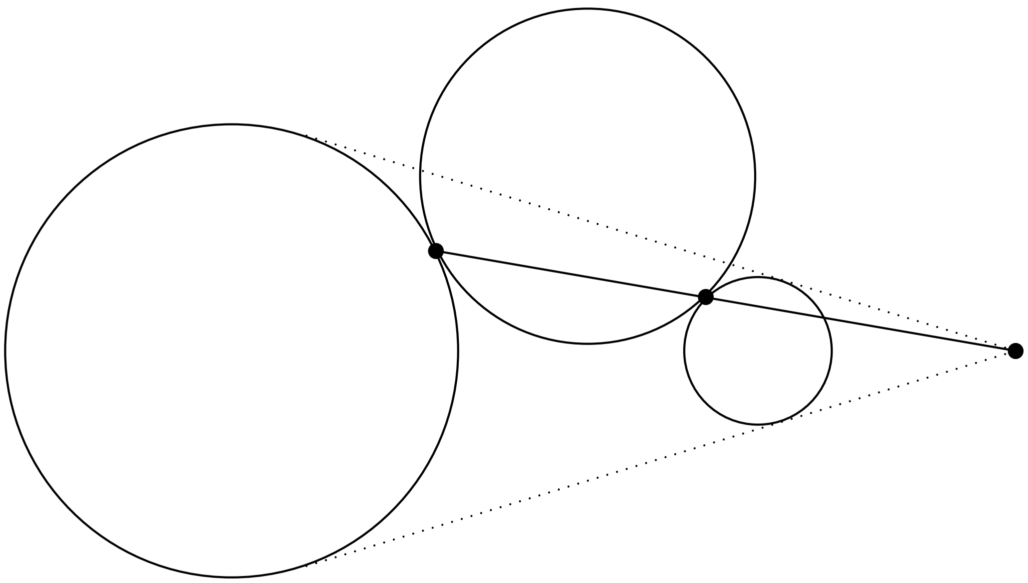
Theorem: The external centres of similitude of three circles, taken in pairs, lie on a line, the (external) *axis of similitude* of the circles.



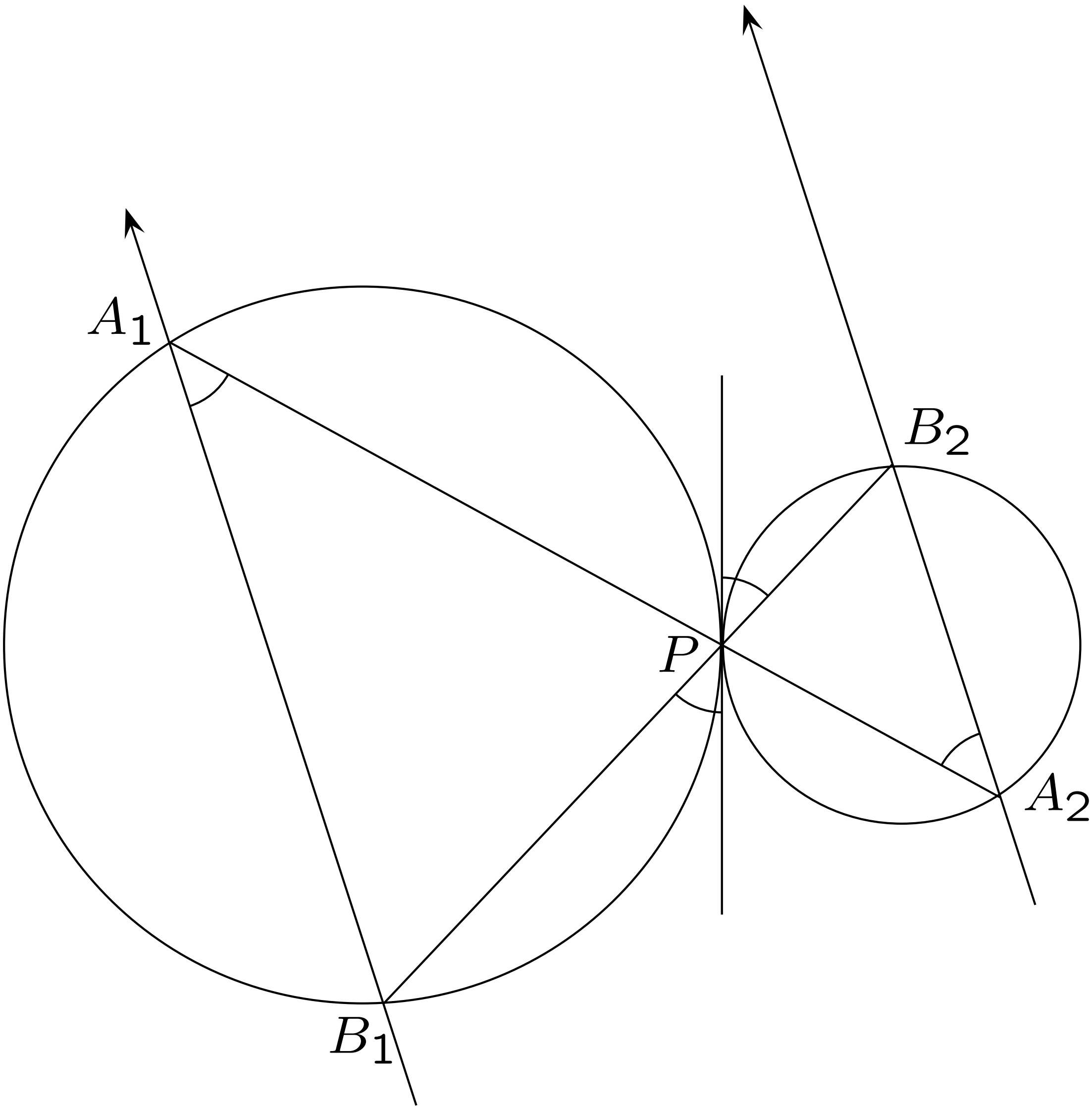
Theorem: Two internal and one external centres of similitude of three circles, taken in pairs, lie on a line, an (internal) *axis of similitude* of the circles.



The four axes of similitude of three circles, one external and three internal.

