Topology 6CCM327A / 7CCM327B Solutions to Exam 09/10

A1

[3] i) X is disconnected if there are non-empty open subsets U & V s.t. $U \cap V = \phi$, $U \cup V = X$.

[3] ii) X is compact if whenever $X = \bigcup_{i \in I} U_i$ with $U_i \subset X$ open subsets, then are finitely many indices $i_1, \dots, i_n \in I$ such that $X = U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$.

[3] iii) Let $\{U_i\}_{i \in I}$ be a collection of open sets covering X, i.e., $X = \bigcup_{i \in I} U_i$. Pick an arbitrary non-empty open set in this collection, say U_{i_0}. By def'n of the finite complement top, $X - U_{i_0}$ is finite. Say $X - U_{i_0} = \{x_1, x_2, \dots, x_r\}$. Since $\{U_i\}$ covers X, there are indices $i_1, i_2, \dots, i_r$ s.t. $x_1 \in U_{i_1}$, $x_2 \in U_{i_2}$, … $x_r \in U_{i_r}$. Then,

$$X = U_{i_0} \cup U_{i_1} \cup \dots \cup U_{i_r}.$$

[3] iv) If Y is non-empty, then $\pi_1(X \times Y) = X$, where π_1 is projection on the first component. Since $\pi_1 : X \times Y \to X$ is continuous, the image of a connected subset under π_1 is connected. By assumption $X \times Y$ is connected $\Longrightarrow$ X connected. Y is connected, similarly.

[4] v) Every finite set is compact. We show that if $(X, \text{discrete})$ is compact, then X is finite. Assume not. Then $X = \bigcup_{x \in X} \{x\}$ is an open covering of X which has no finite subcovering.

A1 Continued

$\boxed{3}$ vi) If you remove 0 from $[0,1)$, the remaining space is connected. But there is no point in $\mathbb{R}$ with this property.

$\boxed{2}$ vii) Consider $f:(0,1] \longrightarrow S^1$ with $f(t)=(\cos 2\pi t, \sin 2\pi t)$. $(0,1]$ is not compact but $f((0,1])=S^1$ which is both infinite & compact.

$\boxed{4}$ viii) Let $C\subset X$ be closed. Since X is compact, C is compact. Since f is continuous, $f(C)$ is compact. Since Y is Hausdorff, $f(C)$ is closed.

A2

i) A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X satisfying:

- Every $x\in X$ appears in one subset in $\mathcal{B}$.

$\boxed{4}$

- If $U_1, U_2 \in \mathcal{B} \Rightarrow U_1\cap U_2$ is a union of subsets in $\mathcal{B}$.

In this example $\mathcal{B}$ is a basis: An arbitrary point (a,b) in $\mathbb{R}^2$ belongs to, for example, U_{a+b-1}.

Also $U_{\lambda_1} \cap U_{\lambda_2} = U_{\max\{\lambda_1,\lambda_2\}}$.

$\mathcal{B}$ is not a topology as $\mathbb{R} \notin \mathcal{B}$.

ii) – Hausdorff? No, every two open subsets $\neq\emptyset$ meet!

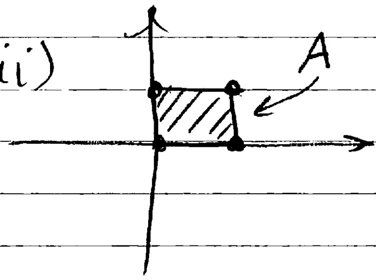
$\boxed{6}$

– Connected? Yes, " " " " " !

– Compact? No, $\{U_{-n}\}_{n\in\mathbb{N}}$ is an open covering which has no finite subcovering.

A2 Continued

– Standard $\supsetneq$ $\mathcal{T}$ as every U_λ is open in the standard top.

iii) 

Interior of $A = \phi$ as no U_λ fits inside A.

5

Closure of $A = \{(x,y) : x+y \leq 1\}$

One way to see this : $\overline{A} = \mathbb{R}^2 - \mathrm{Int}(\mathbb{R}^2 - A)$

$\mathrm{Int}(\mathbb{R}^2 - A) = U_1$ as U_1 is the biggest (union of) U_λ's that fits in $\mathbb{R}^2 - A$.

iv) Let U_λ be an open basis set around $(0,0)$. Then $\lambda < 0$. Therefore

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$$n \geq \frac{-1}{\lambda} \Rightarrow a_n = \left(-\frac{1}{n}, 0\right) \in U_\lambda .$$

same for (-1,1)

2

V) No. $f^{-1}((0,1)) = \{(x,y) \mid 0 < x^2 + y^2 < 1\}$ is not open, as it does not contain any U_λ.

