TG.1 Suppose that G is a topological group, and let $m : G \times G \to G$ and $i : G \to G$ be the multiplication and inversion maps. Then, $i \times id : G \times G \to G \times G$ is continuous (this is the map $(g_1, g_2) \mapsto (g_1^{-1}, g_2)$), and hence so is $\nu = m \circ (i \times id)$.

Conversely, assume that ν is continuous. Then, i is continuous as it is the composite of the homeomorphism $G \to G \times \{e\}$ given by $g \mapsto (g, e)$ (where e is the identity eleemnt of G) followed by the restriction of ν to $G \times \{e\}$. Then, m is continuous as it is the composite $\cancel{m}\,\nu \circ (i \times id)$.

TG.2 We recall that, if A, B are subsets of a topological space X and $f : X \to Y$ is a continuous map, then $f(\bar{A}) \subset \overline{f(A)}$ and $\bar{A} \times \bar{B} = \overline{A \times B}$.

Let H be a subgroup of a topological group G. We want to show that $m(\bar{H} \times \bar{H}) \subset \bar{H}$ and that $i(\bar{H}) \subset \bar{H}$. But

$$i(\bar{H}) \subset \overline{i(H)} \subset \bar{H}$$

because $i(H) \subset H$, and similarly

$$m(\bar{H} \times \bar{H}) = m(\overline{H \times H}) \subset \overline{m(H \times H)} \subset \bar{H}.$$

TG.3 To show that a topological space X has the discrete topology, it is enough to show that every one-point subset is open, since every subset of X is a union of one-point subsets.

Let H be an open subgroup of a group G. If $\{gH\}$ is any point of G/H, it is open if and only if $p^{-1}(\{gH\})$ is open in G, since G/H has the quotient topology ($p : G \to G/H$ is the map that takes each element $g \in G$ to its coset gH). But $p^{-1}(\{gH\}) = l_g(H)$ $(= gH)$ and this is open because H is open and the map $l_g : G \to G$ given by $g' \mapsto gg'$ is a homeomorphism. So G/H has the discrete topology.

Conversely, of G/H has the discrete topology, the point $\{H\} \in G/H$ is an open subset of G/H, so $p^{-1}\{H\}) = H$ is an open subset of G because p is continuous.

TG.4 Assuming the result stated in the Hint, Let $\{U_i\}_{i \in I}$ be an open covering of X. For each $y \in Y$, we can find a finite number of the sets U_i that cover $p^{-1}(\{y\})$, since $p^{-1}(\{y\})$ is compact, say $\{U_i\}_{i \in I_y}$ where I_y is a finite subset of the index set I. Then, $U_y = \bigcup_{i \in I_y} U_i$ is an open set containing $p^{-1}(\{y\})$, so by the Hint there is an open set W_y of Y such that $p^{-1}(W_y) \subset U_y$. Then, $\{W_y\}_{y \in Y}$ is an open covering of Y, so since Y is compact there is a finite subcovering, say $W_{y_1}, \ldots, W_{y_n}$. Let $I' = \bigcup_{k=1}^{n} I_{y_k}$. Then, I' is a finite subset of I and the sets U_i for $i \in I'$ cover X.

To prove the result in the Hint, take $W = Y - p(X - U)$. Since p is a closed map, W is open. Next, $y \in W$ because $y \notin W$ implies $y \in p(X - U)$ so $y = p(x)$ where $x \in X - U$ so $x \in p^{-1}(\{y\}) \cap (X - U)$, which is impossible as $p^{-1}(\{y\}) \subset U$. Finally, $p^{-1}(W) \subset U$ because if $x \in p^{-1}(W)$ then $p(x) \notin p(X - U)$ so $x \notin X - U$, i.e. $x \in U$.

TG.5 (a) First we show that if K is a compact subset of G and U is an open subset of G, containing K there is an open set V of G containing the identity element e such that $VK \subset U$ (VK means $m(V \times K)$). If $x \in K$, then Ux^{-1} is an open set containing e, so by the lemma proved in lectures there is an open set V_x containing e such that $V_x^2 \subset Ux^{-1}$. The open sets $V_x x$ for $x \in K$ cover K, so since K is compact there is a finite subcovering, say $V_{x_1}x_1, \ldots, V_{x_n}x_n$. Let $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then V is an open set containing e and $VK \subset U$, for if $x \in K$ then $x \in V_{x_k}x_k$ for some $k = 1, \ldots, n$, so $Vx \subset VV_{x_k}x_k \subset V_{x_k}^2 x_k \subset U$.

Now let A be closed and B compact. It is enough to show that, if $x \notin AB$, there is an open set W containing x that does not intersect AB. Now, $xB^{-1} \cap A = \emptyset$, so xB^{-1} is contained in the open set $G - A$. By the preceding paragraph, there is an open set V containing e such that $VxB^{-1} \subset G - A$, which means that $Vx \cap AB = \emptyset$. Thus, Vx is an open set containing x that does not intersect AB.

(b) Let A be a closed subset of G. Since G/H has the quotient topology, $p(A)$ is closed if and only if $p^{-1}(p(A))$ is closed. But $p^{-1}(p(A)) = AH$, and this is closed by part (a) and the fact that H is compact.

(c) The map $p : G \to G/H$ is closed (by part (b)), continuous and surjective, and if $gH \in G/H$, $p^{-1}(\{gH\}) = gH$ is compact (because H is compact and $l_g : G \to G$ is a homeomorphism). By 26.12, G is compact.

TG.6 Recall that the component C of e is the largest connected subset of G containing e (i.e. the union of all the connected subsets of G that contain e). Since l_x and $r_{x^{-1}}$ are homeomorphisms (where $r_g(g') = g'g$), $xCx^{-1} = l_x r_{x^{-1}}(C)$ is a connected subset of G and it obviously contains e ($e = xex^{-1}$). So it must be contained in C as C is the largest connected subset of G containing e. This means that C is a normal subgroup.

TG.7 We first show that, if X is a connected topological space and if Y has the discrete topology, every continuous map $f : X \to Y$ is a constant map. In fact, let $x \in X$ and let $U = f^{-1}(\{f(x)\})$. Since Y has the discrete topology, the singleton $\{f(x)\}$ is an open and closed subset of Y. Since f is continuous, U is an open and closed subset of X. Since X is connected, its only open and closed subsets are X and $\emptyset$. Since $x \in U$, we cannot have $U = \emptyset$, so $U = X$. This means that $f(x') = f(x)$ for all $x' \in X$.

Now let H be a discrete normal subgroup of G. Let $h \in H$ and consider the map $f : G \to H$ defined by $f(g) = ghg^{-1}$. Note that $ghg^{-1} \in H$ because H is normal. Because multiplication and inversion are continuous maps, f is continuous. It follows from the preceding paragraph that f is constant. But $f(e) = h$, so for all $g \in G$ we have $f(g) = h$, i.e. $ghg^{-1} = h$, i.e. $gh = hg$.

TG.8 In the obvious notation, $i_{G\times H} = i_G \times i_H$, so $i_{G\times H}$ is continuous if i_G and i_H are continuous. As for the multiplication, let $\sigma : H \times G \to G \times H$ be the map $\sigma(h, g) = (g, h)$. Then, σ is continuous, for if $U \times V$ is a basic open set in $G \times H$ (where U is open in G and V is open in H), then $\sigma^{-1}(U \times V) = V \times U$ is a (basic) open set in $H \times G$. But $m_{G\times H} = (m_G \times m_H) \circ (id_G \times \sigma \times id_H)$, so $m_{G\times H}$ is continuous if m_G and m_H are continuous.