King's College London

University Of London

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Candidate No: .... **Desk No:** ....

MSc Examination

7CCMMS05 (CMMS05) Basic Analysis

Summer 2011

Time Allowed: Two Hours

All questions carry equal marks. Full marks will be awarded for complete answers to FOUR questions. Only the best FOUR questions will count towards grades A or B, but credit will be given for all work done for lower grades.

You are permitted to use a Calculator.
ONLY CALCULATORS APPROVED BY THE COLLEGE MAY BE USED.

TURN OVER WHEN INSTRUCTED

1. **A** Let A be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) x is a limit point of A;

(ii) A is closed;

(iii) $\overline{A}$ is the closure of A.

[4 marks]

B Give an example of two sets $A, B \subset \ell^2$ such that neither A nor B is closed yet $A \cup B$ is closed in ℓ^2. Proof is not required.

[6 marks]

C Let $A \subset CL^2[0, 1]$ be the set

$$A = \{f \mid f(x) = 0 \quad \forall x \in (0, 1/2) \text{ and } |f(x)| \leq 2 \quad \forall x \in [0, 1]\}.$$

Determine the diameter of A in $CL^2[0, 1]$ and prove your claim.

[6 marks]

D Prove that a set A in a metric space (X, ρ) is closed if and only if its complement A^c is open.

[9 marks]

2.

A (i) State precisely what it means to say that a set is countable.

(ii) State precisely what it means to say that a subset A of a metric space (X, ρ) is dense.

(iii) State precisely what it means to say that a normed linear space is separable.

[4 marks]

B Give an example of an uncountable set. Proof is not required.

[5 marks]

C Describe explicitly a dense countable set $A \subset \ell^1$. Prove that your set is indeed dense and countable. You may assume without proof that (i) the set of all rational numbers is countable; (ii) a countable union of countable sets is countable; (iii) a Cartesian product of two countable sets is countable.

[7 marks]

D Prove that ℓ^∞ is not separable. You may use without proof the fact that for any non-empty set A, there is no bijection between A and the set of all subsets of A.

Hint: Construct an uncountable set $X \subset \ell^\infty$ such that the balls $B_{1/2}(x)$ and $B_{1/2}(y)$ are disjoint for distinct $x, y \in X$.

[9 marks]

3. **A** (i) State the axioms of a norm in a linear space.

(ii) For $1 \leq p \leq \infty$, state precisely what it means to say that the sequence $x = (x_1, x_2, \dots)$ belongs to ℓ^p.

(iii) For $1 \leq p \leq \infty$, write down the definition of the norm $\|x\|_p$ of an element $x \in \ell^p$.

[4 marks]

B Give an example of a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of elements of ℓ^1 such that $\|x^{(n)}\|_2 \to 0$ yet $\|x^{(n)}\|_1 \not\to 0$ as $n \to \infty$. Proof is not required.

[6 marks]

C For any $\alpha > 0$, let $x^{(\alpha)}$ be the sequence $x^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, \dots)$, where $x_n^{(\alpha)} = n^{-\alpha}(1 + \log n)^{-2}$. For each p, $1 \leq p \leq \infty$, determine the set

$$A_p = \{\alpha > 0 \mid x^{(\alpha)} \in \ell^p\}$$

explicitly in terms of p and prove your claim.

[6 marks]

D Prove that the space ℓ^1 is complete.

Hint: Assume that $x^{(k)} \in \ell^1$ is Cauchy. Prove that for any $i \in \mathbb{N}$, the sequence $\{x_i^{(k)}\}_{k=1}^{\infty}$ is Cauchy. Use the completeness of $\mathbb{C}$ to define $x_i = \lim_{k\to\infty} x_i^{(k)}$. Now prove that $x = (x_1, x_2, \dots) \in \ell^1$ and $x^{(k)} \to x$ in ℓ^1.

[9 marks]

4. **A** (i) For an index $1 \leq p < \infty$ and a compact interval $[a, b] \subset \mathbb{R}$, state precisely what it means to say that $f \in CL^p[a, b]$.

