King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

BSC AND MSCI EXAMINATION

7CCM327B TOPOLOGY

SUMMER 2011

TIME ALLOWED: THREE HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.
SECTION A CONTRIBUTES ONE THIRD OF THE TOTAL MARKS FOR THE PAPER.
ANSWER ALL QUESTIONS IN SECTION A.
ALL QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN FOUR ARE ATTEMPTED ONLY THE BEST FOUR WILL COUNT.

YOU ARE PERMITTED TO USE A CALCULATOR.
ONLY CALCULATORS APPROVED BY THE COLLEGE MAY BE USED.

TURN OVER WHEN INSTRUCTED

A 1. (a) Show that the only non-empty connected subsets of $\mathbb{Q}$ are the one-point subsets.

(b) Show that the only continuous functions $\mathbb{R} \to \mathbb{Q}$ are the constant functions.

(c) Give an example of a continuous function $\mathbb{Q} \to \mathbb{R}$ that is not the restriction of a continuous function $\mathbb{R} \to \mathbb{R}$. Briefly justify your answer.

(d) What is the fundamental group $\pi_1(\mathbb{Q}, 0)$? Give brief reasons for your answer.

A 2. Let $\mathcal{T}$ be the collection of open intervals (a, ∞), with $a \in \mathbb{R}$, together with the empty set and $\mathbb{R}$ itself.

(a) Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$.

(b) Prove that the standard topology $\mathcal{T}_{st}$ on $\mathbb{R}$ is strictly finer than $\mathcal{T}$.

(c) Give an example of a continuous function $f : (\mathbb{R}, \mathcal{T}_{st}) \to (\mathbb{R}, \mathcal{T})$ that is not continuous when viewed as a function $(\mathbb{R}, \mathcal{T}_{st}) \to (\mathbb{R}, \mathcal{T}_{st})$.

(d) Is $(\mathbb{R}, \mathcal{T})$
  (i) connected;
  (ii) compact;
  (iii) Hausdorff?

Give brief reasons for your answers.

A 3. Let L_n be the straight-line segment in $\mathbb{R}^2$ with endpoints $(0, 0)$ and $(1, \frac{1}{n})$, where $n \in \mathbb{N}$, i.e.

$$L_n = \{(t, \frac{t}{n}) \,|\, 0 \leq t \leq 1\}.$$

Let L_∞ be the line segment with endpoints $(0, 0)$ and $(1, 0)$, and let

$$X = L_\infty \cup \bigcup_{n=1}^{\infty} L_n.$$

Give each L_n (including L_∞) the subspace topology from $\mathbb{R}^2$ and let $\mathcal{T}$ be the collection of subsets U of X such that $U \cap L_n$ is an open subset of L_n for all $n \in \mathbb{N} \cup \{\infty\}$.

(a) Prove that $\mathcal{T}$ is a topology on X that is strictly finer than the subspace topology of X as a subset of $\mathbb{R}^2$ (with its standard topology).

(b) Explain, giving brief reasons, why X is compact in the subspace topology but not in the topology $\mathcal{T}$.

B 4. (a) Explain what is meant by saying that a topological space is
(i) connected;
(ii) path-connected.

(b) Prove that every path-connected topological space is connected.

(c) Let X be a non-empty connected open subset of $\mathbb{R}^n$, where $n \in \mathbb{N}$. Prove that X is path-connected. (Hint: Let $x_0 \in X$ and let Y be the set of points $x \in X$ such that there is a path in X from x_0 to x. Prove that Y is both open and closed in X.)

(d) Let $X = \{(x, y) \in \mathbb{R}^2 \,|\, \text{either } x \text{ or } y \text{ (or both) is rational}\}$. Give X the subspace topology as a subset of $\mathbb{R}^2$. Prove that X is connected.

B 5. (a) Define the fundamental group $\pi_1(X, x_0)$ of a topological space X with respect to a basepoint $x_0 \in X$. Explain (without proofs) the group structure on $\pi_1(X, x_0)$ (you should include in your answer the definitions of the product, the identity element and inverses).

(b) If X is path-connected and $x_0, x_1 \in X$, write down an isomorphism of groups $\pi_1(X, x_0) \to \pi_1(X, x_1)$ (proofs are not required).

(c) Let $f : X \to Y$ be a continuous map, and let $y_0 = f(x_0)$. Define the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and prove that your definition makes sense. Prove that f_* is a homomorphism.

