Problem Sheet 2

Part A

2.1 (Direct sums and complements)

(1) Show that two subspaces W and W' of a vector space V are complementary if and only if V is the (internal) direct sum of W and W'.

(2) Suppose $V = W \oplus W'$. Show that if $\{w_1, \ldots, w_l\}$ is a basis of W and $\{w'_1, \ldots, w'_j\}$ is a basis of W', then $\{w_1, \ldots, w_l, w'_1, \ldots, w'_j\}$ is a basis of V.

(3) Suppose $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$. Show that $\dim V = \dim W_1 + \ldots + \dim W_k$.

2.2 Consider the linear map $\mathbb{C}^3 \to \mathbb{C}^3$ given by the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Write $\mathbb{C}^3$ as direct sum of eigenspaces.

2.3 Recall the definition of an invariant inner product. Let $V = \mathbb{C}^3$ with the usual representation of S_3 on it (which sends $\sigma \in S_3$ to the corresponding permutation of the standard basis vectors). Show that the standard Hermitian inner product on $\mathbb{C}^3$ is invariant. Do the same exercise replacing S_3 and $\mathbb{C}^3$ with S_n and $\mathbb{C}^n$.

Part B

2.4 (Conjugacy classes of S_n) Let $X = \{1, \ldots, n\}$ with the usual action of S_n. For $\sigma \in S_n$ let $H_\sigma =< \sigma >$ be the subgroup of S_n generated by σ, and consider X as an H_σ-set.

The partition of X into H_σ-orbits defines a "**partition of** n", that is, a tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \ldots + \lambda_k = n$. Namely, order the H_σ-orbits in X from largest to smallest $\{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ and define $\lambda_i = |\mathcal{O}_i|$.

(i) For every element σ of S_3 work out the orbits of H_σ in $\{1, 2, 3\}$ and determine the corresponding partition of 3.

(ii) For $\sigma \in S_n$ given in cycle notation, work out how to 'read off' the partition of n that we associated to σ above.

(ii) Show that if σ and σ' are conjugate, then they give rise to the same partition of n. Conversely if σ and σ' give rise to the same partitions then they are conjugate.

(iii) List the conjugacy classes of S_3 and S_4.

2.5 Denote by $C_{(3)}$ the conjugacy class of 3-cycles in S_3. Write down its elements. Suppose V is the vector space with basis indexed by $C_{(3)}$. We denote the basis elements $\{e_c \mid c \in C_{(3)}\}$. Define a representation of S_3 on V by letting $g \cdot e_c = e_{gcg^{-1}}$ and extending linearly. Write this representation out explicitly in terms of matrices. Can you show whether this representation is irreducible, or conversely decompose it into a direct sum of irreducible representations?

2.6 Consider the representation of S_3 on $\mathbb{C}^6$ defined by

$$\begin{aligned} \sigma \cdot e_i &= e_{\sigma(i)} \qquad i = 1,2,3 \\ \sigma \cdot e_{3+i} &= e_{3+\sigma(i)} \quad i = 1,2,3 \end{aligned}$$

for any $\sigma \in S_3$ and where $e_1, \ldots, e_6$ is the standard basis of $\mathbb{C}^6$. Let $\rho : S_3 \to GL_6(\mathbb{C})$ denote the corresponding group homomorphism. Write down the matrices $\rho(\sigma)$ for all $\sigma \in S_3$. How many invariant subspaces can you find at a glance? (LEVEL 7: And after thinking more carefully?)

2.7 Consider the representation ρ of the infinite cyclic group $G = (\mathbb{Z}, +)$ given by $\rho(k) = A^k$ for the $d \times d$-matrix

$$A = \begin{pmatrix} x & 1 & & \\ & x & \ddots & \\ & & \ddots & 1 \\ & & & x \end{pmatrix}$$

made up of a single 'Jordan block'. Work out for which choices of (eigenvalue) x and (degree) d the representation $\rho : \mathbb{Z} \to GL_d(\mathbb{C})$ factors through the group homomorphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to define a representation $\bar{\rho} : \mathbb{Z}/n\mathbb{Z} \to GL_d(\mathbb{C})$ of the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$.

More simply put: For which A as above does $k \mapsto A^k$ actually define a representation of $\mathbb{Z}/n\mathbb{Z}$ for some integer n?

2.8 Let V and W be vector spaces and $\mathrm{Hom}(V, W)$ the set of all linear maps from V to W.

(1) Show that $\mathrm{Hom}(V, W)$ is again a vector space.

(2) Suppose V and W are representations of G. Define a representation of G on $\mathrm{Hom}(V, W)$ using both actions of G. [Make sure to check that $gh \cdot \phi = g \cdot (h \cdot \phi)$ for any $g, h \in G$ and $\phi \in \mathrm{Hom}(V, W)$.]

(3) Show that a homomorphism ϕ is G-equivariant if and only if ϕ is invariant under the action of G on $\mathrm{Hom}(V, W)$ from (2), that is if $g \cdot \phi = \phi$ for all $g \in G$.