King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

BSC AND MSCI EXAMINATION

6CCM351A (CM351A) REPRESENTATION THEORY OF FINITE GROUPS

SUMMER 2010

TIME ALLOWED: TWO HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.

SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER.

ANSWER ALL QUESTIONS IN SECTION A.

ALL QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO ARE ATTEMPTED, THEN ONLY THE BEST TWO WILL COUNT.

NO CALCULATORS ARE PERMITTED.

TURN OVER WHEN INSTRUCTED

SECTION A

A 1. Let G be a finite group.

(i) Define what is meant by a G-set.

(ii) Let X be a G-set and $x \in X$. Define the orbit and the stabilizer of x, and state the orbit-stabilizer lemma.

(iii) Define the space $\mathcal{C}_G$ of class functions. Assuming G is a finite group, prove that $\mathcal{C}_G$ is a vector space of dimension equal to the number of conjugacy classes in G. Define the Hermitian inner product on $\mathcal{C}_G$ that comes up in the orthogonality of characters theorem.

(iv) State Schur's lemma and the orthogonality theorem for characters. For two finite-dimensional complex representations V and W of G, what is the relationship between the space of G-equivariant homomorphisms, $\mathrm{Hom}_G(V, W)$, and the inner product $\langle \chi_V, \chi_W \rangle$ of the characters associated to V and W?

A 2. Suppose X is a finite G-set and $V = \mathbb{C}[X]$ is the permutation representation associated to X. Let χ be the character of the representation V.

(i) Let $G = S_3$ act by permutations on the set $X = \{1, 2, 3\}$. Let ρ be the corresponding permutation representation of G on $\mathbb{C}[\{1, 2, 3\}] = \mathbb{C}^3$. State a formula from lectures for the character of a permutation representation and work out the character of ρ using this formula.

(ii) Let $\mathcal{P}_{1,2}(X)$ be the set of 1-element and 2-element subsets of X. So for the S_3-set $X = \{1, 2, 3\}$ we have

$$\mathcal{P}_{1,2}(X) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}.$$

This set also has an action of S_3, and we let $\mathbb{C}[\mathcal{P}_{1,2}(X)]$ denote the corresponding permutation representation.

The representation of S_3 on $\mathbb{C}^3$ from (**i**) gives rise to a representation of S_3 on the symmetric tensor square $S^2\,\mathbb{C}^3$.

Define an S_3-equivariant isomorphism $\mathbb{C}[\mathcal{P}_{1,2}(X)] \to S^2\,\mathbb{C}^3$.

(iii) State a formula for the character of the symmetric tensor square S^2V of a representation V. Either by using this formula or via the isomorphism from (**ii**), work out the character of the representation $S^2\,\mathbb{C}^3$ of S_3 from (**ii**).

(iv) Define what it means for a representation to be irreducible. Show that $\mathbb{C}[X]$ is not irreducible unless $|X| = 1$.

A 3. Let V denote a finite-dimensional complex vector space and G a group. Suppose we have a representation ρ of G on V.

(i) What is meant by a G-invariant Hermitian inner product on V?

(ii) If G is a finite group, explain how to construct a G-invariant Hermitian inner product on V.

(iii) Define what it means for a representation to be completely reducible and state Maschke's theorem.

(iv) Give an example of a representation of the infinite cyclic group $(\mathbb{Z}, +)$ which is not completely reducible. Justify your answer.

SECTION B

B 4. Consider the cyclic group $C_3 = \langle\, x \mid x^3 = e \,\rangle$.

(i) For any representation V of a group G define its dual representation V^*. Construct all of the irreducible representations of C_3 over $\mathbb{C}$ and explain which representation is dual to which.

(ii) Show that the assignment

$$x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

defines a representation of C_3 on $\mathbb{C}^2$.

(iii) Work out the character of the representation of C_3 from (**ii**) and write it as a sum of irreducible characters.

(iv) Let $V = \mathbb{C}^2$ be the representation of C_3 from (**ii**). Determine explicitly the C_3-invariant subspaces in V and write V as a direct sum of irreducible representations.

(v) Consider the representation of C_3 on $\mathbb{R}^2$ defined by the same assignment,

$$x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Is this representation irreducible or not (as representation on the **real** vector space $\mathbb{R}^2$)?

B 5. Consider the symmetric group, S_3, and the cyclic group with three elements, $C_3 = \langle\, x \,|\, x^3 = e \,\rangle$.

(i) Write down the character tables of S_3 and of C_3.

(ii) Let C be the cyclic subgroup of S_3 generated by the 3-cycle $\sigma = (123)$. Note that C is isomorphic to C_3. For each irreducible character of S_3 work out its restriction to C and decompose the restricted character into irreducible characters of C.

(iii) For a group G, subgroup H and character χ of H, define the induced character $\mathrm{Ind}_H^G \chi$. Consider the linear maps between the spaces of class functions, $\mathrm{Res}_H^G : \mathcal{C}_G \to \mathcal{C}_H$ and $\mathrm{Ind}_H^G : \mathcal{C}_H \to \mathcal{C}_G$ defined by restricting and inducing characters, respectively. What does Frobenius reciprocity say about the relationship between the linear operators Ind_H^G and Res_H^G?

(iv) Using **(iii)** and **(ii)** or otherwise work out $\mathrm{Ind}_C^{S_3} \chi$ for every irreducible character χ of C.

B 6. Consider the quaternion group $Q_8 = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. It has central elements $\mathbf{1}$ and $-\mathbf{1}$, where $\mathbf{1}$ is the identity element, and relations $-\mathbf{1}\mathbf{m} = -\mathbf{m}$, $(-\mathbf{1})(-\mathbf{m}) = \mathbf{m}$ for any $\mathbf{m} = \mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$, as well as

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Recall also that Q_8 has 5 conjugacy classes, which are $\{\mathbf{1}\}, \{-\mathbf{1}\}, \{\pm\mathbf{i}\}, \{\pm\mathbf{j}\}, \{\pm\mathbf{k}\}$. Consider the representation ρ of Q_8 on $V = \mathbb{C}^2$, which is determined by

$$\rho(\mathbf{i}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(\mathbf{j}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

(i) Show that $V \wedge V$ is the trivial representation of Q_8.

(ii) Let $W = V \oplus (V \otimes V)$. Compute the characters of V and $V \otimes V$ and W.

(iii) Define the regular representation for a general group G, and state a formula for its character. For W as in **(ii)** above show that $V \oplus W$ is isomorphic to the regular representation of Q_8.

(iv) Using **(iii)**, show that W contains every irreducible representation of Q_8 as a subrepresentation.

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