(d) Describe (without proofs) an isomorphism

$$\pi_1(S^1, 1) \to \mathbb{Z},$$

where $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ and $\mathbb{Z}$ denotes the group of integers under addition. Write down a path in S^1 based at 1 that corresponds under your isomorphism to the integer $1 \in \mathbb{Z}$.

(e) If $f : S^1 \to S^1$ is given by $f(z) = z^n$, where $n \in \mathbb{Z}$, part (d) implies that f_* corresponds to a homomorphism $\mathbb{Z} \to \mathbb{Z}$. What is this homomorphism?

B 6. Let X be a topological space. A subset A of X is said to be *dense* if $\bar{A} = X$ (where $\bar{A}$ denotes the closure of A in X).

(a) Prove that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

(b) Let X be a Hausdorff topological space. Prove that

$$\Delta_X = \{(x, x') \in X \times X \mid x = x'\}$$

is a closed subset of $X \times X$ (with the product topology).

(c) Let X and Y be topological spaces and assume that Y is Hausdorff. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Let A be a dense subset of X and suppose that $f(x) = g(x)$ for all $x \in A$. Prove that $f = g$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has its standard topology), and suppose that

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that

$$f(nx) = nf(x) \text{ for all } x \in \mathbb{R},\ n \in \mathbb{Z}$$

and deduce that

$$f(qx) = qf(x) \text{ for all } x \in \mathbb{R},\ q \in \mathbb{Q}.$$

Use parts (a) and (c) to deduce that there exists $\lambda \in \mathbb{R}$ such that $f(x) = \lambda x$ for all $x \in \mathbb{R}$.

B 7. If X is a topological space and A is a subset of X, a point $x \in X$ is called a *limit point* of A if every open subset of X that contains x contains a point of A different from x. Let A' denote the set of limit points of A.

(a) Prove that $\bar{A} = A \cup A'$ (where $\bar{A}$ denotes the closure of A in X). Deduce that $A' \subset A$ if and only if A is a closed subset of X.

(b) Let Y be a set with two points, and give Y the indiscrete topology (i.e. the topology for which Y is the only non-empty open set). Let $X = \mathbb{N} \times Y$, where $\mathbb{N}$ has its standard topology as a subset of $\mathbb{R}$ and X has the product topology. Show that every non-empty subset of X has a limit point.

(c) A topological space X is said to be *limit point compact* if every infinite subset of X has a limit point. Prove that if X is compact then it is limit point compact (you may wish to make use of part (a)). Is the converse true? Explain your answer.

B 8. (a) Define the term topological group.

(b) Show that, if G is any group, then G with the discrete topology is a topological group.

(c) If G is a topological group and $g \in G$, let $l_g : G \to G$ be the map defined by $l_g(x) = gx$. Prove that l_g is a homeomorphism.

(d) Prove that, if G is a connected topological group, the only open subgroup of G is G itself.

(e) Let G be a connected topological group and let H be a normal subgroup of G such that the subspace topology of H is the discrete topology. By proving that the map $f(g) = ghg^{-1}$, where $h \in H$ is fixed, is continuous, or otherwise, prove that H is contained in the centre of G (i.e. the set $\{g \in G \,|\, gx = xg \text{ for all } x \in G\}$).

(f) Give an example to show that the result in (e) does not hold if we do not assume that G is connected.

SOLUTIONS

1. (a) It is obvious that every one-point subset (of any topological space) is connected. If $A \subset \mathbb{Q}$ has at least two elements, let $x, y \in A$ with $x < y$ and let z be an irrational number with $x < z < y$. Let $A' = A \cap (-\infty, z)$, $A'' = A \cap (z, \infty)$. Then, $A = A' \cup A''$ is a separation of A.

(b) If $f : \mathbb{R} \to \mathbb{Q}$ is continuous, $f(\mathbb{R})$ is connected since $\mathbb{R}$ is connected. So $f(\mathbb{R})$ must be a single point set, i.e. f is constant.

(c) Take $f(x) = 0$ if $x < \sqrt{2}$, $= 1$ if $x > \sqrt{2}$. Then, f is continuous as it is the restriction to $\mathbb{Q}$ of the function $f : \mathbb{R} - \{\sqrt{2}\} \to \mathbb{R}$ given by the same formulas, and this function is continuous by the pasting lemma. If f were the restriction of a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$, we would have to have $\tilde{f}(x) = f(x)$ if $x \neq \sqrt{2}$, and whatever the value of $\tilde{f}(\sqrt{2})$, $\tilde{f}$ could not be continuous.