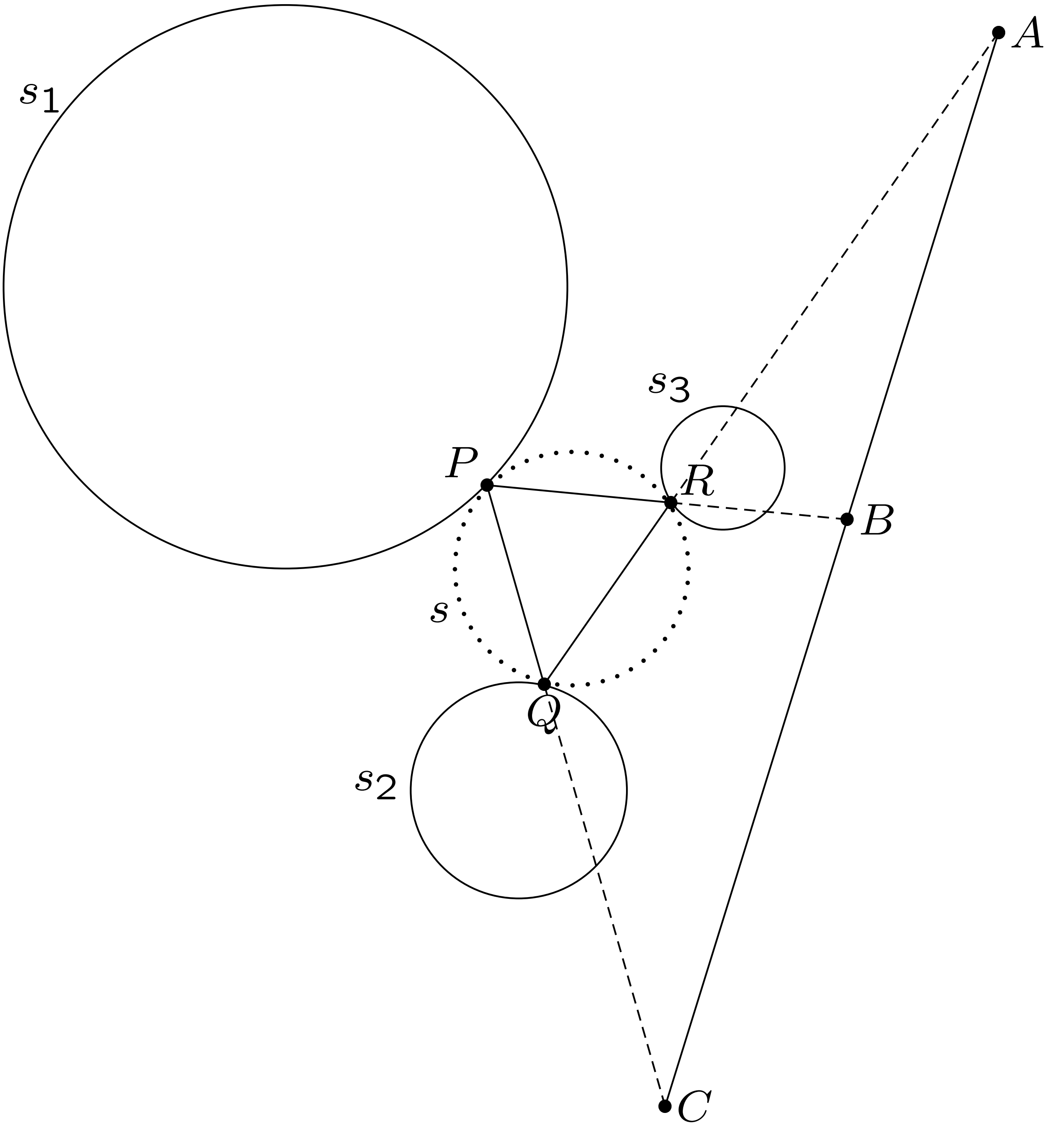


A special case



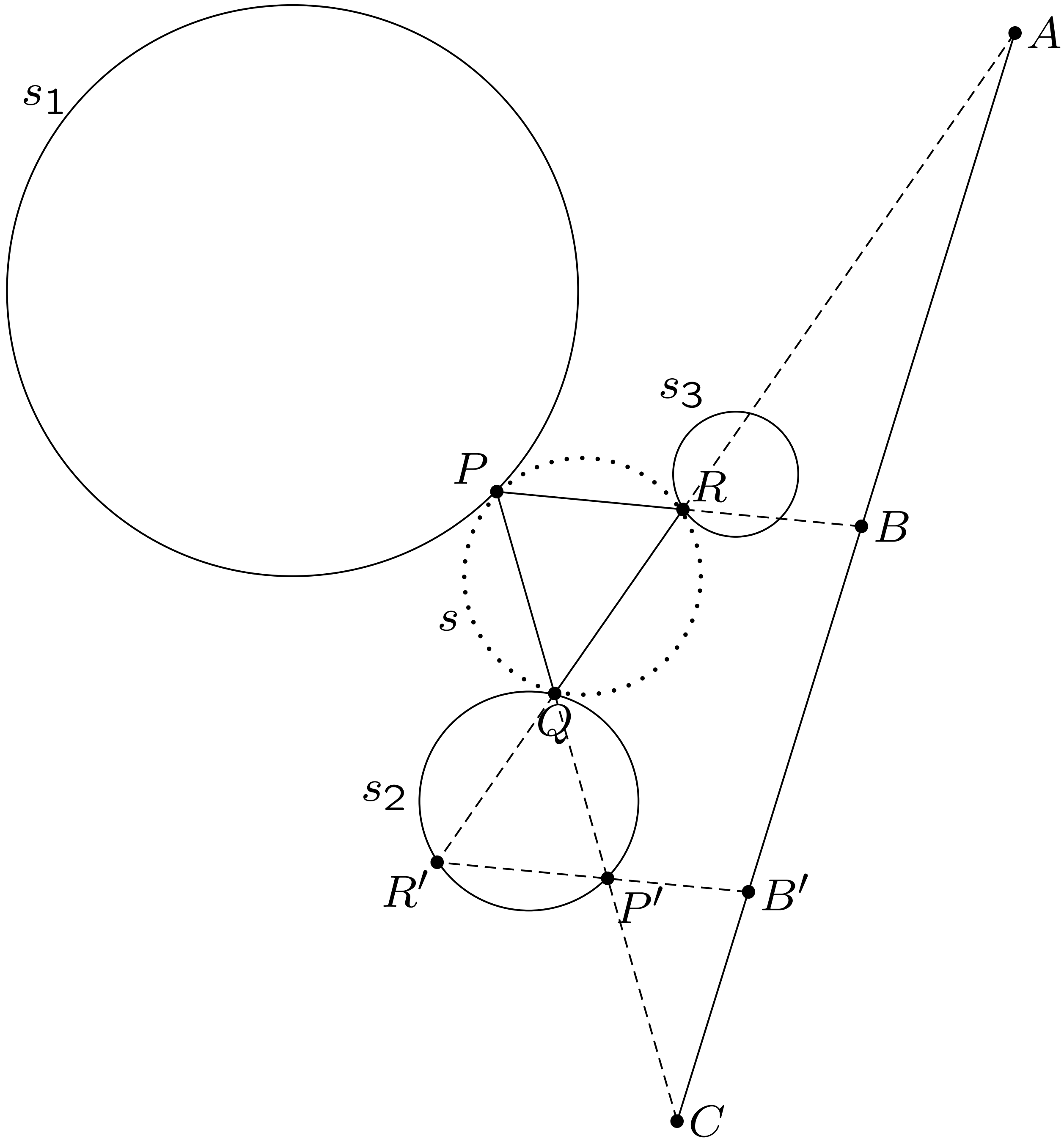
Lemma: Given a circle through A_1, B_1, P, and a second circle touching the first at P, let A_1P, B_1P meet the second circle again at A_2, B_2 respectively. Then A_1B_1 is parallel to A_2B_2.