(ii) Define the space $L^p(a, b)$, $1 \leq p < \infty$.

(iii) State the Hölder inequality for functions $f, g \in C[a, b]$.

[4 marks]

B Give an example of a sequence $f_n \in CL^2[0, 1]$ such that $\|f_n\|_1 \to 0$ yet $\|f_n\|_2 \not\to 0$ as $n \to \infty$. Here $\|f\|_p$ is the norm of f in $CL^p[0, 1]$. Proof is not required.

[6 marks]

C For each of the following sequences, decide whether it is convergent in the given space and if is, determine the limit. Complete proof is not required but you should explain your reasoning.

(i) $f_n(x) = x^n$ in $C[0, 1]$

(ii) $f_n(x) = x^n$ in $C[0, \frac{1}{2}]$

(iii) $f_n(x) = x^n$ in $CL^1[0, 1]$

(iv) $f_n(x) = ne^{-n^2x^2}$ in $CL^2[-1, 1]$

[6 marks]

D Prove that the space $CL^1[-1, 1]$ is not complete.

Hint: Construct a sequence $\{f_n\}_{n=1}^{\infty} \subset CL^1[-1, 1]$ which converges to a discontinuous function in the L^1 norm. Conclude that $\{f_n\}_{n=1}^{\infty}$ cannot converge in CL^1 to an element of this space.

[9 marks]

5. **A** Let K be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) K is compact;

(ii) K is sequentially compact;

(iii) K is totally bounded.

[4 marks]

B Give an example of a closed bounded set $A \subset \ell^1$ and of a sequence of elements $\{x^{(n)}\}_{n=1}^{\infty} \subset A$ such that this sequence has no convergent subsequences. Proof is not required.

[6 marks]

C In ℓ^∞, let K be the following set:

$$K = \{x \mid |x_j| \le s_j, j = 1, 2, \dots\},$$

where s_j is a sequence of non-negative numbers such that $s_j \to 0$ as $j \to \infty$. Prove that K is closed and totally bounded.

Hint: In order to prove total boundedness, fix $\varepsilon > 0$ and consider the map $f_n : \ell^\infty \to \ell^\infty$

$$f_n : (x_1, x_2, \dots, x_n, \dots) \mapsto (x_1, x_2, \dots, x_n, 0, 0, \dots).$$

Choose n sufficiently large (depending on ε). Prove that $f_n(K)$ is totally bounded. Conclude that K is totally bounded.

[6 marks]

D Let (X, ρ) be a compact metric space and let $f : X \to \mathbb{C}$ be a continuous function. Prove that f is uniformly continuous.

[9 marks]

6. **A** Let $(X, \|\cdot\|_X)$ be a Banach space. State precisely what it means to say that

(i) λ is a bounded linear functional on X;

(ii) $\|\lambda\|_{X^*}$ is the norm of λ in X^*;

(iii) X is reflexive.

[4 marks]

B Give an example of a bounded linear functional λ on $C[0,1]$ such that there is no non-zero element $f \in C[0,1]$ with the property $|\lambda(f)| = \|\lambda\|_{(C[0,1])^*} \|f\|_{C[0,1]}$. Proof is not required.

[6 marks]

C Let λ be a linear functional on $CL^3[0,5]$ defined by

$$\lambda(f) = \int_0^5 f(x)dx.$$

Determine the norm of λ and prove your claim. You are allowed to use Hölder inequality but no other theorems from the course.

[6 marks]

D Prove that any bounded linear functional λ on ℓ^1 has the form

$$\lambda(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_1, x_2, \dots) \in \ell^1,$$

with some $y = (y_1, y_2, \dots) \in \ell^\infty$.

Hint: Let $e_n = (\underbrace{0, \dots, 0, 1}_{n}, 0, \dots) \in \ell^1$. Set $y_n = \lambda(e_n)$. Prove that $y \in \ell^\infty$ and that λ is given by the above formula.

[9 marks]