

Full Solutions to Problem Sheet 1

1.1 Show that giving an action of G on X is equivalent to giving a group homomorphism $\alpha : G \rightarrow \text{Perm}(X)$.

Answer: Let $a : G \times X \rightarrow X$ be an action of G . Define the map $\alpha : G \rightarrow \text{Perm}(X)$ to be the map sending a group element $g \in G$ to the “permutation”

$$\alpha(g) : x \mapsto g \cdot x,$$

where $g \cdot x$ is short-hand notation for $a(g, x)$. To check that α is a group homomorphism it suffices to check $\alpha(gh) = \alpha(g) \circ \alpha(h)$. We can check this by applying both sides to some/any $x \in X$:

$$\alpha(gh)(x) = (gh) \cdot x = g \cdot (h \cdot x) = \alpha(g)(h \cdot x) = \alpha(g)(\alpha(h)(x)) = (\alpha(g) \circ \alpha(h))(x).$$

Conversely, given a group homomorphism α as above, we can define the action a by

$$a(g, x) = \alpha(g)(x).$$

Then $a(e, x) = \alpha(e)(x) = x$ and

$$a(gh, x) = \alpha(gh)(x) = (\alpha(g) \circ \alpha(h))(x) = \alpha(g)(\alpha(h)(x)) = a(g, a(h, x)).$$

1.2 Write out formally the definition of an equivariant map between G -sets (i.e. sets X and Y endowed with G -actions $a : G \times X \rightarrow X$ and $b : G \times Y \rightarrow Y$, respectively).

Answer: $f : X \rightarrow Y$ is an equivariant map of G -sets if and only if

$$f(a(g, x)) = b(g, f(x)) \quad \text{for all } g \in G \text{ and } x \in X.$$

1.3 Show that if $f : X \rightarrow Y$ is a bijective morphism of G -sets, then the set-theoretic inverse map is also a morphism of G -sets.

Answer: Since $f : X \rightarrow Y$ is bijective there exists an inverse map of sets $f^{-1} : Y \rightarrow X$ such that $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$. We need to check that f^{-1} is also G -equivariant.

Consider $y \in Y$. Because f is G -equivariant by assumption, we have

$$f(g \cdot (f^{-1}(y))) = b(g, f(f^{-1}(y))) = g \cdot y.$$

Applying f^{-1} to both sides shows that

$$g \cdot f^{-1}(y) = f^{-1}(g \cdot y).$$

This is true for all $g \in G$ and $y \in Y$, and says exactly that f^{-1} is G -equivariant.

1.4 Let X be a G -set. As in class (Lecture 3), define

$$x \sim y \quad : \iff \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y.$$

- (i) Show that $x \sim y$ defines an equivalence relation on X .
- (ii) Let $x \in X$. Show that the subset of X defined as

$$G \cdot x := \{g \cdot x \mid g \in G\}$$

is an equivalence class for \sim .

Answer: (i) We check the three properties of an equivalence relation:

- Reflexive: $e \cdot x = x$ implies $x \sim x$.
- Symmetric: $x \sim y$ means $g \cdot x = y$, some g , which implies $g^{-1} \cdot y = x$ so that $y \sim x$.
- Transitive: $x \sim y$ and $y \sim z$ means $\exists g_1, g_2$ such that $g_1 \cdot x = y$ and $g_2 \cdot y = z$.
Now $g_2 g_1 \cdot x = g_2(g_1 \cdot x) = g_2 \cdot y = z$ implies $x \sim z$.

- (ii) The set $G \cdot x$ consists precisely of those elts of X which are equivalent to x under \sim .
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1.5 Suppose H and H' are conjugate subgroups of G . Show that the G/H and G/H' are isomorphic as G -sets. [Hint: Suppose $H' = xHx^{-1}$. Show that the assignment $gH' \mapsto gxH$ gives a well-defined map from G/H' to G/H and check that this map is an isomorphism of G -sets.]