(d) By the argument in (b), the only (continuous) path $\gamma : [0, 1] \to \mathbb{Q}$ based at 0 is the constant path. So $\pi_1(\mathbb{Q}, 0)$ is the trivial group.

2. (a) $(a, \infty) \cap (b, \infty) = (max(a, b), \infty)$, so $\mathcal{T}$ is closed under finite intersections. If $\{a_i\}$ is any family of real numbers, the union U of the intervals (a_i, ∞) is open in the standard topology of $\mathbb{R}$. It is an interval, because if $x, y \in U$ and $x < y$, then $x \in (a_j, \infty)$, $y \in (a_k, \infty)$ for some j, k, and if $a_j \leq a_k$ (say) then $x, y \in (a_j, \infty)$ so $(x, y) \subset (a_j, \infty) \subset U$. So U is an open interval, and since $(a_i, \infty) \subset U$ for all i, U must be of the form (a, ∞) for some $a \in \mathbb{R}$. So $\mathcal{T}$ is closed under taking arbitrary unions.

(b) Since each $(a, \infty) \in \mathcal{T}_{st}$, it follows that $\mathcal{T} \subset \mathcal{T}_{st}$. The open interval $(-\infty, 0)$ is in $\mathcal{T}_{st}$ but not in $\mathcal{T}$.

(c) $f(x) = 0$ if $x \leq 0$, $= 1$ if $x > 1$.

(d) (i) Yes: no two open intervals (a, ∞), (b, ∞) are disjoint.

(ii) No: the covering $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, \infty)$ by sets open in $\mathcal{T}$ has no finite subcovering.

(iii) No: if $x < y$ any open set in $\mathcal{T}$ that contains x must contain y.

3. (a) Let $\mathcal{T}_{st}$ be the subspace topology of X. If $U \in \mathcal{T}_{st}$, then $U = V \cap X$ for some open set V in $\mathbb{R}^2$, so $U \cap L_n = V \cap L_n$, which is open in L_n by definition of the subspace topology. This proves that $\mathcal{T}_{st} \subset \mathcal{T}$.

The open line segment $L^0_\infty = \{(t, 0) \mid 0 < t < 1\}$ is in $\mathcal{T}$ because $L^0_\infty \cap L_n = \emptyset$ if $n \in \mathbb{N}$, and

$$L^0_\infty \cap L_\infty = L_\infty \cap \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1\}$$

is open in L_∞. But L^0_∞ is not in $\mathcal{T}_{st}$. For, if $L^0_\infty = W \cap X$, where W is open in $\mathbb{R}^2$, then $(\frac{1}{2}, 0) \in W$ so there is an open rectangle $(\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) \times (-\delta, \delta) \subset W$, with $\delta, \epsilon > 0$. Choose $n \in \mathbb{N}$ such that $1/2n < \delta$. Then, $(\frac{1}{2}, \frac{1}{2n}) \in L_n \cap W \subset W \cap X$, but $(\frac{1}{2}, \frac{1}{2n}) \notin L^0_\infty$, a contradiction.

(b) $(X, \mathcal{T}_{st})$ is compact because X is a closed, bounded subset of $\mathbb{R}^2$. But $(X, \mathcal{T})$ is not compact: the infinite open covering formed by the line segments L^0_n ($n \in \mathbb{N} \cup \{\infty\}$) and the open sets

$$X \cap \{(x, y) \in \mathbb{R}^2 \mid x < 1/4\}, \quad X \cap \{(x, y) \in \mathbb{R}^2 \mid x > 3/4\},$$

has no proper subcovering.

4. (a) (i) X is connected if it does not have a separation, i.e. disjoint non-empty open subsets A, B such that $X = A \cup B$.

(ii) X is path-connected if, given any $x_0, x_1 \in X$, there is a path from x_0 to x_1, i.e. a continuous map $\gamma : I = [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

(b) Let X be path-connected and let $X = A \cup B$ be a separation. Let $x_0 \in A$, $x_1 \in B$ and let γ be a path from x_0 to x_1. Then, $\gamma^{-1}(A) \cup \gamma^{-1}(B) = I$ is a separation of I, contradicting the fact that I is connected.

(c) We prove Y is open. Let $x \in Y$, let γ be a path in X from x_0 to x_1 and let $\epsilon > 0$ be such that the open ball $B_\epsilon(x) \subset X$. If $y \in B_\epsilon(x)$, then $y \in Y$ because γ followed by the straight line segment joining x to y is a path in X from x_0 to y. So $B_\epsilon(x) \subset Y$.

