52.1 (a) Take the disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and remove the radial line $x = 0$, $y > 0$. This is star-shaped with centre $(0, 0)$ but is not convex as the line segment joining $(1/2, 1/2)$ to $(-1/2, 1/2)$ crosses the removed line.

(b) If A is star-shaped with centre a_0, every point of A can be joined to a_0 by a straight-line path, so any two points of A can be joined by a path consisting of two line segments. Hence, A is path-connected. Hence, it is enough to show that $\pi_1(A, a_0)$ is the trivial group. If f is a loop in A based at a_0, $H(s, t) = (1 - t)f(s) + ta_0$ is a path-homotopy from f to the constant loop at a_0.

52.2 If f is a loop based at x_0,

$$\begin{aligned}\hat{\gamma}([f]) &= [\tilde{\gamma}] * [f] * [\gamma] \\ &= [\tilde{\beta} * \tilde{\alpha}] * [f] * [\alpha * \beta] \\ &= [\tilde{\beta}] * [\tilde{\alpha}] * [f] * [\alpha] * [\beta] \\ &= \hat{\beta}(\hat{\alpha}([f])).\end{aligned}$$

52.3 Suppose first that $\pi_1(X, x_0)$ is abelian and let α, β be paths from x_0 to x_1. Then $\alpha * \tilde{\beta}$ is a loop based at x_0, so if f is any loop based at x_0, we have

$$[f] * [\alpha * \tilde{\beta}] = [\alpha * \tilde{\beta}] * [f],$$

i.e.

$$[f] * [\alpha] * [\tilde{\beta}] = [\alpha] * [\tilde{\beta}] * [f].$$

Multiply (using $*$) on the left by $[\tilde{\alpha}]$ and on the right by $[\beta]$ and remember that $[\tilde{\alpha}] * [\alpha] = [e_{x_1}]$, $[\beta] * [\tilde{\beta}] = [e_{x_0}]$. This gives

$$[\tilde{\alpha}] * [f] * [\alpha] = [\tilde{\beta}] * [f] * [\beta],$$

i.e. $\hat{\alpha} = \hat{\beta}$.

Conversely, suppose that $\hat{\alpha} = \hat{\beta}$ for all paths α, β from x_0 to x_1. Let f be a loop based at x_0 and let δ be a path from x_0 to x_1. Then, $\Delta = f * \delta$ is another path from x_0 to x_1 so by assumption $\hat{\delta} = \hat{\Delta}$. Thus, if f' is any loop based at x_0, $\hat{\Delta}([f']) = \hat{\delta}([f'])$, i.e.

$$[\tilde{\delta} * \tilde{f}] * [f'] * [f * \delta] = [\tilde{\delta}] * [f'] * [\delta].$$

Multiplying on the left by $[f] * [\delta]$ and on the right by $\tilde{\delta}$ as in the first part gives $[f'] * [f] = [f] * [f']$. Thus, $\pi_1(X, x_0)$ is abelian.

52.4 Let $i : A \to X$ be the inclusion map. Then, $r \circ i$ is the identity map $A \to A$. Hence, $r_* \circ i_*$ is the identity map $\pi_1(A, a_0) \to \pi_1(A, a_0)$. It follows that r_* is surjective (and that i_* is injective).

52.5 Let $H : \mathbb{R}^n \to Y$ be an extension of h and let $i : A \to \mathbb{R}^n$ be the inclusion map. Then, $H \circ i = h$ and so $h_* = H_* \circ i_*$. But the homomorphism $H_* : \pi_1(\mathbb{R}^n, a_0) \to \pi_1(Y, y_0)$ is the trivial homomorphism because $\pi_1(\mathbb{R}^n, a_0)$ is the trivial group (because $\mathbb{R}^n$ is convex, hence simply-connected). Hence h_* is trivial too.

51.1 Let H be a homotopy from h to h' and K a homotopy from k to k'. Then, a homotopy from $k \circ h$ to $k' \circ h'$ is the map $L : X \times I \to Z$ given by $L(x,t) = K(H(x,t),t)$.

51.2(a) If $f : X \to I$ is continuous, $H(x,t) = tf(x)$ is a homotopy from the constant map c that takes every element of X to 0 to f. So every element of $[X, I]$ is equal to the homotopy class of c.

(b) Let f be a path in Y and let $f(0) = y_0$. A homotopy from the constant path e_0 at y_0 to f is $H(s,t) = f(st)$ (note that H is not a path-homotopy). So every element of $[I, Y]$ is equal to the homotopy class of a constant path. If $y_1 \in Y$ and e_1 is the constant path at y_1, let γ be a path from y_0 to y_1. Then $K(s,t) = \gamma(t)$ is a homotopy from e_0 to e_1.

51.3(a) $H(s,t) = s(1-t)$ is a homotopy from i_I to the constant map that takes I to 0. The same homotopy works when $X = \mathbb{R}$.

(b) Let $H : X \times I \to X$ be a homotopy from i_X to the constant map that takes X to x_0, where $x_0 \in X$. Then, if $x_1 \in X$, $\gamma_{x_1}(t) = H(x_1, t)$ is a path from x_1 to x_0. If $x_2 \in X$, $\gamma_{x_1} * \tilde{\gamma}_{x_2}$ is a path from x_1 to x_2.

(c) Let $H : Y \times I \to Y$ be a homotopy from i_Y to the constant map that takes Y to y_0, where $y_0 \in Y$. If $f : X \to Y$ is continuous, $K(x,t) = H(f(x),t)$ is a homotopy from f to the constant map e_0 that takes X to y_0. So every element of $[X, Y]$ is equal to the homotopy class of a constant map. If $y_1 \in Y$ and e_1 is the constant map that takes X to y_1, and if γ is a path in Y from y_0 to y_1 (which exists by part (b)), then $L(x,t) = \gamma(t)$ is a homotopy from e_0 to e_1.

(d) Let $H : X \times I \to X$ be a homotopy from i_X to the constant map that takes X to x_0, where $x_0 \in X$. If $f : X \to Y$ is continuous, $f \circ H$ is a homotopy from f to the constant map that takes X to $f(x_0)$. So every element of $[X, Y]$ is equal to the homotopy class of a constant map. As in part (c), the fact that Y is path connected implies that any two constant maps $X \to Y$ are homotopic.