

## Topology 2010/11

1. Let  $q : X \rightarrow Y$  be a map between sets. Recall that for  $S \subseteq Y$ , we define  $q^{-1}(S) = \{x \in X : q(x) \in S\}$ .
  - (a) Let  $I$  be a set and  $\{S_i\}_{i \in I}$  a collection of subsets of  $Y$  indexed by  $I$ . Show
    - (i)  $q^{-1}(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} q^{-1}(S_i)$ .
    - (ii)  $q^{-1}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} q^{-1}(S_i)$ .
    - (iii)  $q^{-1}(S^c) = (q^{-1}(S))^c$ .
  - (b) Assume  $q : X \rightarrow Y$  is onto. Define a relation on  $X$  given by  $x_1 \sim_q x_2$  iff  $q(x_1) = q(x_2)$ .
    - (i) Show that  $\sim_q$  is an equivalence relation on  $X$ .
    - (ii) Show that every equivalence class with respect to  $\sim_q$  is of the form  $q^{-1}(\{y\})$  for some element  $y \in Y$ .
    - (iii) Show that there is a one-to-one correspondence between  $Y$  and the set of equivalence classes of  $\sim_q$ .
    - (iv) Prove that for every equivalence relation  $\sim$  on  $X$ , one can find a  $Y$  and an onto map  $q : X \rightarrow Y$  such that  $\sim = \sim_q$ .
2. Let  $A_1, A_2$ , and  $A_3$  be sets. Let  $\{X_i\}_{i \in I}$  be a collection of sets.
  - (a) Show that there is a one-to-one correspondence between  $A_1 \times A_2 \times A_3$  and  $(A_1 \times A_2) \times A_3$ .
  - (b) Let  $\pi_1 : A_1 \times A_2 \rightarrow A_1$  be given by  $\pi_1(a_1, a_2) = a_1$ . This map is called the projection onto the first factor. Similarly, we can define  $\pi_2$ , the projection onto the second factor. Let  $V_1 \subseteq A_1$ , and  $V_2 \subseteq A_2$ . Show that
 
$$\pi_1^{-1}(V_1) = V_1 \times A_2, \quad \pi_1^{-1}(V_1) \cap \pi_2^{-1}(V_2) = V_1 \times V_2.$$

In particular, in both cases, the set in question can be written as a product of subsets of  $A_1$  and  $A_2$ .
  - (c) In general, show that for all  $j \in I$  there is a map  $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$  which is the “projection onto the  $j$ -th factor”.
  - (d) Let  $i_1, i_2, \dots, i_n$  be indices in  $I$ , and assume that for each  $i \in I$  we are given  $U_i \subseteq X_i$ . Show that  $\bigcap_{1 \leq r \leq n} \pi_{i_r}^{-1}(U_{i_r})$  can be described as a product of subsets of  $X_i$ 's.
  - (e) Let  $S$  be the set of all sequences  $\{a_n\}_n$  of real numbers satisfying:  $|a_1| < 1$ ,  $|a_3| < 2$ ,  $|a_4| < 1$ . Express  $S$  as a Cartesian product of subsets of the real numbers.
  - (f) Let  $S'$  be the set of all sequences  $\{a_n\}_n$  of real numbers satisfying:  $a_1 = a_3$ . Can we express  $S'$  as a Cartesian product of subsets of the real numbers?
3. Let  $X = \{a, b, c\}$  be a 3-element set. Write down all possible topologies on  $X$ .
4. Let  $X$  be a topological space. Let  $A$  be a subset of  $X$  such that for each  $x \in A$  there is an open set  $U$  containing  $x$  such that  $U \subseteq A$ . Show that  $A$  is open in  $X$ .
5. Let  $\{\mathcal{T}_i\}_{i \in I}$  be a collection of topologies on a set  $X$ . Show that  $\bigcap_{i \in I} \mathcal{T}_i$  is a topology on  $X$ .
6. Let  $X$  be an infinite set. Which of the following collections of subsets of  $X$  defines a topology on  $X$ ?
 
$$\mathcal{T}_c = \{U \in \mathcal{P}(X) \mid X - U \text{ is at most countable or all of } X\}$$

$$\mathcal{T}_\infty = \{U \in \mathcal{P}(X) \mid X - U \text{ is infinite or empty or all of } X\}$$

7. Let  $(X, d)$  be a metric space. From definitions, show that

$$\mathcal{B} = \{B_\epsilon(x) : \epsilon < 1, x \in X\}$$

forms a basis for a topology on  $X$ .

8. (a) Compare the following topologies on  $\mathbb{R}$ , whenever possible. (i.e. decide which ones are finer and which ones are coarser)

$\mathcal{T}_1$  = the standard topology.

$\mathcal{T}_2$  = the finite complement topology.

