7CCMMS05 (CMMS05) Basic Analysis, Summer 2011
Exam solutions

Syllabus, lecture notes, past years exam papers can be found at the course webpage http://www.mth.kcl.ac.uk/courses/cmms05.html

Each question has the following structure, with some minor variations:

A Definition [4 marks]

B Example [6 marks]

C Calculation [6 marks]

D Proof [9 marks]

QUESTION 1

A Let A be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) x is a limit point of A;

(ii) A is closed;

(iii) $\overline{A}$ is the closure of A.

[4 marks]

Solution (bookwork):

(i) x is a limit point of A if there exists a sequence of elements $\{x_n\}_{n=1}^{\infty} \subset A$ which converges to x. [2 marks]

(ii) A set A is closed if it contains all of its limit points. [1 mark]

(iii) The closure of A is the intersection of all closed sets B such that $A \subset B$. [1 mark]

B Give an example of two sets $A, B \subset \ell^2$ such that neither A nor B is closed yet $A \cup B$ is closed in ℓ^2. Proof is not required.

[6 marks]

Solution (unseen): Let

$$A = \{x \mid x_1 \in (-1, 2], x_n = 0 \text{ for } n = 2, 3, \dots\},$$
$$B = \{x \mid x_1 \in [-2, 1), x_n = 0 \text{ for } n = 2, 3, \dots\};$$
$$\text{then } A \cup B = \{x \mid x_1 \in [-2, 2], x_n = 0 \text{ for } n = 2, 3, \dots\} \text{ is closed.}$$

[6 marks]

C Let $A \subset CL^2[0,1]$ be the set

$$A = \{f \mid f(x) = 0 \quad \forall x \in (0, 1/2) \text{ and } |f(x)| \leq 2 \quad \forall x \in [0, 1]\}.$$

Determine the diameter of A in $CL^2[0,1]$ and prove your claim.
[6 marks]

Solution (unseen): The diameter of A is $2\sqrt{2}$. [1 mark]

Indeed, for any $f, g \in A$ we have

$$\|f - g\|_{L^2}^2 \leq \int_0^1 |f(x) - g(x)|^2 dx$$
$$= \int_{1/2}^1 |f(x) - g(x)|^2 dx \leq \int_{1/2}^1 4^2 dx = 16 \times \frac{1}{2} = 8,$$

and so $\operatorname{diam} A \leq \sqrt{8} = 2\sqrt{2}$. [2 marks]

On the other hand, for any $\varepsilon \in (0, 1/4)$ let

$$f_\varepsilon(x) = \begin{cases} 0 & 0 \leq x \leq 1/2, \\ (x - 1/2)/\varepsilon & 1/2 \leq x \leq 1/2 + 2\varepsilon, \\ 2 & 1/2 + 2\varepsilon \leq x \leq 1 \end{cases}$$

and $g_\varepsilon(x) = -f_\varepsilon(x)$. Then $f_\varepsilon, g_\varepsilon \in A$ and

$$\|f_\varepsilon - g_\varepsilon\|_{L^2}^2 = \frac{4}{\varepsilon^2} \int_{1/2}^{1/2+2\varepsilon} (x - 1/2)^2 dx + \int_{1/2+2\varepsilon}^1 4^2 dx$$
$$= \frac{4}{\varepsilon^2} \int_0^{2\varepsilon} x^2 dx + 16(\tfrac{1}{2} - 2\varepsilon) = \frac{4}{\varepsilon^2} \frac{8\varepsilon^3}{3} + 8 - 32\varepsilon = 32\varepsilon/3 + 8 - 32\varepsilon = 8 - 64\varepsilon/3.$$

Since ε can be taken arbitrary small, $\|f_\varepsilon - g_\varepsilon\|_{L^2}$ can be made to be arbitrary close to $2\sqrt{2}$. Thus, $\operatorname{diam} A \geq 2\sqrt{2}$. [3 marks]

D Prove that a set A in a metric space (X, ρ) is closed if and only if its complement A^c is open.
[9 marks]