Proof: Put in the common tangent at P, and use the alternate segment theorem, twice.


A
s1
s3
P
R
B
s
Q
s2
C

We are given circles s_1, s_2, s_3, and we have to construct the circle, s, or (equivalently) we have to find the points of contact P, Q, R.

First construct the axis of similitude ABC of s_1, s_2, s_3, and note that QR, RP, PQ pass through the three centres of similitude A, B, C respectively.


A
s1
s3
P
R
B
s
Q
s2
R′
P′
B′
C

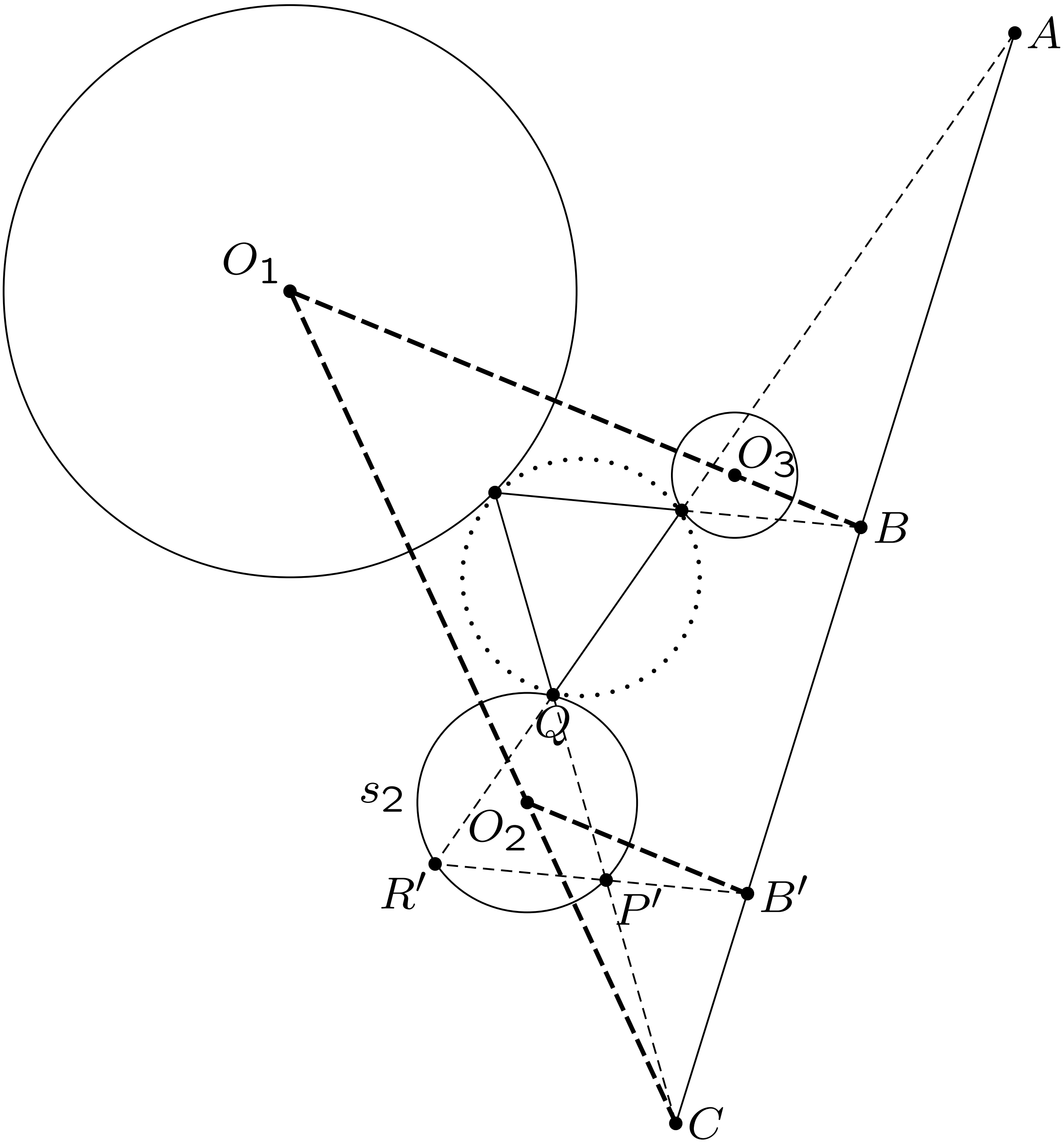
Let PQ meet s_2 again at P', and let RQ meet s_2 again at R'. Let $R'P'$ meet the axis of similitude AC at B'.