Vi) Consider $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad F((x,y)) = x+y$

Since $(x,y) \sim (x',y') \Rightarrow F((x,y)) = F((x',y'))$,

F induces $f: \mathbb{R}^2/\sim \longrightarrow \mathbb{R}$ $\quad f([(x,y)]) = x+y$

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We show f is continuous, open, and bijective.

To show f is continuous, it is enough to show F is continuous: $F^{-1}((a,\infty)) = U_a$ which is open in $(\mathbb{R}^2, \mathcal{T})$.

f is clearly surjective. f is injective: $f([(x,y)]) = f([(x',y')]) \Rightarrow x+y = x'+y' \Rightarrow [(x,y)] = [(x',y')]$

A3 Continued.

Finally we show f is open. V is open in $\mathbb{R}^2/\sim$ if for $q: \mathbb{R}^2 \to \mathbb{R}^2/\sim$, we have $q^{-1}(V)$ is open in $(\mathbb{R}^2, T)$. This shows that a basis for the quotient top. on $\mathbb{R}^2/\sim$ is $\{q(U_\lambda)\}_{\lambda \in \mathbb{R}}$.

But $f(q(U_\lambda)) = F(U_\lambda) = (\lambda, \infty)$ which is open in $\mathbb{R}$ with the given topology.

B3

4

i) The fundamental group of X based at x_0 is the group of all equivalence classes of loops in X based at x_0 modulo path homotopy, where the group operation is given by

$$[\alpha] * [\beta]\ (t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

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ii) Enough to show $\pi_1(X, (0,0)) = 0$.
Let α be a loop in X based at $(0,0)$
We can write a line homotopy between α and the constant loop based at $(0,0)$.

$F: [0,1] \times [0,1] \longrightarrow X$

$$F(s,t) = (1-t)\,\alpha(s) + t\,(0,0)$$

F Continuous
$F(s,0) = \alpha(s) \quad \forall s$
$F(s,1) = (0,0) \quad \forall s \quad \Rightarrow \checkmark$.
$F(0,t) = (0,0) \quad \forall t$
$F(1,t) = (0,0) \quad \forall t$

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iii) $P: E \to B$ is a covering map if for any $x \in B$ there is an open set U containing x, s.t.
$p^{-1}(U) =$ disjoint union of $V_i : i \in I \neq \phi$.
and s.t. $\forall i \in I$, $P|_{V_i} : V_i \longrightarrow U$ is a homeomorphism

iv) Let $x = (\cos\theta, \sin\theta) \in S^1$. Consider

$$U = \{ (\cos\alpha, \sin\alpha) \mid \theta - \frac{\pi}{2} < \alpha < \theta + \frac{\pi}{2} \}$$

Then:

$$f^{-1}(U) = \{ (\cos x, \sin x) \mid \frac{\theta}{2} - \frac{\pi}{4} < x < \frac{\theta}{2} + \frac{\pi}{4} \} \cup$$

$\boxed{6}$

$$\{ (\cos x, \sin x) \mid \frac{\theta}{2} + \frac{3\pi}{4} < x < \frac{\theta}{2} + \frac{5\pi}{4} \}$$

$$= V_1 \cup V_2$$

First note $V_1 \cap V_2 = \phi$. We need to show $f : V_i \to U$ is a homeomorphism for $i = 1, 2$. We show it for $i = 1$. In fact consider $f : \overline{V_1} \to \overline{U}$. This is a map between compact Hausdorff spaces which is continuous and bijective. Hence it is a homeomorphism. Hence its restriction to V_1 is also a homeomorphism

v) We need

a continuous map $\tilde{\alpha} : [0,1] \longrightarrow S^1$ such that

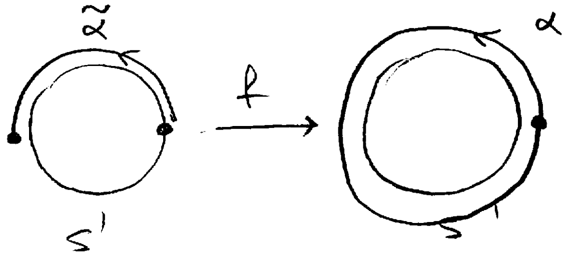
$\boxed{5}$

$$f \circ \tilde{\alpha} = \alpha$$

Define $\tilde{\alpha}$ as follows:

$$\tilde{\alpha}(t) = (\cos(\pi t), \sin(\pi t))$$

and check that $f \circ \tilde{\alpha} = \alpha$.



f wraps S^1 twice around S^1.