Solution (bookwork): 1. Suppose A is open and let $x \in A$. Then there exists a ball $B_r(x) \subset A$. This ball does not contain elements of A^c. Thus, if a sequence of elements of A^c converges, it cannot converge to x. It follows that all limits of convergent sequences of elements of A^c lie in A^c, so A^c is closed. [4 marks]

2. Suppose A is closed and let $x \in A^c$. Consider the balls $B_{1/n}(x)$. Suppose that each of these balls contains an element of A. Then we can choose a sequence of elements of A which converges to x. Since A is closed, it follows that $x \in A$; this contradicts to the choice $x \in A^c$. Thus, there exists n such that $B_{1/n}(x)$ does not contain elements of A; then $B_{1/n}(x) \subset A^c$. Since x is arbitrary, it follows that A^c is open. [5 marks]

QUESTION 2

A

(i) State precisely what it means to say that a set is countable.

(ii) State precisely what it means to say that a subset A of a metric space (X, ρ) is dense.

(iii) State precisely what it means to say that a normed linear space is separable.

[4 marks]

Solution (bookwork):

(i) A set S is countable if there is a bijection between S and $\mathbb{N}$. [1 mark]

(ii) A subset A of a metric space (X, ρ) is dense if $\overline{A} = X$. [1 mark]

(iii) A normed linear space is separable if there exists a countable dense subset of this space. [2 marks]

B Give an example of an uncountable set. Proof is not required.

[5 marks]

Solution (bookwork):

The set of all subsets of $\mathbb{N}$ is uncountable. Alternatively: $\mathbb{R}$ is uncountable. [5 marks]

C Describe explicitly a dense countable set $A \subset \ell^1$. Prove that your set is indeed dense and countable. You may assume without proof that (i) the set of all rational numbers is countable; (ii) a countable union of countable sets is countable; (iii) a Cartesian product of two countable sets is countable.

[7 marks]

Solution (bookwork): Let $\mathbb{Q}_{\mathbb{C}} = \{x + iy \mid x, y \in \mathbb{Q}\} \subset \mathbb{C}$. For $n \in \mathbb{N}$, let

$$A_n = \{x \in \ell^1 \mid x_j \in \mathbb{Q}_{\mathbb{C}} \text{ for } j = 1, \dots, n \text{ and } x_j = 0 \text{ for } j > n\}.$$

Let $A = \cup_{n=1}^{\infty} A_n$. Let us prove that A is countable. There is an obvious bijection between $\mathbb{Q}_{\mathbb{C}}$ and $\mathbb{Q} \times \mathbb{Q}$, so $\mathbb{Q}_{\mathbb{C}}$ is countable.

By induction on n, the Cartesian product $\mathbb{Q}_{\mathbb{C}}^n = \mathbb{Q}_{\mathbb{C}} \times \mathbb{Q}_{\mathbb{C}} \times \cdots \times \mathbb{Q}_{\mathbb{C}}$ (n terms) is countable. There is an obvious bijection between A_n and $\mathbb{Q}_{\mathbb{C}}^n$, so A_n is countable.

Finally, A is countable as a countable union of countable sets A_n. [3 marks]

Let us prove that A is dense in ℓ^1. For $x \in \ell^1$ and a given $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} |x_k| < \varepsilon/2$.

Next, for each $k = 1, \dots, n$, choose $y_k \in \mathbb{Q}_\mathbb{C}$ such that $|y_k - x_k| < \varepsilon/(2n)$. Then $y = (y_1, y_2, \dots, y_n, 0, 0, \dots) \in A_n \subset A$ and $\|y - x\|_1 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. It follows that A is dense. [4 marks]

D Prove that ℓ^∞ is not separable. You may use without proof the fact that for any non-empty set A, there is no bijection between A and the set of all subsets of A.

Hint: Construct an uncountable set $X \subset \ell^\infty$ such that the balls $B_{1/2}(x)$ and $B_{1/2}(y)$ are disjoint for distinct $x, y \in X$.