It would be sufficient to determine the points P', Q and R', on s_2. Note that QR' goes through A, $R'P'$ goes through B', and $P'Q$ goes through C. We have A and C (and B), but can we locate B'?

Note that PR and $P'R'$ are parallel. So $\triangle CBP$ is similar to $\triangle CB'P'$, and thus

$$CB : CB' = CP : CP' = r_1 : r_2$$

where r_i is the radius of s_i, for each i.



A
O_1
O_3
B
Q
s_2
O_2
R'
P'
B'
C

So

$$CB : CB' = r_1 : r_2$$

and also

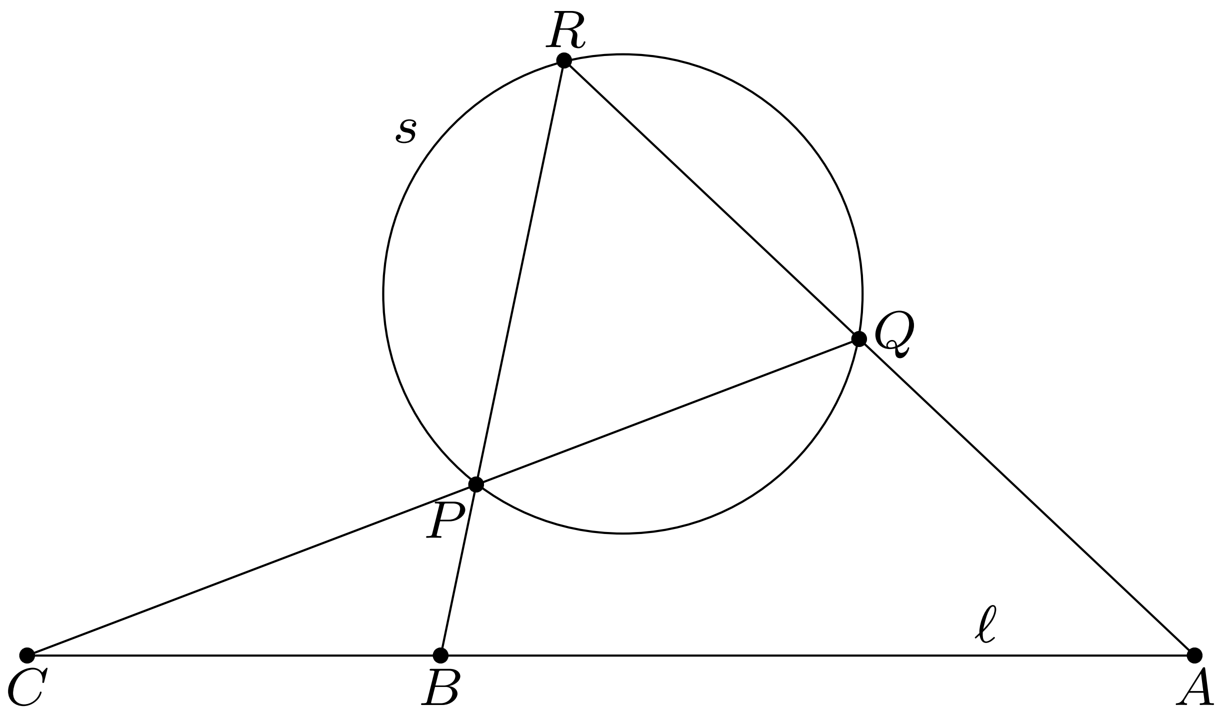
$$CO_1 : CO_2 = r_1 : r_2$$

where O_i is the centre of s_i, for each i.

It follows that $\triangle CO_1B$ and $\triangle CO_2B'$ are similar, and hence O_1B is parallel to O_2B', that is, O_1O_3 is parallel to O_2B'.

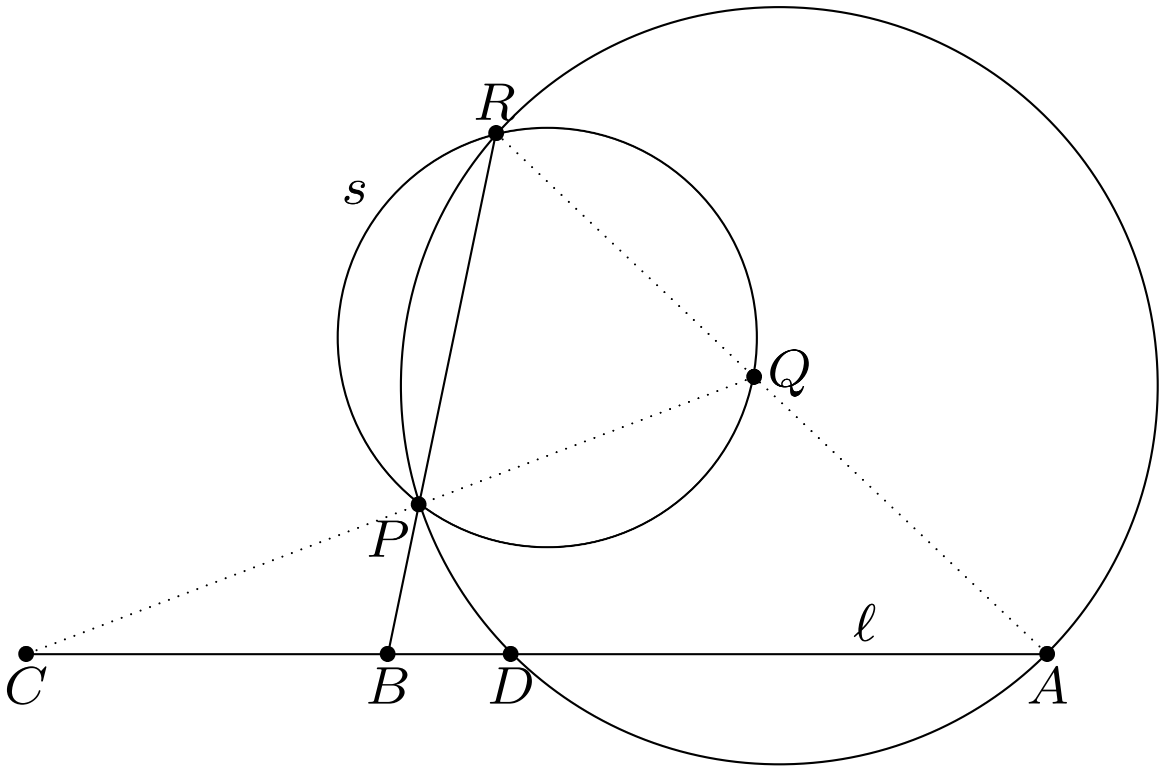
So to locate B', construct a line through O_2 parallel to O_1O_3, and this line meets the axis of similitude AC at B'.

