Problem Sheet 1

Let G be a finite group, and X a set (not necessarily finite).

Part A

1.1 Show that giving an action of G on X is equivalent to giving a group homomorphism $\alpha : G \to Perm(X)$.

1.2 Write out formally the definition of an equivariant map between G-sets (i.e. sets X and Y endowed with G-actions $a : G \times X \to X$ and $b : G \times Y \to Y$, respectively).

1.3 Show that if $f : X \to Y$ is an isomorphism of G-sets (so, a bijective G-equivariant map), then the set-theoretic inverse map f^{-1} from Y to X is automatically G-equivariant.

1.4 Let X be a G-set. As in class (Lecture 3), define

$$x \sim y \quad :\iff \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y.$$

(i) Show that $x \sim y$ defines an equivalence relation on X (i.e. is symmetric in x and y, reflexive: $x \sim x$, and is transitive: if $x \sim y$ and $y \sim z$ then $x \sim z$.).

(ii) Let $x \in X$. Show that the subset of X defined as

$$G \cdot x := \{g \cdot x \mid g \in G\}$$

is an equivalence class for $\sim$.

Note: *The set $G \cdot x$ is called the "G-orbit of x".*

Part B

1.5 Suppose H and H' are conjugate subgroups of G. Show that the G/H and G/H' are isomorphic as G-sets. [Hint: Suppose $H' = xHx^{-1}$. Show that the assignment $gH' \mapsto gxH$ gives a well-defined map from G/H' to G/H and check that this map is an isomorphism of G-sets.]

1.6 Let $X = \{1, \ldots, n\}$, and denote by $\mathcal{P}(X)$ the power set, that is, the set of all subsets, of X. Then the standard action of S_n on X induces an action of S_n on $\mathcal{P}(X)$. Describe the decomposition of $\mathcal{P}(X)$ into a union of S_n-orbits:

(1) Enumerate the S_n-orbits.
(2) Determine their cardinalities.
(3) Choose a point in each orbit and describe its stabilizer subgroup.

Verify the orbit-stabilizer theorem in this example.

1.7 Prove "Cayley's theorem": Any finite group G is isomorphic to a subgroup of the symmetric group S_n, for large enough integer n.

1.8 Let G be the dihedral group $D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ srs = r^{-1} \rangle$. Construct a homogeneous G-set X such that $H = \{1, s\}$ is the stabilizer subgroup G_x of a point $x \in X$. What is the cardinality of X?