King's College London

University Of London

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Candidate No: .... **Desk No:** ....

MSc Examination

7CCMMS05 (CMMS05) Basic Analysis

Summer 2010

Time Allowed: Two Hours

All questions carry equal marks. Full marks will be awarded for complete answers to FOUR questions. Only the best FOUR questions will count towards grades A or B, but credit will be given for all work done for lower grades.

You are permitted to use a Calculator.
ONLY CALCULATORS APPROVED BY THE COLLEGE MAY BE USED.

TURN OVER WHEN INSTRUCTED

1. **A** Let A be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) x is an interior point of A;

(ii) $\text{int}(A)$ is the interior of A;

(iii) A is open.

[4 marks]

B Give an example of two sets $A_1, A_2 \in C[0,1]$ such that neither A_1 nor A_2 is open yet $A_1 \cup A_2$ is open in $C[0,1]$. Proof is not required.

[6 marks]

C Consider the set $A \subset C[0,1]$ defined by

$$A = \{f \mid f(0) = f(1) = 0 \text{ and } |f(x)| < 1 \text{ for all } x \in [0,1]\}.$$

Determine the diameter of A in $C[0,1]$ and prove your claim.

[6 marks]

D Let $S \subset C[0,1]$ be the set which consists of all functions $f \in C[0,1]$ with the following property. For each $f \in S$ there exists $n \in \mathbb{N}$ (n may depend on f) such that for all $x \in [0, 2^{-n}]$ one has $|f(x)| < 2^n$. Prove that S is open in $C[0,1]$.

[9 marks]

2.

A Let (X, ρ) be a metric space. State precisely what it means to say that

(i) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ);

(ii) (X, ρ) is complete;

(iii) a metric space (Y, d) is a completion of (X, ρ).

[4 marks]

B Give an example of a metric ρ on $\mathbb{R}$ such that the metric space $(\mathbb{R}, \rho)$ is incomplete. Proof is not required.

[6 marks]

C Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$, $n \in \mathbb{N}$, given by

$$f_n(x) = \begin{cases} n, & \text{if } x \in [0, \frac{1}{n}], \\ 0, & \text{if } x \in (\frac{1}{n}, 1]. \end{cases}$$

Determine whether $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_1$ given by

$$\|f\|_1 = \int_0^1 |f(x)| dx.$$

Prove your claim.

[6 marks]

D Prove that the linear space $C[0, 1]$ equipped with the norm $\|\cdot\|_1$ as in part **C** of this question is incomplete.

[9 marks]

3. **A** Let K be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) K is compact;

(ii) K is sequentially compact;

(iii) K is totally bounded.

[4 marks]

B Give an example of a closed bounded set $A \subset C[0,1]$ and of a sequence of elements $f_n \in A$, $n \in \mathbb{N}$, such that this sequence has no convergent subsequences in $C[0,1]$. Proof is not required.

[6 marks]

C Let $\mathbb{R}^n_\infty$ be the linear space $\mathbb{R}^n$ equipped with the norm

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

Let $B_1[0]$ be the unit ball in $\mathbb{R}^n_\infty$. Prove that $B_1[0]$ is totally bounded. Proceed as follows: for a given $\varepsilon > 0$ construct explicitly a finite set $S \subset \mathbb{R}^n$ such that $B_1[0] \subset \cup_{x \in S} B_\varepsilon(x)$.

[6 marks]

D Prove that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to the standard Euclidean norm $\|\cdot\|_2$. You may use without proof the following facts:

(i) Any closed bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is compact;

(ii) Any continuous function on a compact set attains its maximum and its minimum.

[9 marks]

4. **A** Let $(X, \|\cdot\|_X)$ be a Banach space and $\mathcal{B}(X)$ be the set of all bounded linear operators from X to X. State precisely what it means to say that

(i) a sequence of linear operators $T_n \in \mathcal{B}(X)$ converges to an operator $T \in \mathcal{B}(X)$ in the operator norm;

(ii) a sequence of linear operators $T_n \in \mathcal{B}(X)$ converges to an operator $T \in \mathcal{B}(X)$ strongly.

[4 marks]

B Give an example of a sequence of bounded operators T_n in ℓ^2 such that as $n \to \infty$, the sequence T_n converges strongly but not in the operator norm. Proof is not required.

[6 marks]

C Let the linear operator $T_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, be defined by

$$(T_n f)(x) = n \int_0^{1/n} e^{x-y} f(y) dy.$$

Find the strong limit of the sequence T_n as $n \to \infty$ and prove your claim.

[6 marks]

D Let $X = \mathbb{R}^d$ with the usual Euclidean norm. Prove that if a sequence of operators $T_n \in \mathcal{B}(X)$ converges strongly to an operator $T \in \mathcal{B}(X)$, then it also converges to T in the operator norm.

[9 marks]

5. **A** Let $(X, \|\cdot\|_X)$ be a normed linear space. State precisely what it means to say that

(i) λ is a bounded linear functional on X;

(ii) $\|\lambda\|_{X^*}$ is the norm of λ in X^*;

(iii) X is reflexive.

[4 marks]

B Give an example of a Banach space $(X, \|\cdot\|_X)$ and of a bounded linear functional $\lambda \in X^*$ such that there is no non-zero element $x \in X$ with the property $|\lambda(x)| = \|\lambda\|_{X^*}\|x\|_X$. Proof is not required.

[6 marks]

C Let $CL^2(0,1)$ be the linear space $C[0,1]$ equipped with the norm

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}.$$

Let λ be a linear functional on $CL^2(0,1)$ defined by

$$\lambda(f) = \int_0^1 e^x f(x) dx.$$

Prove that λ is a bounded functional on $CL^2(0,1)$. Determine the norm of λ and prove your claim.

[6 marks]

D Prove that any bounded linear functional λ on ℓ^1 has the form

$$\lambda(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_1, x_2, \dots) \in \ell^1,$$

with some $y = (y_1, y_2, \dots) \in \ell^\infty$.

[9 marks]

6. **A** Let $(X, \|\cdot\|_X)$ be a normed linear space and $T : X \to X$ be a linear map. State precisely what it means to say that

(i) T is a bounded;

(ii) T is continuous;

(iii) $\|T\|$ is the norm of T.

[4 marks]

B Give an example of a bounded linear operator $T : \ell^\infty \to \ell^\infty$ such that $\|T\mathbf{1}\| < \|T\|$. Here $\mathbf{1}$ is the element of ℓ^∞ given by $\mathbf{1} = (1, 1, 1, \dots)$. Proof is not required.

[6 marks]

C Let the linear operator $T : C[0, a] \to C[0, a]$ be defined by $(Tf)(x) = \int_0^x f(t)dt$. Prove that T is bounded and determine the norm of T.

[6 marks]

D Prove that a linear operator on a normed linear space X is bounded if and only if it is continuous.

[9 marks]