

SKELETON NOTES FOR REPRESENTATION THEORY OF FINITE GROUPS

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LECTURES 1 AND 2

Review of familiar groups (see also Gordon revision notes linked to course webpage).

Course overview.

LECTURE 3

GROUP ACTIONS.

Defined **group actions** and **G -sets**.

Examples:

- (1) Trivial action of G on X .
- (2) Tautological action of S_n on $\{1, \dots, n\}$.
- (3) Action of D_{12} on solid hexagon.

Defined **G -equivariant** maps, and **isomorphisms** of G -sets.

Defined **orbits** and did examples again.

Defined **homogeneous spaces**, or **transitive actions**. Gave example, namely (2) above.

Defined **stabilizer subgroups** and did examples again. Noted connection between size of stabilizer subgroup and size of orbit in the D_{12} example.

LECTURE 4

Actions related to group theory.

- (1) Example of left multiplication action on $X = G/H$. Answered 3 questions:
 - What are the orbits?
 - What is stabilizer of special point eH ?
 - What is stabilizer of more general point gH ?
- (2) Example of $X = G$ and G acting by conjugation. Answered 3 questions:
 - Is this action transitive?
 - What are the orbits? (conjugacy classes)
 - What are the stabilizer subgroups? (commutators)

Lemma 1. Suppose X is a G -set and $x, y \in X$ and $g \in G$. If $g \cdot x = y$ then $G_y = gG_xg^{-1}$.

Corollary 2. If X is a G -set and x, y lie in the same orbit of G , then the stabilizer subgroups G_x and G_y are isomorphic.

Lemma 3. If X is a G -set and $x \in X$. Then the orbit $G \cdot x = \{g \cdot x \mid g \in G\}$ is itself a G -set, and is clearly homogeneous. In fact, there is an isomorphism of G -sets

$$\phi : G/G_x \rightarrow G \cdot x$$

given by

$$gG_x \mapsto g \cdot x.$$

Corollary 4 (The orbit-stabilizer lemma). If X is a G -set and $x \in X$. Then the cardinality of the orbit through x is given by the formula

$$|G \cdot x| = \frac{|G|}{|G_x|}.$$

Remark. Suppose X is a homogeneous G -set, and $x \in X$. Then by the definitions $G \cdot x = X$, and the previous lemma implies that

$$X \cong G/G_x$$

as G -set. Thus every homogeneous G -set is isomorphic to a coset space of G .

LECTURE 5

Defined **fixed point set** $Fix_X(g)$ and stated

Proposition 5 (Cauchy-Frobenius orbit counting Lemma). For a finite group G and a finite G -set X ,

$$\# \text{ orbits} = \frac{1}{|G|} \sum_{g \in G} |Fix_X(g)|.$$

Did two examples with one orbit:

- D_{12} acting on the vertices of a regular hexagon. (Reflections have 2 fixed points, rotations have 0 fixed points. There are as many rotations as reflections in D_{12} , so the average number of fixed points is 1, as expected).
- S_n acting on $\{1, \dots, n\}$. Homogeneous, so one orbit. Cauchy Frobenius Lemma says in this case that the average number of fixed points of a permutation is 1.

Proved the Cauchy-Frobenius Lemma. Idea: multiply both sides of the equation by $|G|$, then show that left and right hand sides both count (in different ways) the number of points in the subset $\{(g, x) \mid g \cdot x = x\}$ of $G \times X$.

Non-examinable extra: If k is the finite field with q elements, then the n -dimensional vector space k^n has q^n elements. We used Cauchy-Frobenius lemma to check that there are $\frac{q^n - 1}{q - 1}$ lines (one-dimensional linear subspaces) in k^n . For this we took $X = k^n \setminus \{0\}$ and $G = k^\times$, acting by scalar multiplication. (See also Level 7 Bonus Problem 1 on the course web page).