[9 marks]

Solution (bookwork)

For any $X \subset \mathbb{N}$, let $e^X \in \ell^\infty$ be defined by the following rule: $e_n^X = 1$ if $n \in X$ and $e_n^X = 0$ if $n \notin X$. Then for any $X \neq Y$ we have $\|e^X - e^Y\|_\infty = 1$.

Suppose that ℓ^∞ has a dense countable subset D. Consider the collection of balls $B_{1/2}(e^X)$ over all $X \subset \mathbb{N}$. Since $2^\mathbb{N}$ is uncountable, this collection of balls is uncountable. On the other hand, these balls are disjoint and there is an element of D in each of these balls. Thus, we have a bijection between a subset of D (which is countable) and the set of all such balls. This is a contradiction.

[9 marks]

QUESTION 3

A

(i) State the axioms of a norm in a linear space.

(ii) For $1 \le p \le \infty$, state precisely what it means to say that the sequence $x = (x_1, x_2, \dots)$ belongs to ℓ^p.

(iii) For $1 \le p \le \infty$, write down the definition of the norm $\|x\|_p$ of an element $x \in \ell^p$.

[4 marks]

Solution (bookwork):

(i) A norm is a function $\|\cdot\|$ from a linear space X to $[0, \infty)$ such that:

(1) $\|x\| > 0$ if $x \ne 0$;

(2) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and all $\alpha \in \mathbb{C}$;

(3) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$.

[2 mark]

(ii) If $p < \infty$, then $x = (x_1, x_2, \dots) \in \ell^p$ iff

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

If $p = \infty$, then $x \in \ell^\infty$ iff $\sup_{n=1,2,\dots} |x_n| < \infty$. [1 mark]

(iii) If $p < \infty$, then

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}.$$

If $p = \infty$, then

$$\|x\|_\infty = \sup_{n=1,2,\dots} |x_n|.$$

[1 marks]

B Give an example of a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of elements of ℓ^1 such that $\|x^{(n)}\|_2 \to 0$ yet $\|x^{(n)}\|_1 \not\to 0$ as $n \to \infty$. Proof is not required.

[6 marks]

Solution (unseen):

$$x^{(n)} = (\underbrace{1/n, 1/n, \dots, 1/n}_{n}, 0, 0, \dots).$$

Then $\|x^{(n)}\|_1 = 1$, and $\|x^{(n)}\|_2 = \sqrt{n(1/n)^2} = n^{-1/2} \to 0$ as $n \to \infty$. [6 marks]

C For any $\alpha > 0$, let $x^{(\alpha)}$ be the sequence $x^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, \dots)$, where $x_n^{(\alpha)} = n^{-\alpha}(1 + \log n)^{-2}$. For each p, $1 \le p \le \infty$, determine the set

$$A_p = \{\alpha > 0 \mid x^{(\alpha)} \in \ell^p\}$$

explicitly in terms of p and prove your claim.

[6 marks]

Solution (unseen; similar question appeared in 2008/09 exam paper): Let $p < \infty$. Then $x^{(\alpha)} \in \ell^p$ iff

$$\sum_{n=1}^{\infty} n^{-\alpha p}(1 + \log n)^{-2p} < \infty.$$

The above series converges iff $\alpha p \ge 1$. Thus, $A_p = [1/p, \infty)$ for $p < \infty$.

Finally, if $p = \infty$, then $x^{(\alpha)} \in \ell^\infty$ iff

$$\sup_n n^{-\alpha}(1 + \log n)^{-2} < \infty.$$

This is true for all $\alpha > 0$. Thus, $A_\infty = (0, \infty)$.

[6 marks]

D Prove that the space ℓ^1 is complete.

Hint: Assume that $x^{(k)} \in \ell^1$ is Cauchy. Prove that for any $i \in \mathbb{N}$, the sequence $\{x_i^{(k)}\}_{k=1}^{\infty}$ is Cauchy. Use the completeness of $\mathbb{C}$ to define $x_i = \lim_{k\to\infty} x_i^{(k)}$. Now prove that $x = (x_1, x_2, \dots) \in \ell^1$ and $x^{(k)} \to x$ in ℓ^1.

