7CCMMS05 (CMMS05) Basic Analysis, Summer 2010
Exam solutions

Syllabus, lecture notes, past years exam papers can be found at the course webpage http://www.mth.kcl.ac.uk/courses/cmms05.html

Each question has the following structure, with some minor variations:

A Definition [4 marks]

B Example [6 marks]

C Calculation [6 marks]

D Proof [9 marks]

QUESTION 1

A Let A be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) x is an interior point of A;
(ii) $\text{int}(A)$ is the interior of A;
(iii) A is open.
[4 marks]

Solution (bookwork):

(i) x is an interior point of A if there exists $r > 0$ such that $B_r(x) \subset A$. [2 marks]

(ii) the interior of A is the collection of all interior points of A. [1 mark]

(iii) A is open if A coincides with the interior of A. [1 mark]

B Give an example of two sets $A_1, A_2 \in C[0,1]$ such that neither A_1 nor A_2 is open yet $A_1 \cup A_2$ is open in $C[0,1]$. Proof is not required.
[6 marks]

Solution (unseen): Let

$$A_1 = \{f \in C[0,1] \mid 1 < \|f\|_C \le 3\},$$
$$A_2 = \{f \in C[0,1] \mid 2 \le \|f\|_C < 4\};$$
$$\text{then } A_1 \cup A_2 = \{f \in C[0,1] \mid 1 < \|f\|_C < 4\} \text{ is open.}$$

[6 marks]

C Consider the set $A \subset C[0,1]$ defined by

$$A = \{f \mid f(0) = f(1) = 0 \text{ and } |f(x)| < 1 \text{ for all } x \in [0,1]\}.$$

Determine the diameter of A in $C[0,1]$ and prove your claim.

[6 marks]

Solution (unseen): The diameter of A is 2. [1 mark]

Indeed, for any $f, g \in A$ we have

$$\|f - g\|_C \leq \|f\|_C + \|g\|_C = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |g(x)| < 1 + 1 = 2$$

by the definition of A. This proves that $\operatorname{diam} A \leq 2$. [2 marks]

On the other hand, given $\varepsilon \in (0,1)$, let $f(x) = (1-\varepsilon)\sin(\pi x)$, and $g(x) = -f(x)$. Then $f, g \in A$ and

$$\|f - g\|_C = 2\|f\|_C = 2(1-\varepsilon).$$

This proves that $\operatorname{diam} A \geq 2 - 2\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrary small, we obtain that $\operatorname{diam} A \geq 2$. [3 marks]

D Let $S \subset C[0,1]$ be the set which consists of all functions $f \in C[0,1]$ with the following property. For each $f \in S$ there exists $n \in \mathbb{N}$ (n may depend on f) such that for all $x \in [0, 2^{-n}]$ one has $|f(x)| < 2^n$. Prove that S is open in $C[0,1]$.

[9 marks]

Solution (unseen): For $n \in \mathbb{N}$, let $S_n \subset C[0,1]$ be

$$S_n = \{f \in C[0,1] \mid |f(x)| < 2^n \text{ for all } x \in [0, 2^{-n}]\}.$$

We have $S = \cup_{n=1}^{\infty} S_n$. Since the union of open sets is open, it suffices to prove that S_n is open for all $n \in \mathbb{N}$. [3 marks]

Let us prove that S_n is open. Given $f \in S_n$, let

$$\varepsilon = \frac{1}{2} \sup_{x\in[0,2^{-n}]} (2^n - |f(x)|).$$

We claim that $B_\varepsilon(f) \subset S_n$. Indeed, if $\|f - g\| < \varepsilon$ then for any $x \in [0, 2^{-n}]$ we have

$$|g(x)| \leq |g(x) - f(x)| + |f(x)| < \varepsilon + |f(x)| < 2^n,$$

as required. [6 marks]

QUESTION 2

A Let (X,ρ) be a metric space. State precisely what it means to say that
(i) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X,ρ);
(ii) (X,ρ) is complete;
(iii) a metric space (Y,d) is a completion of (X,ρ).
[4 marks]

Solution (bookwork):

(i) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X,ρ) iff for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n, m > N$ one has $\rho(x_n, x_m) < \varepsilon$. [1 mark]

(ii) (X,ρ) is complete iff any Cauchy sequence in X converges to an element $x \in X$. [1 mark]

