King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

MSC EXAMINATION

7CCMMS20 (CMMS20) ALGEBRAIC GEOMETRY

SUMMER 2011

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS. FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO FOUR QUESTIONS. ONLY THE BEST FOUR QUESTIONS WILL COUNT TOWARDS GRADES A OR B, BUT CREDIT WILL BE GIVEN FOR ALL WORK DONE FOR LOWER GRADES.

NO CALCULATORS ARE PERMITTED.

TURN OVER WHEN INSTRUCTED

1. **a)** Let K be a field with $\operatorname{char} K \neq 2$. Find a parametrization of the conic $x^2 + y^2 = 2$ on $\mathbb{A}^2_K$ by considering a variable line through the point $(1, 1)$.

b) Suppose $K = \mathbb{Q}$. Show that the conic $x^2 + y^2 = 2$ has infinite number of points.

c) Prove that for each collection of points $P_1, ..., P_5$ on $\mathbb{A}^2_{\mathbb{C}}$ there is at least one conic C that contains these points.

d) Let C be a non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$ and let $F_d(X, Y, Z)$ be a homogeneous polynomial of degree d. Prove that either the curve C is contained in the curve $F_d = 0$, or the intersection of C with $F_d = 0$ has at most $2d$ points. (You may assume that C is given by the equation $XZ - Y^2 = 0$.)

2. In this problem K denotes an algebraically closed field.

a) (i) Define the *affine algebraic variety* $V(I) \subset \mathbb{A}^n_K$ determined by an ideal $I \subset K[x_1, x_2, ..., x_n]$.

(ii) Define the ideal $I(X)$ of a set $X \subset \mathbb{A}^n_K$.

(iii) Take $X \subset \mathbb{A}^1_{\mathbb{C}}$ consisting of all integer numbers, $X = \mathbb{Z}$. What is $I(X)$?

b) Find an ideal $J \subset K[x, y]$ such that $V(J)$ is the union of the points $(0, 0)$, $(1, 1)$ and the line $x + y = 1$ in $\mathbb{A}^2_K$.

c) Let V_1, $V_2 \subset \mathbb{A}^n_K$ be affine algebraic varieties. Prove that $V_1 \cup V_2$ is also an affine algebraic variety.

d) Let $J = (x^2(x-1) - y, y)$ be an ideal in $K[x, y]$. Find $V(J)$ and find $f \in I(V(J)) \setminus J$. Give reasons for your answer.

3. **a)** Explain what is meant by a morphism $f : V \to W$ between two affine varieties over an algebraically closed field K. When are V and W called isomorphic?

b) Consider the affine variety $V = V(x^2 - y, x^3 - z)$ in $\mathbb{A}^3_{\mathbb{C}}$. Show that V is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$.

c) Let V be an irreducible affine variety in $\mathbb{A}^n_{\mathbb{C}}$. Give the definition of the coordinate ring of V.

d) Construct a bijective morphism from $\mathbb{A}^1_{\mathbb{C}}$ to $V(x^2 - y^3) \subset \mathbb{A}^2_{\mathbb{C}}$.

e) Show that the affine varieties $\mathbb{A}^1_{\mathbb{C}}$ and $V(x^2 - y^3)$ are not isomorphic.

4. **a)** What does it mean to say that a commutative ring is Noetherian?

b) Give an example of a ring that is not Noetherian.

c) State and prove Hilbert's Basis Theorem.

d) If $\mathbb{Z}_{57}$ denotes the ring of integers modulo 57, is $\mathbb{Z}_{57}[X]$ a Noetherian ring? Give reasons for your answer.

5. In a)-c) V denotes an irreducible hypersurface in $\mathbb{A}^n_K$.

a) Define the tangent space $T_P(V)$ to an irreducible hypersurface $V \subset \mathbb{A}^n_K$ in affine space at a point $P \in V$.

b) What does it mean to say that P is a singular point of V?

c) Show that if the ground field K is algebraically closed with $\mathrm{char\,K} = 0$ then the set of non-singular points of V is non-empty. [You may assume Nullstellensatz and the fact that $K[X_1, ..., X_n]$ is an UFD.]

d) Determine, giving your reasoning, the set of singular points of the hypersurface $V \subset \mathbb{A}^3_{\mathbb{C}}$ defined by

$$f = (x-1)^2x^2 + x^2y^2 + z^2.$$