To finish, we must solve the following: given the circle s_2, and three collinear points A, B', C, locate P', Q, R' on s_2 so that QR', $R'P'$, $P'Q$ pass through A, B', C respectively.



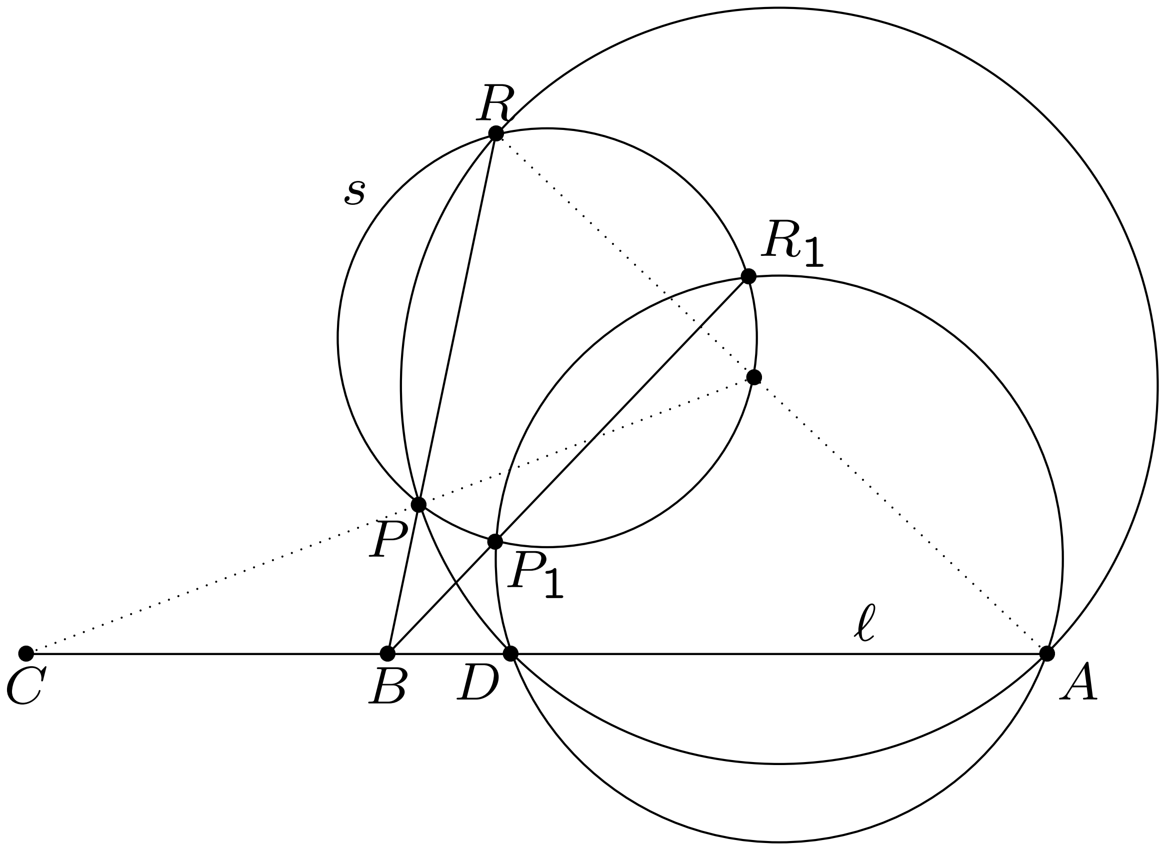
Cleaning up the notation, we have circle s, line ℓ, and points A, B, C on ℓ. We must construct P, Q, R on s so that A lies on QR, B lies on RP, and C lies on PQ.

(s, B, P, R were previously s_2, B', P', R', respectively.)



Consider the circle through P, R, and A, and suppose this circle meets ℓ again at D.

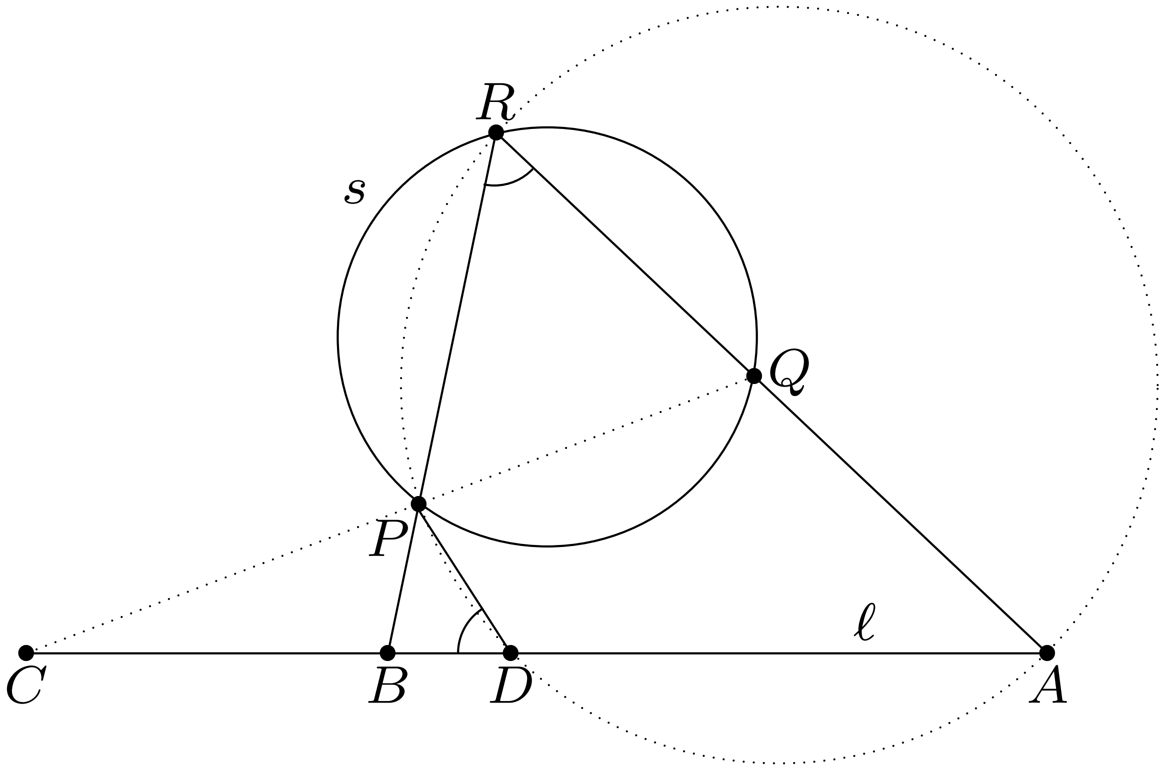
(But remember, we don't actually know where P and R—or Q—are, yet.)