(iii) (Y,d) is a completion of (X,ρ) iff (Y,d) is complete and there exists a dense subset $Y_0 \subset Y$ such that (Y_0,d) is isometric to (X,ρ). [2 marks]

B Give an example of a metric ρ on $\mathbb{R}$ such that the metric space $(\mathbb{R},\rho)$ is incomplete. Proof is not required.
[6 marks]

Solution (similar question was given as an exercise):
$\rho(x,y) = \tan^{-1}|x-y|$. [6 marks]

C Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, given by

$$f_n(x) = \begin{cases} n, & \text{if } x \in [0, \frac{1}{n}], \\ 0, & \text{if } x \in (\frac{1}{n}, 1]. \end{cases}$$

Determine whether $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_1$ given by

$$\|f\|_1 = \int_0^1 |f(x)| dx.$$

Prove your claim.
[6 marks]

Solution (unseen): Let $m > n$. One has

$$\begin{aligned} \|f_n - f_m\|_1 &= \int_0^{1/m} |f_n(x) - f_m(x)| dx + \int_{1/m}^{1/n} |f_n(x) - f_m(x)| dx \\ &= \frac{1}{m}(m-n) + (\frac{1}{n} - \frac{1}{m})n = 2 - \frac{2n}{m}. \end{aligned}$$

[3 marks]

Thus, for any $n \in \mathbb{N}$ we have

$$\sup_{m>n} \|f_n - f_m\|_1 = \sup_{m>n} (2 - \frac{2n}{m}) = 2,$$

and so $\{f_n\}_{n=1}^{\infty}$ is not a Cauchy sequence. [3 marks]

D Prove that the linear space $C[0,1]$ equipped with the norm $\|\cdot\|_1$ as in part **C** of this question is incomplete.

[9 marks]

Solution (bookwork; this was discussed in the lectures for the interval $[-1, 1]$):

Let $CL^1(0,1)$ be the linear space $C[0,1]$ equipped with the norm $\|\cdot\|_1$. Consider the sequence

$$f_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ n(x - \frac{1}{2}), & x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}] \\ 1, & x \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

We claim that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $CL^1(0,1)$ but doesn't converge in this space. [2 marks]

Let $m > n$. Then

$$\|f_n - f_m\|_1 = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - f_m(x)| dx \le 2 \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} dx = \frac{2}{n}.$$

This shows that for a given $\varepsilon > 0$ taking $N > 2/\varepsilon$ we get $\|f_n - f_m\|_1 \le \varepsilon$ for any $n, m > N$. Thus, $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence. [3 marks]

Suppose that $\|f_n - f\|_1 \to 0$ as $n \to \infty$ for some $f \in C[0,1]$. Consider two cases.

(a) $f(1/2) \neq 0$. Then by the continuity of f there exists $a > 0$ such that $|f(x) - f(1/2)| < \frac{1}{2}|f(1/2)|$ for all $x \in [\frac{1}{2} - a, \frac{1}{2}]$. It follows that

$$\begin{aligned} \|f - f_n\|_1 \ge \int_{\frac{1}{2}-a}^{\frac{1}{2}} |f(x) - f_n(x)| dx &= \int_{\frac{1}{2}-a}^{\frac{1}{2}} |f(x)| dx \\ &\ge \int_{\frac{1}{2}-a}^{\frac{1}{2}} \frac{1}{2} |f(1/2)| dx = \frac{a}{2} |f(1/2)| > 0. \end{aligned}$$

This contradicts the assumption $\|f_n - f\|_1 \to 0$.

(b) $f(1/2) = 0$. Then by the continuity of f there exists $a > 0$ and $\delta > 0$ such that $|f(x)| \le \delta$ for all $x \in [\frac{1}{2}, \frac{1}{2} + a]$. Taking $n > 2/a$ we obtain

$$\begin{aligned}\|f - f_n\|_1 \ge \int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+a} |f(x) - f_n(x)|dx &= \int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+a} |f(x) - 1|dx \\ &\ge (1-\delta)(a - \frac{1}{n}) \ge (1-\delta)\frac{a}{2} > 0.\end{aligned}$$

Again, this contradicts the assumption $\|f_n - f\|_1 \to 0$. [4 marks]

QUESTION 3

A Let K be a subset of a metric space (X, ρ). State precisely what it means to say that

(i) K is compact;

(ii) K is sequentially compact;

(iii) K is totally bounded.