We prove $X - Y$ is open. Let $x \in X - Y$ and let $B_\epsilon(x) \subset X$. If $y \in B_\epsilon(x) \cap Y$, there is a path γ in X from x_0 to y. Then, γ followed by the straight line segment from y to x is a path in X from x_0 to x, contradicting $x \notin Y$. So $B_\epsilon(x) \cap Y = \emptyset$, i.e. $B_\epsilon(x) \subset X - Y$.

Since X is connected, the only open and closed subsets of X are $\emptyset$ and X. Since $x_0 \in Y$ we must have $Y = X$. If x_1 and x_2 are in X, there are paths γ_1, γ_2 in X from x_0 to x_1 and x_2, respectively. Then, the reverse of γ_1 followed by γ_2 is a path from x_1 to x_2. So X is path-connected.

(d) If $x \in \mathbb{Q}$, $y \in \mathbb{R}$, the straight line segment with endpoints (x, y) and $(x, 0)$ is a path in X, as is the straight line segment with endpoints $(x, 0)$ and $(0, 0)$. Hence, there is a path consisting of two line segments from (x, y) to $(0, 0)$. Similarly if $x \in \mathbb{R}$, $y \in \mathbb{Q}$. Since any two points of X can be joined to $(0, 0)$ by a path, they can be joined to each other.

5. (a) A path in X based at x_0 is a continuous map $\gamma : I \to X$ such that $\gamma(0) = \gamma(1) = x_0$ ($I = [0, 1]$). Two such paths γ_0, γ_1 are path-homotopic if there is a continuous map $H : I \times I \to X$ such that

$$H(s, 0) = \gamma_0(s),\ H(s, 1) = \gamma_1(s),\ H(0, t) = H(1, t) = x_0,$$

for all $s, t \in I$. Path-homotopy is an equivalence relation on the set of all paths based at x_0, and the equivalence classes are the elements of $\pi_1(X, x_0)$. The equivalence class of γ is denoted $[\gamma]$.

The product is

$$[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2],$$

where $\gamma_1 * \gamma_2(s) = \gamma_1(2s)$ if $s \in [0, 1/2]$, and $= \gamma_2(2s - 1)$ if $s \in [1/2, 1]$. The identity element is $[e_{x_0}]$, where $e_{x_0}(s) = x_0$ for all $s \in I$. The inverse of $[\gamma]$ is $[\tilde{\gamma}]$, where $\tilde{\gamma}(s) = \gamma(1 - s)$.

(b) Let α be a path from x_0 to x_1. Define $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}[\gamma] = [\tilde{\alpha} * \gamma * \alpha].$$

(c) $f_*[\gamma] = [f \circ \gamma]$. This makes sense because if H is a path-homotopy from γ_1 to γ_2, then $f \circ H$ is a path-homotopy from $f \circ \gamma_1$ to $f \circ \gamma_2$. f_* is a homomorphism because

$$f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2).$$

(d) Let $f : \mathbb{R} \to S^1$ be the continuous map $f(t) = e^{2\pi it}$. If γ is a path in S^1 starting at 1, there is a unique path $\tilde{\gamma}$ in $\mathbb{R}$ starting at 0 such that $f \circ \tilde{\gamma} = \gamma$. Since γ ends at 1 and $f^{-1}(\{1\}) = \mathbb{Z}$, the endpoint of $\tilde{\gamma}$ is an integer $\nu(\gamma)$. The isomorphism $\pi_1(S^1, 1) \to \mathbb{Z}$ takes $[\gamma]$ to $\nu(\gamma)$.

A suitable path is $\gamma(t) = e^{2\pi it}$.

(e) If $\gamma(t) = e^{2\pi it}$, $f_*[\gamma] = [f \circ \gamma]$ and $(f \circ \gamma)(t) = e^{2\pi int}$, so $\widetilde{(f \circ \gamma)}(t) = nt$ and $\nu(f \circ \gamma) = n$. Hence, the homomorphism $\mathbb{Z} \to \mathbb{Z}$ corresponding to f_* takes 1 to n, and is therefore $m \mapsto mn$.

6. (a) Let $x \in \mathbb{R}$. If U is an open set in $\mathbb{R}$ containing x, then U contains an open interval (a, b) containing x. Let $q \in \mathbb{Q} \cap (a, b)$. Then, $q \in U$. Hence, every open subset of $\mathbb{R}$ containing x contains a point of $\mathbb{Q}$. This proves $x \in \bar{\mathbb{Q}}$. Hence, $\bar{\mathbb{Q}} = \mathbb{R}$.

