King's College London

UNIVERSITY OF LONDON

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BSc AND MSci EXAMINATION

6CCM351A (CM351A) REPRESENTATION THEORY OF FINITE GROUPS

SUMMER 2009

TIME ALLOWED: TWO HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.

SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER.

ANSWER ALL QUESTIONS IN SECTION A.

ALL QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO ARE ATTEMPTED, THEN ONLY THE BEST TWO WILL COUNT.

NO CALCULATORS ARE PERMITTED.

TURN OVER WHEN INSTRUCTED

SECTION A

A 1. Let G be a group and V a finite-dimensional vector space over a field $\mathbb{K}$.

(i) Define what is meant by a representation ρ of G on V.

(ii) Let ρ be a representation of G on V, and let W be a G-invariant subspace in V. Define what is meant by a G-invariant complement to W.

(iii) Suppose that $\mathbb{K} = \mathbb{C}$, the group G is finite and V is a representation of G with a G-invariant subspace W. State Maschke's theorem in this setting and outline its proof.

(iv) State Schur's Lemma.

A 2. Let G be a finite group and V a finite-dimensional representation of G over $\mathbb{C}$.

(i) Define the Hermitian inner product on the space of class functions on G, and state the orthogonality theorem for characters.

(ii) Define the regular representation of G and write down its character.

(iii) Write down the character table for C_4, the cyclic group of order 4.

(iv) Write down the relationship between the character χ_V of the representation V and the character χ_{V^*} of its dual representation V^*. Determine the dual representation for each of the irreducible representations of C_4 using the character table from **(iii)**.

A 3. **(i)** Let $X = \{1, 2, 3\}$ with the natural action of S_3, and let $W = \mathbb{C}[X]$ be the associated permutation representation. Work out the character of W and the character of $W \otimes W$.

(ii) Let V be a complex vector space with basis $v_1, \ldots, v_n$. Write down a basis of $\bigwedge^2 V$.

(iii) If V is a representation of G with character χ, show that the character of $\bigwedge^2 V$ is given by

$$\chi_{\bigwedge^2 V}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)),$$

where χ is the character of V.

(iv) Let $G = S_3$ and let W be the permutation representation $\mathbb{C}[\{1, 2, 3\}]$ as in **(i)** above. Work out the character of $\bigwedge^2 W$.

SECTION B

B 4. Consider the quaternion group $Q_8 = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. It has central elements $\mathbf{1}$ and $-\mathbf{1}$, where $\mathbf{1}$ is the identity element, and relations $-\mathbf{1}\mathbf{m} = -\mathbf{m}$, $(-\mathbf{1})(-\mathbf{m}) = \mathbf{m}$ for any $\mathbf{m} = \mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$, as well as

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Recall also that Q_8 has 5 conjugacy classes, which are $\{\mathbf{1}\}, \{-\mathbf{1}\}, \{\pm\mathbf{i}\}, \{\pm\mathbf{j}\}, \{\pm\mathbf{k}\}$.

(i) Compute the values of the character χ of the two-dimensional representation ρ of Q_8, which is determined by

$$\rho(\mathbf{i}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(\mathbf{j}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Show that the representation ρ is irreducible.

(ii) State any relationships you know of between the order of a finite group G and the dimensions of its irreducible representations.

(iii) How many irreducible representations does Q_8 have and what are their dimensions?

(iv) What is the multiplicity of the representation ρ of Q_8 from **(i)** in the regular representation $\mathbb{C}[Q_8]$?

B 5. Suppose X is a finite G-set and $V = \mathbb{C}[X]$ is the permutation representation associated to X. Let χ be the character of V.

(i) Suppose $x \in X$. Define the orbit of x and the stabilizer subgroup G_x of x.

(ii) Prove that the character of the permutation representation $\mathbb{C}[X]$ is given by the formula

$$\chi(g) = |\{x \in X \mid g \cdot x = x\}| = |Fix_X(g)|.$$

(iii) State the Cauchy-Frobenius Lemma on the number of orbits in X.

(iv) Let m be the multiplicity of the trivial representation in $\mathbb{C}[X]$. Express m using characters. Use **(ii)** and **(iii)** to show that m is equal to the number of orbits in X.

B 6. Let H be a subgroup of G, and V a representation of H.

(i) Define the induced representation $\mathrm{Ind}_H^G V$.

(ii) Let G be the dihedral group of order 8 and H the cyclic subgroup of all rotations,

$$\begin{aligned} D_8 &= \langle r, s \mid r^4 = s^2 = e,\ srs^{-1} = r^{-1} \rangle, \\ H &= \{e, r, r^2, r^3\}. \end{aligned}$$

Let $\underline{\mathbb{C}}_H$ be the trivial representation of H. Either by constructing the induced representation directly or via the character formula for induced representations, work out the character of $\mathrm{Ind}_H^{D_8} \underline{\mathbb{C}}_H$.

(iii) State Frobenius reciprocity.