[4 marks]

Solution (bookwork):

(i) K is compact iff any open cover of K contains a finite subcover. [1 mark]

(ii) K is sequentially compact iff any sequence of elements of K contains a convergent subsequence. [1 mark]

(iii) K is totally bounded iff for any $\varepsilon > 0$ there exist elements $x_1, x_2, \dots x_n \in X$ such that $K \subset \cup_{k=1}^{n} B_\varepsilon(x_k)$. [2 marks]

B Give an example of a closed bounded set $A \subset C[0,1]$ and of a sequence of elements $f_n \in A$, $n \in \mathbb{N}$, such that this sequence has no convergent subsequences in $C[0,1]$. Proof is not required.

[6 marks]

Solution (bookwork):

Let A be the closed ball in $C[0,1]$ with the radius 1 centered at 0. Let f_n be defined as follows. Choose a sequence of disjoint intervals $\delta_n \subset [0,1]$, for example, $\delta_n = (2^{-n}, 2^{-n+1})$, $n \in \mathbb{N}$. Let f_n be supported on δ_n and such that $0 \le f_n(x) \le 1$ and $f_n(x) = 1$ for some $x \in \delta_n$. Explicitly, one can take

$$f_n(x) = \begin{cases} 2^{n+1}(x - 2^{-n}) & \text{if } x \in (2^{-n}, 2^{-n} + 2^{-n-1}), \\ 2^{n+1}(2^{-n+1} - x) & \text{if } x \in (2^{-n} + 2^{-n-1}, 2^{-n+1}), \\ 0 & \text{otherwise.} \end{cases}$$

[6 marks]

C Let $\mathbb{R}^n_\infty$ be the linear space $\mathbb{R}^n$ equipped with the norm

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

Let $B_1[0]$ be the unit ball in $\mathbb{R}^n_\infty$. Prove that $B_1[0]$ is totally bounded. Proceed as follows: for a given $\varepsilon > 0$ construct explicitly a finite set $S \subset \mathbb{R}^n$ such that $B_1[0] \subset \cup_{x \in S} B_\varepsilon(x)$.

[6 marks]

Solution (unseen): Choose $m \in \mathbb{N}$ such that $2^{-m-1} < \varepsilon$. Note that

$$\forall x \in [-1,1] \quad \exists j \in \mathbb{Z},\ |j| \le 2^m, \qquad \text{such that } |x - j2^{-m}| \le 2^{-m-1}. \tag{1}$$

For any $\mathbf{j} \in \mathbb{Z}^n$ let

$$x_{\mathbf{j}} = (j_1 2^{-m}, j_2 2^{-m}, \ldots, j_n 2^{-m}) \in \mathbb{R}^n.$$

Consider

$$S = \{x_{\mathbf{j}} \mid |j_i| \le 2^m \text{ for all } i = 1, \ldots, n\}.$$

Clearly, S is finite. By (1), for any $x \in B_1[0]$ we have

$$\exists \mathbf{j} \in \mathbb{Z}^n \quad \text{such that } |x_i - j_i 2^{-m}| \le 2^{-m-1} \text{ and } |j_i| \le 2^m \text{ for all } i = 1, \ldots, n$$

and so $x \in B_{2^{-m-1}}[x_{\mathbf{j}}]$. It follows that $B_1[0] \subset \cup_{a \in S} B_\varepsilon(a)$, as required. [6 marks]

D Prove that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to the standard Euclidean norm $\|\cdot\|_2$. You may use without proof the following facts:

(i) Any closed bounded set in $(\mathbb{R}^n, \|\cdot\|_2)$ is compact;

(ii) Any continuous function on a compact set attains its maximum and its minimum.

