King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

BSC AND MSCI EXAMINATION

6CCM327A (CM327Z) TOPOLOGY

SUMMER 2010

TIME ALLOWED: 2 HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.
SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER.
ANSWER ALL QUESTIONS.

NO CALCULATORS ARE PERMITTED.

TURN OVER WHEN INSTRUCTED

SECTION A

In section A, each question carries 25 points.

In the following, **R**, **Q**, and **N**, denote, respectively, the real numbers, the rational numbers, and the positive integers.

A 1. In the following, X and Y denote topological spaces.

i. Define what it means for X to be *disconnected*.

ii. Define what it means for X to be *compact*.

iii. Show that the finite complement topology on any set is compact.

iv. Assume X, Y are non-empty. Show that if $X \times Y$ is connected, then both X and Y are connected.

v. Let X be a set with discrete topology. Show that X is compact if and only if X is finite.

vi. Prove that $\mathbf{R}$ and $[0, 1)$ are not homeomorphic (both considered with standard topology).

vii. Give an example of a non-compact space whose image under a continuous map is both compact and infinite.

viii. Show that if X is compact and Y is Hausdorff, then any continuous map $f : X \to Y$ is closed (that is, the image of any closed subset of X is a closed subset of Y).

A 2. For a real number λ, define $U_\lambda = \{(x, y) \in \mathbf{R}^2 : \ x + y > \lambda\}$. Consider the following collection of subsets of $\mathbf{R}^2$.

$$\mathcal{B} = \{U_\lambda : \lambda \in \mathbf{R}\}.$$

i. Define what it means for a collection of subsets of a set to be a *basis for a topology* . Show that $\mathcal{B}$ is a basis for a topology on $\mathbf{R}^2$. Is $\mathcal{B}$ a topology?

ii. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$.

- Is $\mathcal{T}$ Hausdorff? why?
- Is $\mathcal{T}$ connected? why?
- Is $\mathcal{T}$ compact? why?
- Compare $\mathcal{T}$ with the standard topology on $\mathbf{R}^2$.

iii. Let A be unit square in $\mathbf{R}^2$ (the square whose vertices are the points $(0, 0), (0, 1), (1, 0), (1, 1)$). Find the closure and interior of A in the topology $\mathcal{T}$. Explain your reasoning.

iv. Consider the sequence $(a_n = (-1/n, 0))_{n\in\mathbf{N}}$. Is $(0, 0)$ a limit of this sequence? Is $(-1, 1)$ a limit of this sequence? Why?

v. Let $f : (\mathbf{R}^2, \mathcal{T}) \to (\mathbf{R}, \text{standard})$ be given by $f((x, y)) = x^2 + y^2$. Is f continuous? why?

vi. Let $\sim$ be an equivalence relation on $(\mathbf{R}^2, \mathcal{T})$ defined as follows:

$$(x, y) \sim (x', y') \ \text{ if } \ x + y = x' + y'.$$

Show that $\mathbf{R}^2/\!\sim$ (considered with the quotient topology induced from $\mathcal{T}$) is homeomorphic to $\mathbf{R}$ equipped with the topology generated by the basis $\{(a, \infty) | a \in \mathbf{R}\}$.

SECTION B

In section B, each question carries 25 points.

B 3. Let X be a topological space.

i. Define $\pi_1(X, x_0)$, the *fundamental group* of X with respect to a base point x_0, and briefly describe the group structure.

ii. Let X be the union of the x-axis and the y-axis in $\mathbf{R}^2$. Show that $\pi_1(X, x) = \{0\}$ for all $x \in X$.

iii. Define what it means for a map $E \to B$ to be a *covering map*.

iv. Show that $f : S^1 \to S^1$ defined by $f((\cos\theta, \sin\theta)) = (\cos 2\theta, \sin 2\theta)$ is a covering map.

v. Find a lifting, under f, of the loop $\alpha : [0, 1] \to S^1$ given by

$$\alpha(t) = (\cos(2\pi t), \sin(2\pi t)).$$

B 4. Let (X, d) be a metric space. Let A be a non-empty closed subset of X. For $x \in X$ define

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

If K is any subset of X, define

$$d(K, A) = \inf\{d(x, A) : x \in K\}.$$

Remark: to solve any part of this problem, you can assume all the parts that come before it.

i. Show that if $x \notin A$, then $d(x, A) > 0$.

ii. Prove that the map $d_A : X \to \mathbf{R}$ defined by $d_A(x) = d(x, A)$ is continuous.

iii. Prove that if K is compact, then there is a point $k \in K$ such that $d(K, A) = d(k, A)$.

iv. Prove that if K is a compact subset of X which is disjoint from A, then $d(K, A) > 0$.

v. Give an example of two non-empty disjoint closed subsets C_1, C_2 in a metric space such that $d(C_1, C_2) = 0$.

vi. Let C denote the cylinder in $\mathbf{R}^3$ defined by

$$C = \{(x, y, z) | x^2 + y^2 = 1\},$$

considered with the subspace topology. For any real number h, let S_h denote the circle $x^2 + y^2 = 1$ on the plane $z = h$ (i.e., the intersection of the cylinder C with the plane $z = h$). Let U be an open subset of C which contains S_0. Prove that there is $\epsilon > 0$ such that U contains S_h for $-\epsilon < h < \epsilon$.