If we *guess* P_1, R_1 for the positions of P, R, and put in the circle through P_1, R_1 and A, we get the *same* intersection, D, with ℓ !

So we can do this to find the position of D, even though we still don't know the positions of P and R.

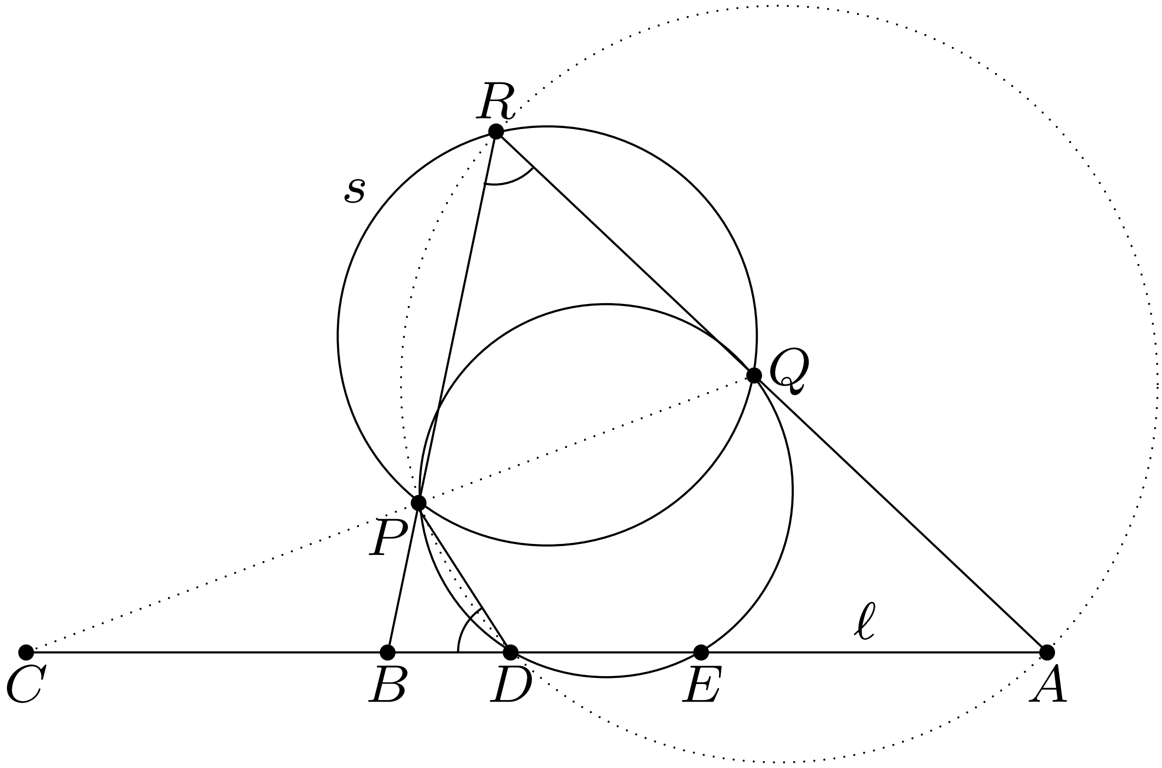
(B is the radical centre of the three circles.)



To recap: we know where A, B, C, D are, but not P, Q, or R.