[9 marks]

Solution (bookwork): Let $e_1 \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. Then we have

$$\|x\| = \|\sum_{i=1}^n x_i e_i\| \le \sum_{i=1}^n |x_i| \|e_i\| \le C\|x\|_2, \tag{2}$$

where $C^2 = \|e_1\|^2 + \cdots + \|e_n\|^2$. [3 marks]

In order to prove the opposite inequality, consider the function $f(x) = \|x\|$ on the unit sphere $K = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$. The function f is continuous by (2). The sphere K is compact in $(\mathbb{R}^n, \|\cdot\|_2)$ since it is a bounded and closed set. Thus, f attains a minumum m on K, i.e. there exists $x_0 \in K$ such that $\|x\| \ge \|x_0\| = m$ for all $x \in K$. Since $x_0 \ne 0$, by the axioms of the norm $\|x_0\| = m > 0$. Thus, $\|x\| \ge m > 0$ for all $x \in K$. By rescaling, we get $\|x\| \ge m\|x\|_2$ for all $x \in \mathbb{R}^n$, which completes the proof. [6 marks]

QUESTION 4

A Let $(X, \|\cdot\|_X)$ be a Banach space and $\mathcal{B}(X)$ be the set of all bounded linear operators from X to X. State precisely what it means to say that

(i) a sequence of linear operators $T_n \in \mathcal{B}(X)$ converges to an operator $T \in \mathcal{B}(X)$ in the operator norm;

(ii) a sequence of linear operators $T_n \in \mathcal{B}(X)$ converges to an operator $T \in \mathcal{B}(X)$ strongly.

[4 marks]

Solution (bookwork):

(i) $T_n \to T$ in the operator norm if

$$\sup_{x \in X, \|x\|_X = 1} \|T_n x - Tx\|_X \to 0$$

as $n \to \infty$. [2 marks]

(ii) $T_n \to T$ in strongly if $\|T_n x - Tx\|_X \to 0$ as $n \to \infty$ for all $x \in X$. [2 marks]

B Give an example of a sequence of bounded operators T_n in ℓ^2 such that as $n \to \infty$, the sequence T_n converges strongly but not in the operator norm. Proof is not required.

[6 marks]

Solution (bookwork):

Let $T_n : \ell^2 \to \ell^2$ be defined by

$$T_n : (x_1, x_2, \dots) \mapsto (x_n, x_{n+1}, x_{n+2}, \dots).$$

Then $T_n \to 0$ strongly but not in the operator norm.

[6 marks]

C Let the linear operator $T_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, be defined by

$$(T_n f)(x) = n \int_0^{1/n} e^{x-y} f(y) dy.$$

Find the strong limit of the sequence T_n as $n \to \infty$ and prove your claim.

[6 marks]

Solution (unseen; a similar problem was given as an exercise): The strong limit is the linear operator T given by

$$(Tf)(x) = f(0)e^x.$$

[2 marks]

Indeed,

$$(T_n f)(x) - (Tf)(x) = (f(0) - n \int_0^{1/n} e^{-y} f(y)dy)e^x,$$

and so

$$\|T_n f - Tf\|_C = \left| f(0) - n \int_0^{1/n} e^{-y} f(y)dy \right| e.$$

It remains to prove that

$$\left| f(0) - n \int_0^{1/n} e^{-y} f(y)dy \right| \to 0 \quad \text{as } n \to \infty. \tag{3}$$

Since f is continuous at 0, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $|y| < \delta$ we have $|e^{-y} f(y) - f(0)| < \varepsilon$. It follows that for any $n > 1/\delta$ we have

$$\left| f(0) - n \int_0^{1/n} e^{-y} f(y)dy \right| \leq n \int_0^{1/n} |e^{-y} f(y) - f(0)|dy \leq n \int_0^{1/n} \varepsilon dy = \varepsilon.$$

This proves (3). [4 marks]

D Let $X = \mathbb{R}^d$ with the usual Euclidean norm. Prove that if a sequence of operators $T_n \in \mathcal{B}(X)$ converges strongly to an operator $T \in \mathcal{B}(X)$, then it also converges to T in the operator norm.

[9 marks]

Solution (unseen):

Let $e_1, \ldots, e_d$ be the standard basis in $\mathbb{R}^d$, so that any element $x \in \mathbb{R}^d$ can be represented as $x = x_1 e_1 + \cdots + x_d e_d$.