[9 marks]

Solution (bookwork):

Let $x^{(k)}$ be a Cauchy sequence in ℓ^1. Fix $i \in \mathbb{N}$. Since $|x_i| \le \|x\|_1$, we see that $x_i^{(k)}$ is a Cauchy sequence of complex numbers. Since $\mathbb{C}$ is complete, we see that $x_i^{(k)} \to x_i$ as $k \to \infty$.

Since $x^{(k)}$ is Cauchy, for a given ε there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sum_{i=1}^{n} |x_i^{(k)} - x_i^{(m)}| \le \varepsilon, \qquad \forall k, m \ge N_\varepsilon, \quad \forall n \in \mathbb{N}.$$

Let $m \to \infty$ here; then we get

$$\sum_{i=1}^{n} |x_i^{(k)} - x_i| \le \varepsilon, \qquad \forall k \ge N_\varepsilon, \quad \forall n \in \mathbb{N}.$$

Since n is arbitrary, we get the convergence of the series and the inequality

$$\sum_{i=1}^{\infty} |x_i^{(k)} - x_i| \leq \varepsilon, \qquad \forall k \geq N_\varepsilon. \tag{1}$$

For any given k this means that the sequence $x_i^{(k)} - x_i$ is an element of ℓ^1. Since $x_i^{(k)}$ is also an element of ℓ^1 and ℓ^1 is a linear space, we get that $x = (x_1, x_2, \dots) \in \ell^1$.

Now (1) can be rewritten as $\|x^{(k)} - x\|_1 \leq \varepsilon$, $k \geq N_\varepsilon$. Since ε can be taken arbitrary small, we get that $\|x^{(k)} - x\|_1 \to 0$ as $k \to \infty$.

[9 marks]

QUESTION 4

A (i) For an index $1 \le p < \infty$ and a compact interval $[a,b] \subset \mathbb{R}$, state precisely what it means to say that $f \in CL^p[a,b]$.

(ii) Define the space $L^p(a,b)$, $1 \le p < \infty$.

(iii) State the Hölder inequality for functions $f, g \in C[a,b]$.

[4 marks]

Solution (bookwork):

(i)$f \in CL^p[a,b]$ if $f : [a,b] \to \mathbb{C}$ is continuous. [1 mark]

(ii) $L^p[a,b]$ is the completion of $CL^p[a,b]$ with respect to the norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}.$$

[1 mark]

(iii) The Hölder inequality is

$$\left| \int_a^b f(x)g(x)dx \right| \le \|f\|_p \|g\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. [2 marks]

B Give an example of a sequence $f_n \in CL^2[0,1]$ such that $\|f_n\|_1 \to 0$ yet $\|f_n\|_2 \not\to 0$ as $n \to \infty$. Here $\|f\|_p$ is the norm of f in $CL^p[0,1]$. Proof is not required.

[6 marks]

Solution (unseen):

$$f_n(x) = \frac{\sqrt{n}}{1 + x^2 n^2}.$$

Then

$$\|f_n\|_1 = \int_0^1 \frac{\sqrt{n}}{1 + x^2 n^2} dx = \frac{1}{\sqrt{n}} \int_0^n \frac{dx}{1 + x^2} = O(1/\sqrt{n}),$$

$$\|f_n\|_2^2 = \int_0^1 \frac{n}{(1 + x^2 n^2)^2} dx = \int_0^n \frac{dx}{(1 + x^2)^2} \to \int_0^\infty \frac{dx}{(1 + x^2)^2},$$

as $n \to \infty$. [6 marks]

C For each of the following sequences, decide whether it is convergent in the given space and if is, determine the limit. Complete proof is not required but you should explain your reasoning.