B4 i) If $x \notin A$, then $x \in X - A$ which is an open subset of X. Hence, there is $\varepsilon > 0$ such that

$\boxed{4}$ $$B_\varepsilon(x) \subset X - A \quad \text{or} \quad B_\varepsilon(x) \cap A = \phi.$$

Therefore, the distance between x and any point of A is bigger than or equal to ε.

$\Rightarrow d(x, A) \geqslant \delta > 0$.

ii) Assume $d_A(x_0) = d$. To show d_A is Continuous at x_0, we prove that for any $\varepsilon > 0$ $\exists\ \delta > 0$ s.t. $x \in B_\delta(x_0) \Rightarrow d - \varepsilon < d_A(x) < d + \varepsilon$.

We show $\delta = \frac{\varepsilon}{2}$ works.

$\boxed{6}$ $x \in B_\varepsilon(x_0) \Rightarrow d(x, x_0) < \varepsilon \Rightarrow \forall\ a \in A$:

$$d(x_0, a) - d(x_0, x) < d(x, a) < d(x_0, a) + d(x, x_0)$$

$$d(x_0, a) - \varepsilon/2 < d(x, a) < d(x_0, a) + \varepsilon/2$$

Taking $\inf\limits_{a \in A}$ from all sides:

$$d - \varepsilon < d - \varepsilon/2 \leq d(x, A) \leq d + \frac{\varepsilon}{2} < d + \varepsilon$$

iii) Since $d_A : X \to \mathbb{R}$ is continuous, its restriction to K is also continuous, i.e. $d_A : K \to \mathbb{R}$ is continuous. Since K is compact, d_A attains a minimum over K:

$\boxed{4}$ $$\exists\ k \in K \quad \inf\{d_A(x) \mid x \in K\} = d_A(k)$$

in other words $d(K, A) = d_A(k) = d(k, A)$.

iv) Assume not. Then $d(K,A)=0$. By previous part, $\exists\, k\in K$ $d(k,A)=d(K,A)=0$.

$\Rightarrow \inf\{d(k,a) \mid a\in A\} = 0$ $\boxed{4}$

$\Rightarrow \exists$ a sequence $a_n\in A$ s.t. $d(k,a_n) \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \lim_{n\to\infty} a_n = k \Rightarrow k\in \bar{A}=A \Rightarrow k\in A\cap K=\phi$

Which is a contradiction.

v) $X=\mathbb{R}^2$ with Euclidean metric. $\boxed{2}$

$C_1 =$ x-axis

$C_2 =$ graph of $y=\frac{1}{x}$

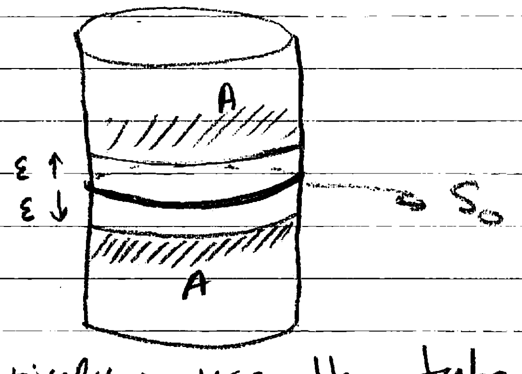
vi) Consider $A=C-U$, $K=S_0$. A is closed and K is compact, and by assumption $A\cap K=\phi$. Hence by part iv):

$$\delta = d(K,A) > 0$$ $\boxed{5}$

$\Rightarrow$ taking $\varepsilon=\delta$ we find that

$$S_h \cap A = \phi \quad \text{for } -\varepsilon < h < \varepsilon$$

or equivalently $S_h \subset U$ for $-\varepsilon < h < \varepsilon$.



- Alternatively: use the tube lemma after proving that subspace top on C is the product top on $S^1\times\mathbb{R}$.