Let T be the strong limit of T_n. Then $\|T_n e_k - Te_k\| \to 0$ as $n \to \infty$ for all $k = 1, \ldots d$. It follows that

$$C_n = \left(\sum_{k=1}^{d} \|T_n e_k - Te_k\|^2 \right)^{1/2} \to 0 \quad \text{as } n \to \infty. \tag{4}$$

Using the Cauchy-Schwartz inequality, we get

$$\begin{aligned} \|T_n x - Tx\| &= \|\sum_{k=1}^{d} x_k (T_n e_k - Te_k)\| \\ &\leq \sum_{k=1}^{d} |x_k| \|T_n e_k - Te_k\| \leq C_n \left(\sum_{k=1}^{d} |x_k|^2 \right)^{1/2} = C_n \|x\|. \end{aligned}$$

By (4), it follows that $\|T_n - T\| \to 0$ as $n \to \infty$, as required. [9 marks]

QUESTION 5

A Let $(X, \|\cdot\|_X)$ be a normed linear space. State precisely what it means to say that

(i) λ is a bounded linear functional on X;

(ii) $\|\lambda\|_{X^*}$ is the norm of λ in X^*;

(iii) X is reflexive.

[4 marks]

Solution (bookwork):

(i) λ is a bounded linear functional on X, if λ is linear:

$$\lambda(\alpha x + \beta y) = \alpha\lambda(x) + \beta\lambda(y) \quad \forall x, y \in X, \quad \forall \alpha, \beta \in \mathbb{C}$$

and λ is bounded, i.e. there exists $C > 0$ such that $|\lambda(x)| \le C\|x\|_X$ for any $x \in X$. [1 mark]

(ii) $\|\lambda\|_{X^*}$ is defined as

$$\|\lambda\|_{X^*} = \sup\left\{\frac{|\lambda(x)|}{\|x\|_X} \mid x \in X, x \neq 0\right\}.$$

[1 mark]

(iii) X is reflexive if the map J defined below is a surjection. The map $J : X \to X^{**}$, $x \mapsto \lambda_x$ is defined by $\lambda_x(\ell) = \ell(x)$, $\ell \in X^*$. [2 marks]

B Give an example of a Banach space $(X, \|\cdot\|_X)$ and of a bounded linear functional $\lambda \in X^*$ such that there is no non-zero element $x \in X$ with the property $|\lambda(x)| = \|\lambda\|_{X^*}\|x\|_X$. Proof is not required.

[6 marks]

Solution (a similar question appeared in 2008/2009 exam paper):

$X = \ell_1$, $\lambda(x) = \sum_{n=1}^{\infty}(1 - \frac{1}{n})x_n$. Then $\|\lambda\| = 1$ but for all $x \in \ell^1$ one has $|\lambda(x)| < \|x\|_{\ell^1}$. [6 marks]

C Let $CL^2(0,1)$ be the linear space $C[0,1]$ equipped with the norm

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}.$$

Let λ be a linear functional on $CL^2(0,1)$ defined by

$$\lambda(f) = \int_0^1 e^x f(x) dx.$$

Prove that λ is a bounded functional on $CL^2(0,1)$. Determine the norm of λ and prove your claim.

[6 marks]

Solution (unseen):

The norm of T is

$$\varkappa := \left(\int_0^1 e^{2x} \, dx \right)^{1/2} = \left(\frac{e^2 - 1}{2} \right)^{1/2}.$$

Indeed, by Cauchy-Schwartz,

$$|\lambda(f)| \le \left(\int_0^1 e^{2x} dx \right)^{1/2} \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} = \varkappa \|f\|_2,$$

and so $\|\lambda\| \le \varkappa$. [3 marks]

On the other hand, taking $f(x) = e^x$, we get $\|f\|_2 = \varkappa$ and

$$\lambda(f) = \int_0^1 e^x e^x \, dx = \varkappa^2.$$

It follows that $|\lambda(f)| = \varkappa \|f\|_2$ and so $\|\lambda\| \ge \varkappa$. [3 marks]

D Prove that any bounded linear functional λ on ℓ^1 has the form

$$\lambda(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_1, x_2, \dots) \in \ell^1,$$

with some $y = (y_1, y_2, \dots) \in \ell^\infty$.