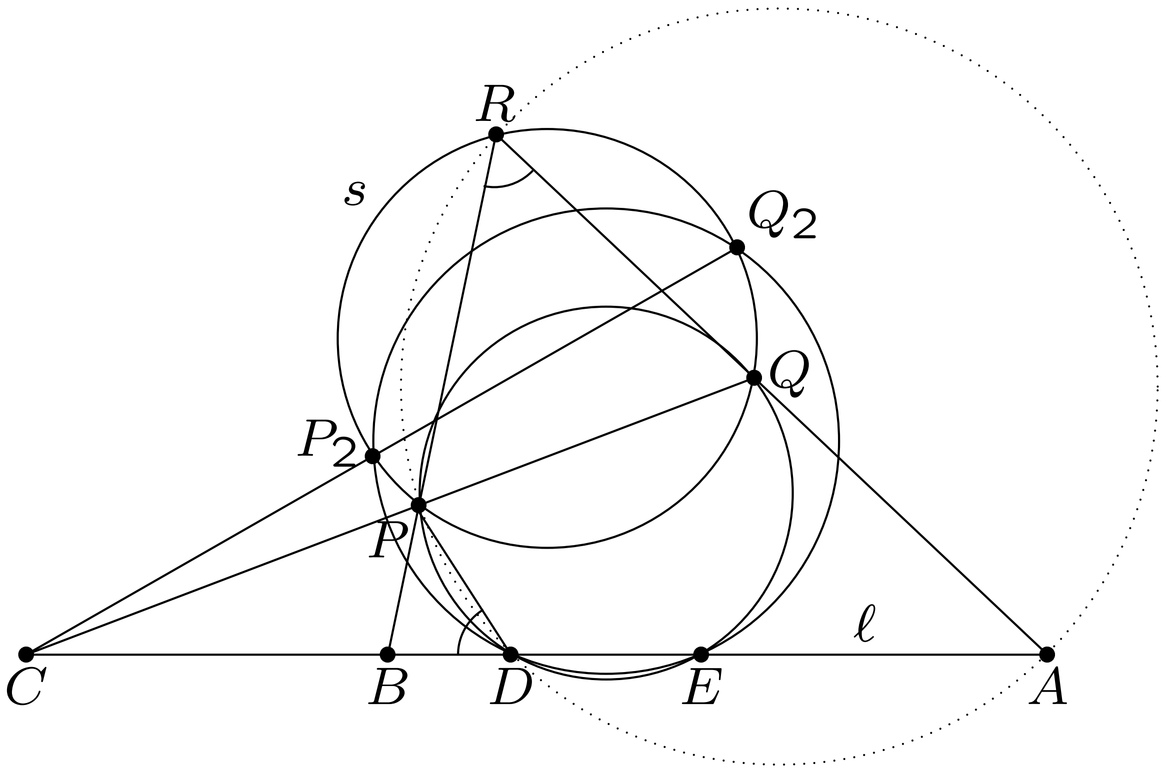
Note that, because P, R, A, D lie on a circle, we have $\angle PRA = \angle PDB$.

Exercise: If DP meets s again at S, then SQ is parallel to ℓ.



Now consider the circle through P, Q, and D, and suppose this circle meets ℓ again at E.

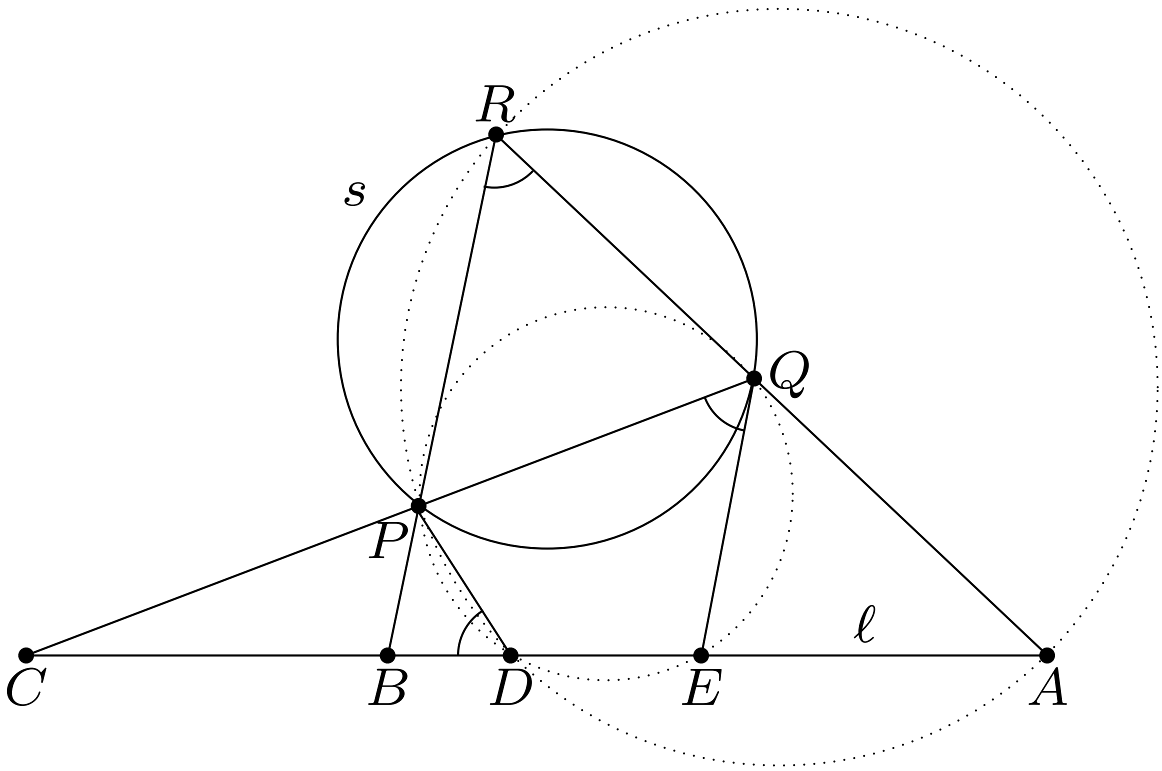
Even though we don't yet know where P and Q are, we can use the same trick to find E that previously we used to find D:



If we *guess* P_2, Q_2 for the positions of P, Q, and put in the circle through P_2, Q_2 and D, we get the *same* intersection, E, with ℓ.

So we can do this to find the position of E, even though we still don't know the positions of P and Q.

(C is the radical centre of the three circles—the solid ones, that is.)