(i) $f_n(x) = x^n$ in $C[0,1]$

(ii) $f_n(x) = x^n$ in $C[0, \frac{1}{2}]$

(iii) $f_n(x) = x^n$ in $CL^1[0,1]$

(iv) $f_n(x) = ne^{-n^2x^2}$ in $CL^2[-1,1]$

[6 marks]

Solution (unseen):

(i)f_n diverges. In fact, f_n converges pointwise to a discontinuous function:

$$f(x) = \begin{cases} 0, & x \in [0,1), \\ 1, & x = 1. \end{cases}$$

Thus, f_n cannot converge to a continuous function in $C[0,1]$. [2 marks]

(ii) f_n converges to 0. Indeed, $|f_n(x)| \leq (1/2)^n \to 0$, $n \to \infty$. [1 mark]

(iii) f_n converges to 0:

$$\|f_n\|_1 = \int_0^1 x^n dx = \frac{1}{n+1} \to 0, \quad n \to \infty.$$

[1 mark]

(iv) f_n diverges. In fact, $f_n(x)$ converges to 0 for all $x \neq 0$, yet

$$\|f_n\|_1 = \int_{-1}^1 ne^{-n^2x^2} dx = \int_{-n}^n e^{-x^2} dx \to \int_{-\infty}^{\infty} e^{-x^2} dx \neq 0.$$

[2 marks]

D Prove that the space $CL^1[-1,1]$ is not complete.

Hint: Construct a sequence $\{f_n\}_{n=1}^{\infty} \subset CL^1[-1,1]$ which converges to a discontinuous function in the L^1 norm. Conclude that $\{f_n\}_{n=1}^{\infty}$ cannot converge in CL^1 to an element of this space.

[9 marks]

Solution (bookwork):

Let $f_n \in CL^1[-1,1]$ be defined by

$$f_n(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq -1/n; \\ nx & \text{if } -1/n \leq x \leq 1/n; \\ 1 & \text{if } 1/n \leq x \leq 1. \end{cases}$$

Then f_n is continuous. For $m > n$ we have

$$\|f_n - f_m\|_1 = \int_{-1/n}^{1/n} |f_n(x) - f_m(x)| dx \le \int_{-1/n}^{1/n} 2dx = \frac{4}{n}.$$

Thus, $\{f_n\}$ is a Cauchy sequence.

Let us prove that f_n does not converge in $CL^1[-1, 1]$. Let g be the discontinuous function,

$$g(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

By the same argument as above one has

$$\|f_n - g\|_1 = \int_{-1/n}^{1/n} |f_n(x) - g(x)| dx \le \frac{2}{n} \to 0$$

as $n \to \infty$.

Now suppose that there exists $f \in CL^1[-1, 1]$ such that $\|f_n - f\|_1 \to 0$. Since f is continuous and g is discontinuous, it is not difficult to see that $\|f - g\|_1 > 0$. By the triangle inequality for the norm $\|\cdot\|_1$, we get that $\|f - g\|_1 \le \|f - f_n\|_1 + \|f_n - g\|_1 \to 0$ as $n \to \infty$ — contradiction.

[9 marks]

QUESTION 5

A Let K be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) K is compact;

(ii) K is sequentially compact;

(iii) K is totally bounded.

[4 marks]

Solution (bookwork):

(i) K is compact iff any open cover of K contains a finite subcover. [1 mark]

(ii) K is sequentially compact iff any sequence of elements of K contains a convergent subsequence. [1 mark]

(iii) K is totally bounded iff for any $\varepsilon > 0$ there exist elements $x_1, x_2, \dots x_n \in X$ such that $K \subset \cup_{k=1}^{n} B_\varepsilon(x_k)$. [2 marks]

B Give an example of a closed bounded set $A \subset \ell^1$ and of a sequence of elements $\{x^{(n)}\}_{n=1}^{\infty} \subset A$ such that this sequence has no convergent subsequences. Proof is not required.

[6 marks]

Solution (bookwork):

Let $A = B_1[0]$, and $x^{(n)} = (\underbrace{0, 0, \dots, 0, 1}_{n}, 0, 0 \dots)$. [6 marks]

C In ℓ^∞, let K be the following set:

$$K = \{x \mid |x_j| \leq s_j, \quad j = 1, 2, \dots\},$$

where s_j is a sequence of non-negative numbers such that $s_j \to 0$ as $j \to \infty$. Prove that K is closed and totally bounded.