[9 marks]

Solution (bookwork; this question appeared in 2008/09 exam):

Let $y_n = \lambda(e_n)$, where e_n are the vectors of the standard basis in ℓ^1. Then $|y_n| \le \|\lambda\| \|e_n\|_1 = \|\lambda\|$ and so $y \in \ell^\infty$. Consider the linear functional

$$\lambda_y(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x \in \ell^1.$$

We have $|\lambda_y(x)| \leq \|y\|_\infty \|x\|_1$ and so λ_y is bounded. For any $x \in \ell^1$ with finitely many non-zero coordinates we have

$$\lambda(x) = \lambda\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n \lambda(e_n) = \sum_{n=1}^{\infty} x_n y_n = \lambda_y(x).$$

Now for any $x \in \ell^1$ let $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Then, by the above calculation, $\lambda(x^{(n)}) = \lambda_y(x^{(n)})$ for any n. Since $\|x^{(n)} - x\|_1 \to 0$ as $n \to \infty$ by the continuity of λ and λ_y we have $\lambda(x^{(n)}) \to \lambda(x)$ and $\lambda_y(x^{(n)}) \to \lambda_y(x)$. Thus, $\lambda(x) = \lambda_y(x)$, as required. [9 marks]

QUESTION 6

A Let $(X, \|\cdot\|_X)$ be a normed linear space and $T : X \to X$ be a linear map. State precisely what it means to say that

(i) T is a bounded;

(ii) T is continuous;

(iii) $\|T\|$ is the norm of T.

[4 marks]

Solution (bookwork):

(i) T is bounded iff there exists $C > 0$ such that for any $x \in X$ one has $\|Tx\|_X \leq C\|x\|_X$. [1 mark]

(ii) T is continuous iff for any $\alpha \in X$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - \alpha\|_X < \delta$ then $\|Tx - T\alpha\|_X < \varepsilon$. [2 marks]

(iii) The norm of T is

$$\|T\| = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} \mid x \in X,\ x \neq 0\right\}.$$

[1 mark]

B Give an example of a bounded linear operator $T : \ell^\infty \to \ell^\infty$ such that $\|T\mathbf{1}\| < \|T\|$. Here $\mathbf{1}$ is the element of ℓ^∞ given by $\mathbf{1} = (1, 1, 1, \dots)$. Proof is not required.

[6 marks]

Solution (unseen):

$$T : (x_1, x_2, x_3, \dots) \mapsto (x_1 - x_2, x_3, x_4, \dots).$$

Then $\|T\| = 2$ but $\|T\mathbf{1}\| = 1 < 2$. [6 marks]

C Let the linear operator $T : C[0, a] \to C[0, a]$ be defined by $(Tf)(x) = \int_0^x f(t)dt$. Prove that T is bounded and determine the norm of T.

[6 marks]

Solution (this question was given as an exercise):

The norm of T is a. [2 marks]

Indeed,

$$|(Tf)(x)| \leq \int_0^x |f(t)|dt \leq \|f\|_C \int_0^x dt = \|f\|_C x \leq \|f\|_C a,$$

and therefore $\|T\| \leq a$. [2 marks]

On the other hand, taking $f(x) = 1$, we get $\|f\|_C = 1$ and

$$\sup_{x\in[0,a]} |(Tf)(x)| = \sup_{x\in[0,a]} \left| \int_0^x dt \right| = a.$$

It follows that $\|T\| \geq a$. [2 marks]

D Prove that a linear operator on a normed linear space X is bounded if and only if it is continuous.

[9 marks]

Solution (bookwork): Assume first that T is bounded, that is, $\|Tx\|_X \leq C\|x\|_X$. Let $\alpha \in X$ and $\varepsilon > 0$. Take $\delta = C^{-1}\varepsilon$. Then, for all $x \in X$ satisfying $\|x - \alpha\|_X \leq \delta$, we have

$$\|Tx - T\alpha\|_X = \|T(x - \alpha)\|_X \leq C\|x - \alpha\|_X \leq \varepsilon.$$

This implies that T is continuous. [4 marks]

Assume now that T is continuous. Then T is continuous at 0, and therefore there exists $\delta > 0$ such that

$$\|Tx_0 - 0\|_X = \|Tx_0\|_X \leq 1 \quad \text{whenever} \quad \|x_0 - 0\|_X = \|x_0\|_X \leq \delta\,. \qquad (5)$$

If $x \in X$, let us denote $c = \delta\|x\|_X^{-1}$ and $x_0 = c\,x$. Then $\|x_0\|_X = \delta$. Since T is a linear map, (5) implies $\|Tx\|_X = c^{-1}\|Tx_0\|_X \leq c^{-1} = \delta^{-1}\|x\|_X$, which means that T is bounded. [5 marks]