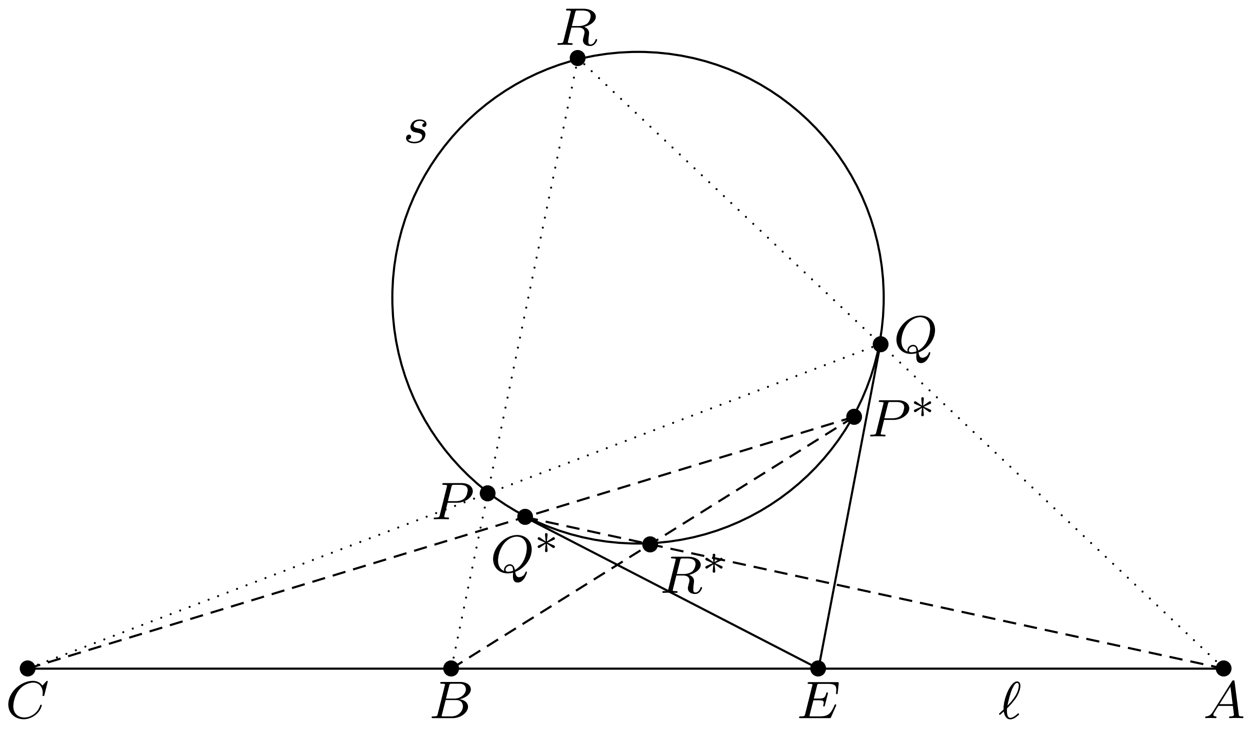


Note that, because P, Q, E, D lie on a circle, we have $\angle PQE = \angle PDB$.

But this means $\angle PQE = \angle PRQ$, so that QE must be the tangent to s at Q, by the alternate segment theorem.

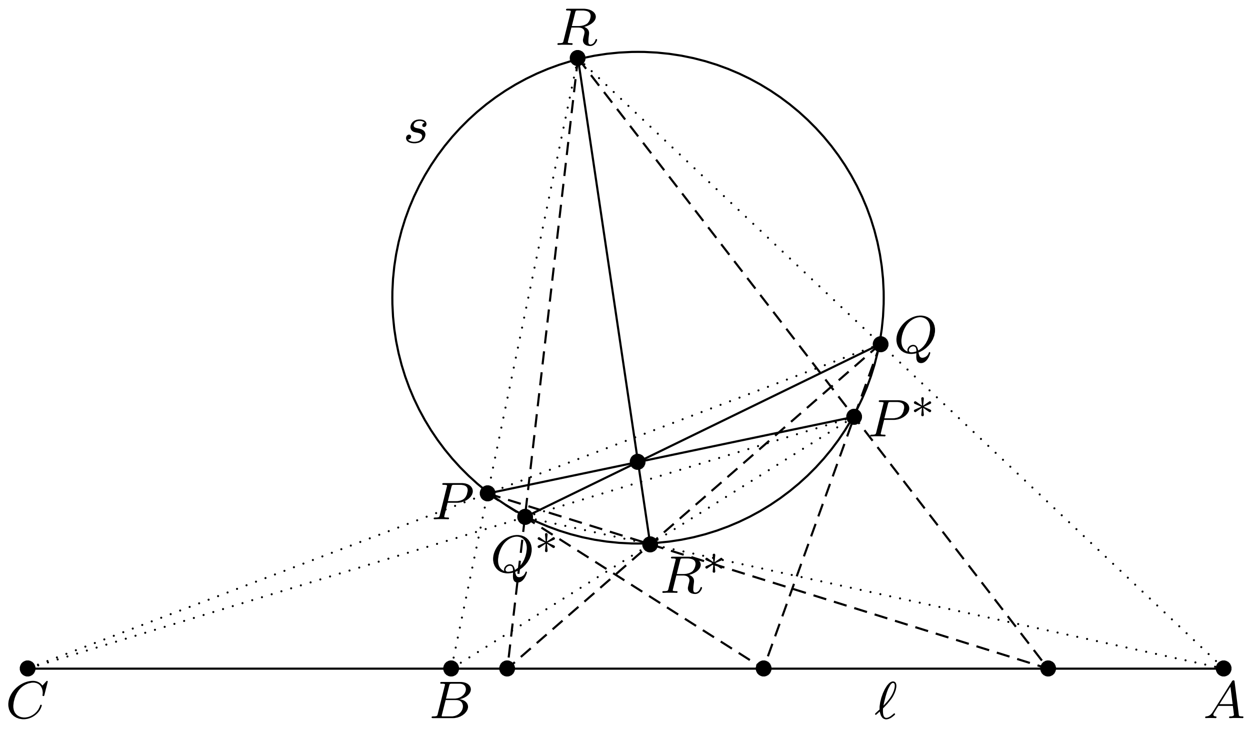
Since we know where E is, we can now construct this tangent to find the position of Q.

Joining Q to A and C gives us the positions of R and P, and we have finished.



Of course, there are *two* tangents from E to the circle s, and the second one, EQ^*, gives rise to a second set of points on s: P^*, Q^*, R^*, such that A lies on Q^*R^*, B lies on R^*P^*, and C lies on P^*Q^*.

This means that ℓ, which was one of the axes of similitude of the original three circles, gives rise to two circles touching the original three circles; and likewise in general we expect two such solutions from each of the four axes of similitude, and hence eight solutions in all.



A few more facts about this diagram:

The lines PP^*, QQ^*, RR^* meet at a point.

The lines PQ^*, QP^* meet on ℓ; and so do the lines QR^*, RQ^*; and also the lines RP^*, PR^*.

We know the tangents at Q and Q^* meet on ℓ; and so the tangents at P and P^* must also meet on ℓ; and also the tangents at R and R^*. (These six tangents are *not* shown in the diagram.)