Hint: In order to prove total boundedness, fix $\varepsilon > 0$ and consider the map $f_n : \ell^\infty \to \ell^\infty$

$$f_n : (x_1, x_2, \dots, x_n, \dots) \mapsto (x_1, \dots, x_n, 0, 0, \dots).$$

Choose n sufficiently large (depending on ε). Prove that $f_n(K)$ is totally bounded. Conclude that K is totally bounded.

[6 marks]

Solution (similar example was considered in the lectures)

The set K is closed because it can be represented as an intersection of closed sets

$$K = \cap_{n=1}^{\infty} K_n, \quad K_n = \{x \in \ell^\infty \mid |x_n| \le s_n\}.$$

Choose n so that $s_m \le \epsilon/2$ for all $m \ge n+1$. Consider the map $f_n : \ell^\infty \to \mathbb{C}^n$,

$$f_n : x = (x_1, x_2, \dots, x_n, \dots) \mapsto f_n(x) = (x_1, x_2, \dots, x_n).$$

The set $f_n(K)$ is totally bounded (as a bounded set in a finite dimensional space). Thus, there exist points $x^{(\alpha)} \in \mathbb{C}^n$, $\alpha = 1, \dots, N$, such that $f_n(K) \subset \cup_{\alpha=1}^{N} B_{\varepsilon/2}(x^{(\alpha)})$.

For $x^{(\alpha)} = (x_1^{(\alpha)}, \dots, x_n^{(\alpha)})$, let $y^{(\alpha)} = (x_1^{(\alpha)}, \dots, x_n^{(\alpha)}, 0, \dots) \in \ell^\infty$. If $x \in K$, then $f_n(x) \in B_{\varepsilon/2}(x^{(\alpha)})$ for some α. Then, as it is straightforward to see,

$$\|x - y^{(\alpha)}\|_\infty \le \frac{\varepsilon}{2} + \|f_N(x) - x^{(\alpha)}\|_{\mathbb{C}^n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so $x \in B_\varepsilon(y^{(\alpha)})$. It follows that $K \subset \cup_{\alpha=1}^{N} B_\varepsilon(y^{(\alpha)})$.

[6 marks]

D Let (X, ρ) be a compact metric space and let $f : X \to \mathbb{C}$ be a continuous function. Prove that f is uniformly continuous.

[9 marks]

Solution (bookwork):

Let $\varepsilon > 0$. Since f is continuous, for every point $x \in X$ there exists $\delta_x > 0$ such that

$$|f(y) - f(x)| \le \varepsilon/2 \quad \text{whenever} \quad \rho(y, x) \le \delta_x. \tag{2}$$

Let $J_x = B_{\delta_x/2}(x)$. Since $x \in J_x$, the collection of open balls $\{J_x\}_{x \in X}$ is an open cover of X. Since X is compact, it has a finite subcover, that is, there exists a finite collection of points $x_1, x_2, \dots, x_n$ such that $X = \cup_{k=1}^{n} J_{x_k}$. Denote $\delta = \frac{1}{2} \min\{\delta_{x_1}, \dots \delta_{x_n}\}$. Since the number of points x_k is finite, we have $\delta > 0$.

Let $x, y \in X$ and $\rho(x, y) \le \delta$. Since $X = \cup_{k=1}^{n} J_{x_k}$, there exists k such that $x \in J_{x_k}$, that is, $\rho(x, x_k) \le \delta_{x_k}/2$. By the triangle inequality

$$\rho(y, x_k) \le \rho(x, x_k) + \rho(x, y) \le \delta_{x_k}/2 + \delta \le \delta_{x_k}$$

and, in view of (2), $|f(y) - f(x)| \le |f(y) - f(x_k)| + |f(x_k) - f(x)| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus we have proved that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \le \varepsilon$ whenever $\rho(x, y) \le \delta$.

[9 marks]

QUESTION 6

A Let $(X, \|\cdot\|_X)$ be a Banach space. State precisely what it means to say that

(i) λ is a bounded linear functional on X;

(ii) $\|\lambda\|_{X^*}$ is the norm of λ in X^*;

(iii) X is reflexive.