(b) Let $W = (X \times X) - \Delta_X$. We show that W is open in $X \times X$. If $x \neq x'$ there are disjoint open sets U, V such that $x \in U$, $x' \in V$. Then, $U \times V$ is an open set in $X \times X$ that contains (x, x'), and $U \times V \subset W$ because $U \cap V = \emptyset$.

(c) The map $f \times g : X \to Y \times Y$ given by $(f \times g)(x) = (f(x), g(x))$ is continuous, so $(f \times g)^{-1}(\Delta_Y)$ is closed by (b). But $A \subset (f \times g)^{-1}(\Delta_Y)$, so $\bar{A} \subset (f \times g)^{-1}(\Delta_Y)$. Since A is dense, $(f \times g)^{-1}(\Delta_Y) = X$, i.e. $f = g$.

(d) $f(nx) = nf(x)$ for $n \in \mathbb{N}$ is proved by induction on n. Similarly $f(-nx) = -nf(x)$ for $n \in \mathbb{N}$. And $f(0) = f(0 + 0) = f(0) + f(0)$ so $f(0) = 0$.

If $q = m/n$ with $m, n \in \mathbb{Z}$,

$$nf(qx) = f(nqx) = f(mx) = mf(x),$$

so $f(qx) = qf(x)$.

Let $\lambda = f(1)$ and let $g(x) = \lambda x$. Then, $f(q) = g(q)$ for all $q \in \mathbb{Q}$, so by (a) and (c) $f = g$.

7. (a) If $x \in A'$, every open subset of X that contains x intersects A, so $x \in \bar{A}$. Hence, $A' \subset \bar{A}$ and since $A \subset \bar{A}$ we have $A \cup A' \subset \bar{A}$. Conversely, let $x \in \bar{A} - A$. If U is an open set containing x, then U intersects A (as $x \in \bar{A}$) in a point y, say. Then, $y \neq x$ as $x \notin A$. So $x \in A'$. This proves $\bar{A} \subset A \cup A'$.

A is closed iff $A = \bar{A}$ iff $A = A \cup A'$ iff $A' \subset A$.

(b) The non-empty open sets in $\mathbb{N} \times Y$ are the sets $Z \times Y$, where Z is any non-empty subset of $\mathbb{N}$. Let A be a non-empty subset of $\mathbb{N} \times Y$ and let $(n, y) \in A$ ($n \in \mathbb{N}$, $y \in Y$). Let $y' \in Y$, $y' \neq y$. Then, $(n, y') \in A'$ since any open set $Z \times Y$ containing (n, y') also contains (n, y).

(c) Suppose A is an infinite subset of a compact space X, and suppose for a contradiction that $A' = \emptyset$. By (a), A is closed. If $a \in A$, a is not a limit point of A, so there is an open subset U_a of X such that $U_a \cap A = \{a\}$. Then, $\{U_a\}_{a \in A}$ together with $X - A$ is an infinite open covering of X which has no proper subcovering. This contradicts the fact that X is compact.

The converse is false, for the example in (b) is limit point compact (as every non-empty subset has a limit point), but is not compact as the infinite open covering $\{\{n\} \times Y\}_{n \in \mathbb{Z}}$ has no proper subcovering.

8. (a) A topological group is a group G that is also a topological space such that the product $m : G \times G \to G$ ($m(g, g') = gg'$) and inverse $i : G \to G$ ($i(g) = g^{-1}$) are continuous maps.

(b) If G has the discrete topology so does $G \times G$ (since every point is an open set), so every map $G \to G$ and $G \times G \to G$ is continuous.

(c) l_g is the composite of the homeomorphism $G \to \{g\} \times G$ given by $h \mapsto (g,h)$ followed by m, so l_g is continuous. It is a homeomorphism because $l_g^{-1} = l_{g^{-1}}$.

(d) Let G be connected and let U be an open subgroup of G, $U \neq G$. Let $\{g_iU\}$ be the distinct cosets of U - since $U \neq G$ there are at least two distinct cosets. Since l_{g_i} is a homeomorphism, each coset $g_iU = l_{g_1}(U)$ is open. This contradicts G being connected.

(e) Since H is normal, f is a well-defined map $f : G \to H$. The map f is continuous as it is the composite $m \circ (id \times (l_h \circ i))$. As G is connected and H is discrete, f is constant, i.e. $f(g) = f(e) = h$ for all $g \in G$, i.e. $gh = hg$. This means that h is in the centre of G.

(f) For a counterexample, take G to be any non-abelian group, give G the discrete topology, and take $H = G$.