

Definitions from Representation Theory

1. A **G -set** is set X with an **action** of G on it, i.e. a map $G \times X \rightarrow X : (g, x) \mapsto g \cdot x$, such that $(gh) \cdot x = g \cdot (h \cdot x)$ and $e \cdot x = x$ for all $g, h \in G$ and $x \in X$.

2. A G -set is called **homogeneous** if the action of G on X is **transitive**, that is for any two elements $x, y \in X$ there exists a group element $g \in G$ such that $g \cdot x = y$.

3. An **orbit** in a G -set X is an equivalence class for the equivalence relation on X defined by

$$x \sim y : \iff y = g \cdot x \text{ for some } g \in G.$$

For $x \in X$ the orbit through x is the subset $G \cdot x = \{g \cdot x \mid g \in G\}$ of X .

4. The **fixed point set** of an element $g \in G$ in a G -set X is the set

$$\text{Fix}_X(g) = \{x \in X \mid g \cdot x = x\}.$$

5. If x is an element of a G -set, then the **stabilizer** of x is the *subgroup* of G ,

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

6. For a group G and a subgroup H , a **left coset** is a subset of G of the form $gH = \{gh \mid h \in H\}$. We denote by G/H the set of all left cosets. [Note that this is a homogeneous G -set under the action of multiplication from the left.]

7. Let W be a linear subspace of a vector space V . A **complement** to W is another subspace W' such that $W + W' = V$ and $W \cap W' = \{0\}$.

8. Let W_1, \dots, W_k be subspaces of a vector space V . We say that V is the (internal) **direct sum** of W_1, \dots, W_k if every element $v \in V$ can be written in a unique way as a sum $v = w_1 + \dots + w_k$ where $w_1 \in W_1, \dots, w_k \in W_k$.

9. Let V and W be vector spaces over \mathbb{K} . A linear map $V \rightarrow W$ is also called a **homomorphism** from V to W . The set of all homomorphisms from V to W forms a vector space [by $(f + \lambda g) : v \mapsto f(v) + \lambda g(v)$], which is denoted by $\text{Hom}(V, W)$.

10. For a vector space V over \mathbb{K} the **dual space** V^* is defined to be the vector space $V^* = \text{Hom}(V, \mathbb{K})$.

11. A **projection operator** is a linear map $\pi : V \rightarrow V$ such that $\pi \circ \pi = \pi$. If $W = \text{Im}(\pi)$ and $W' = \text{ker}(\pi)$ then π is also called the projection onto W along W' . (For a projection $\pi : V \rightarrow V$ one always has $\text{Im}(\pi) \oplus \text{ker}(\pi) = V$).

12. For vector spaces V_1, \dots, V_n over \mathbb{K} the (external) **direct sum** $V_1 \oplus V_2 \oplus \dots \oplus V_n$ is defined to be the vector space of ‘tuples’

$$V_1 \oplus V_2 \oplus \dots \oplus V_n = \{(v_1, \dots, v_n) \mid v_i \in V_i, \text{ all } i = 1, \dots, n\}$$

[Note: same notation as for (internal) direct sum!]

13. A **representation** of a group G (over a field \mathbb{K}) is a group homomorphism $\rho : G \rightarrow GL(V)$ where V is a vector space (over \mathbb{K}). [Recall that this defines an action of G on V , and we will often write $g \cdot v$ for $\rho(g)(v)$. We may also say “ V is a representation of G ” for short, thinking of the representation as a vector space V with extra structure, namely a group homomorphism ρ , or $\rho_V : G \rightarrow GL(V)$.]

14. An **invariant subspace** in a representation V of G is a linear subspace W such that for all $w \in W$ we have $g \cdot w \in W$ for all $g \in G$.

15. An **irreducible representation** is a representation that has no non-trivial invariant subspaces. (An invariant subspace of a representation V is called *trivial* if it is either $\{0\}$ or V itself.)

16. An **invariant complement** to an invariant subspace W in a representation V , is an invariant subspace W' such that $W + W' = V$ and $W \cap W' = \{0\}$. [Equivalently, W' is an invariant subspace such that $W \oplus W' = V$.]

17. If V is a complex representation of G a **invariant Hermitian inner product** on V is a Hermitian inner product $\langle \cdot, \cdot \rangle$ such that $\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$ for all $g \in G$.

18. The **trivial representation** of G (over \mathbb{K}) is the one-dimensional representation of G given by $\rho(g) = id_{\mathbb{K}}$ for all $g \in G$. The one-dimensional vector space \mathbb{K} endowed with the trivial representation of G on it will also be denoted by $\underline{\mathbb{K}}_G$.

19. Let V and W be representations of G over \mathbb{K} . A **G -equivariant homomorphism** from V to W is a linear map $\phi : V \rightarrow W$ such that $\phi(g \cdot v) = g \cdot (\phi(v))$. The G -equivariant homomorphisms form a linear subspace of the vector space $\text{Hom}(V, W)$. This linear subspace is denoted by $\text{Hom}_G(V, W)$.

20. A representation V of G is called **faithful** if the associated group homomorphism $\rho_V : G \rightarrow GL(V)$ is injective. [Equivalently, this is if the kernel $\text{ker}(\rho_V) := \{g \in G \mid \rho_V(g) = e\}$ is the trivial subgroup $\{e\}$.]