[4 marks]

Solution (bookwork):

(i) λ is a bounded linear functional on X, if λ is linear:

$$\lambda(\alpha x + \beta y) = \alpha\lambda(x) + \beta\lambda(y) \quad \forall x, y \in X, \quad \forall \alpha, \beta \in \mathbb{C}$$

and λ is bounded, i.e. there exists $C > 0$ such that $|\lambda(x)| \leq C\|x\|_X$ for any $x \in X$. [1 mark]

(ii) $\|\lambda\|_{X^*}$ is defined as

$$\|\lambda\|_{X^*} = \sup\left\{\frac{|\lambda(x)|}{\|x\|_X} \mid x \in X, x \neq 0\right\}.$$

[1 mark]

(iii) X is reflexive if the map J defined below is a surjection. The map $J : X \to X^{**}$, $x \mapsto \lambda_x$ is defined by $\lambda_x(\ell) = \ell(x)$, $\ell \in X^*$. [2 marks]

B Give an example of a bounded linear functional λ on $C[0,1]$ such that there is no non-zero element $f \in C[0,1]$ with the property $|\lambda(f)| = \|\lambda\|_{(C[0,1])^*}\|f\|_{C[0,1]}$. Proof is not required.

[6 marks]

Solution (unseen):

$$\lambda(f) = \int_0^{1/2} f(x)dx - \int_{1/2}^1 f(x)dx$$

[6 marks]

C Let λ be a linear functional on $CL^3[0,5]$ defined by

$$\lambda(f) = \int_0^5 f(x)dx.$$

Determine the norm of λ and prove your claim. You are allowed to use Hölder inequality but no other theorems from the course. [6 marks]

Solution (unseen, but the isometry $(L^p)^* \approx L^q$ was mentioned):

We have $\|\lambda\| = 5^{2/3}$. Indeed,

$$|\lambda(f)| = \left|\int_0^5 f(x)dx\right| \le \|f\|_3 \left(\int_0^5 1^{3/2}dx\right)^{2/3} = 5^{2/3}\|f\|_3,$$

and so $\|\lambda\| \le 5^{2/3}$. On the other hand, taking $f(x) \equiv 1$, we get $\|f\|_3 = \left(\int_0^5 1^3 dx\right)^{1/3} = 5^{1/3}$, and $|\lambda(f)| = 5$, and so

$$\frac{|\lambda(f)|}{\|f\|_3} = \frac{5}{5^{1/3}} = 5^{2/3}.$$

Thus, $\|\lambda\| \ge 5^{2/3}$.

D Prove that any bounded linear functional λ on ℓ^1 has the form

$$\lambda(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_1, x_2, \dots) \in \ell^1,$$

with some $y = (y_1, y_2, \dots) \in \ell^\infty$.

[9 marks]

Solution (bookwork; this question appeared in 2008/2009 and 2009/2010 exam papers):

Let $y_n = \lambda(e_n)$, where e_n are the vectors of the standard basis in ℓ^1. Then $|y_n| \le \|\lambda\|\|e_n\|_1 = \|\lambda\|$ and so $y \in \ell^\infty$. Consider the linear functional

$$\lambda_y(x) = \sum_{n=1}^{\infty} x_n y_n, \qquad x \in \ell^1.$$

We have $|\lambda_y(x)| \le \|y\|_\infty \|x\|_1$ and so λ_y is bounded. For any $x \in \ell^1$ with finitely many non-zero coordinates we have

$$\lambda(x) = \lambda\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n \lambda(e_n) = \sum_{n=1}^{\infty} x_n y_n = \lambda_y(x).$$

Now, for any $x \in \ell^1$ let $x^{(n)} = (x_1, x_2, \dots, x_n, 0, 0, \dots)$. Then, by the above calculation, $\lambda(x^{(n)}) = \lambda_y(x^{(n)})$ for any n. Since $\|x^{(n)} - x\| \to 0$ as $n \to \infty$, by the continuity of λ and λ_y we obtain $\lambda(x^{(n)}) \to \lambda(x)$ and $\lambda_y(x^{(n)}) \to \lambda_y(x)$. Thus, $\lambda(x) = \lambda_y(x)$, as required.

[9 marks]