Answer: We have to show, for $H' = xHx^{-1}$, that the assignment $gH' \mapsto gxH$ gives a well-defined map $\phi : G/H' \rightarrow G/H$. The potential problem is that two elements g_1 and g_2 of G can represent the same coset. Suppose this is the case, so $g_1H' = g_2H'$. Then $g_1 = g_2h'$ for some $h' \in H'$. By our assumption that $H' = xHx^{-1}$, we have $h' = xhx^{-1}$, for an element $h \in H$. Therefore

$$g_1xH = g_2h'xH = g_2(xhx^{-1})xH = g_2xhH = g_2xH.$$

This shows that the map ϕ is well defined. It is immediate that ϕ is a map of G -sets, since any $g \in G$ acts by multiplication from the left, and

$$\phi(gg_1H') = gg_1xH = g\phi(g_1H').$$

1.6 Let $X = \{1, \dots, n\}$, and denote by $\mathcal{P}(X)$ the power set, that is, the set of all subsets, of X . Then the standard action of S_n on X induces an action of S_n on $\mathcal{P}(X)$. Describe the decomposition of $\mathcal{P}(X)$ into a union of S_n -orbits:

- (1) Enumerate the S_n -orbits.
- (2) Determine their cardinalities.
- (3) Choose a point in each orbit and describe its stabilizer subgroup.

Verify the orbit-stabilizer theorem in this example.

Answer: Suppose Y and Y' are two elements of $\mathcal{P}(X)$, i.e. subsets of $\{1, \dots, n\}$, for which there is a permutation σ taking Y to Y' . Then Y and Y' must have the same number of elements. So we may consider $\mathcal{P}_k(X) \subset \mathcal{P}(X)$, the set of k -element subsets of X , where $k = 0, \dots, n$. Now for any two elements in $\mathcal{P}_k(X)$ it is possible to find a permutation σ taking one to the other. Therefore each one of these is an orbit.

- We have found $n + 1$ orbits: $\mathcal{P}_0(X), \dots, \mathcal{P}_n(X)$.
- The orbit $\mathcal{P}_k(X)$ has $\binom{n}{k}$ elements.
- Consider the element $x_k := \{1, \dots, k\}$ of $\mathcal{P}_k(X)$. Its stabilizer G_{x_k} is isomorphic to $S_k \times S_{n-k}$. Namely, it consists of permutations of $\{1, \dots, k\}$ times permutations of the set $\{k + 1, \dots, n\}$ of $n - k$ elements.

According to the orbit-stabilizer theorem the order, $\binom{n}{k}$, of the orbit $\mathcal{P}_k(X) = S_n \cdot x_k$ should be given by

$$\frac{|S_n|}{|G_{x_k}|} = \frac{n!}{k!(n-k)!}.$$

But this is precisely the standard formula for $\binom{n}{k}$, verifying the orbit-stabilizer theorem in this example.

1.7 Prove “Cayley’s theorem”: Any finite group G is isomorphic to a subgroup of the symmetric group S_n , for large enough integer n .

Answer: If X is a set with N elements, then $\text{Perm}(X) \cong S_N$. If G is a finite group, then the action a of G on itself by multiplication from the left defines a group homomorphism $\alpha : G \rightarrow \text{Perm}(G)$ (as in 1.1). The kernel of the map α is just the trivial subgroup $\{e\}$ (In other words if $\alpha(g)$ is the identity permutation, then $g = e$). Therefore α is injective. So it is an isomorphism onto its image. Identifying $\text{Perm}(G) \cong S_N$, where $N = |G|$, we see that G is isomorphic to a subgroup of a symmetric group.

1.8 Let G be the dihedral group $D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$. Construct a homogeneous G -set X such that $H = \{1, s\}$ is the stabilizer subgroup G_x of a point $x \in X$. What is the cardinality of X ?

Answer: One solution is just to let $X = G/H$, the set of H -cosets with the usual action of multiplication from the left. This is a homogeneous G -set, and H is the stabilizer of the element $x := eH$. For the cardinality of X we get

$$|X| = \frac{|G|}{|H|} = \frac{2n}{2} = n.$$

Another possible solution is to recall that D_{2n} can be identified with the group of symmetries of a regular n -gon P_n . The elements r^k act by rotation and, for a chosen fixed vertex x of P_n , we can let s act as the reflection across the line through x and the centre of P_n . Then we may choose for X the set of vertices of P_n . Because the rotations can be used to take x to any other vertex, this is a homogeneous G -set. The stabilizer subgroup G_x of x is precisely equal to H . The cardinality of X is the number of vertices of P_n , that is, n .