$\mathcal{T}_3$  = the topology generated by  $\mathcal{B}_1$  (the upper-limit topology on  $\mathbb{R}$ ).

$\mathcal{T}_4$  = the topology generated by  $\mathcal{B}_2$ .

(b) Find the closures of  $K = \{1/n | n \in \mathbb{N}\}$  and  $A = (0, 1)$  in each of the above topologies.

(c) Decide which one of these topologies satisfy the Hausdorff axiom.

9. (i) Show that the countable collection  $\{(a, b) | a < b, a, b \in \mathbb{Q}\}$  is a basis that generates the usual topology of  $\mathbb{R}$ .

(ii) Show that the collection  $\{[a, b) | a < b, a, b \in \mathbb{Q}\}$  is a basis that generates a topology different from the lower-limit topology on  $\mathbb{R}$ .

10. Criticize the following “proof” that  $\overline{\cup A_\alpha} \subseteq \cup \overline{A_\alpha}$ : if  $\{A_\alpha\}$  is a collection of subsets in  $X$  and if  $x \in \overline{\cup A_\alpha}$ , then every open neighborhood  $U$  of  $x$  intersects  $\cup A_\alpha$ . Thus  $U$  must intersect some  $A_\alpha$ , so that  $x$  must belong to the closure of some  $A_\alpha$ . Therefore  $x \in \cup \overline{A_\alpha}$ .

11. Show that the composition of two continuous maps is continuous: that is, if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are both continuous, then  $g \circ f : X \rightarrow Z$  is also continuous. (Hint: first show that  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  for any subset  $V$  of  $Z$ ).

12. Let  $X, Y$  be topological spaces. A map  $f : X \rightarrow Y$  is locally constant, if for every  $x_0 \in X$  there is an open neighborhood  $U_{x_0}$  of  $x_0$  such that  $f(x) = f(x_0)$  for all  $x \in U_{x_0}$  (which means  $f$  is constant on  $U_{x_0}$ ). Assume that  $Y$  has the discrete topology. Prove that  $f$  is continuous if and only if  $f$  is locally constant.

13. (The pasting lemma) Let  $\{A_i\}_{i \in I}$  be a collection of subsets of a topological space  $X$  such that  $X = \cup_i A_i$ . Let  $f : X \rightarrow Y$  be a map of topological spaces such that the restriction of  $f$  to each  $A_i$  (viewed with the subspace topology) is continuous.

(i) Show that if each  $A_i$  is an open subset of  $X$ , then  $f$  is continuous.

(ii) Show that if each  $A_i$  is closed, and, furthermore, the collection  $\{A_i\}$  is finite then  $f$  is continuous. (Hint: use the following characterization of continuity.  $F : X \rightarrow Y$  is continuous if for every closed subset  $D$  of  $Y$ , we have that  $f^{-1}(D)$  is closed.

(iii) Find an example where the collection  $\{A_i\}$  is countable and each  $A_i$  is closed, but  $f$  is not continuous.

14. Let  $(X, \mathcal{T})$  be a topological space. Let  $Y$  be a subset of  $X$  (equipped with the subspace topology,  $\mathcal{T}_Y$ ).

(i) Consider  $\mathbb{Z}$  inside  $(\mathbb{R}, \text{standard topology})$ . Show that the subspace topology on  $\mathbb{Z}$  is the discrete topology.

- (ii) Let  $X$  denote  $\mathbb{R}^2$  with standard topology. Think of  $\mathbb{R}$  as a subset  $Y$  of  $X$  (identify  $\mathbb{R}$  with the “ $x$ -axis” in  $\mathbb{R}^2$ ). Show that the subspace topology on  $Y$  is the same as the standard topology of  $\mathbb{R}$ .
  - (iii) Let  $S^1$  denote the unit circle in  $\mathbb{R}^2$ . Describe the subspace topology of  $S^1$ .
  - (iv) Show that the map  $f : S^1 \rightarrow [0, 1)$  which sends  $z = \cos \theta + i \sin \theta$  ( $0 \leq \theta < 2\pi$ ) to  $f(z) = \frac{\theta}{2\pi}$  is not continuous. (Here  $S^1$  is viewed with its subspace topology inside  $\mathbb{R}^2$ .)
  - (v) Show that a subset  $C \subseteq Y$  is closed in  $Y$  if and only if  $C = D \cap Y$  where  $D$  is closed in  $X$ .
  - (vi) Let  $S = \{1/n : n \in \mathbb{N}\}$ . Show that  $S$  is not closed in  $\mathbb{R}$  (with standard topology) but it is closed in  $(0, 1]$  with its subspace topology induced from  $\mathbb{R}$ . (Use the previous part).
  - (vii) Let  $A \subset Y$ . Show that the subspace topology on  $A$  induced from  $Y$  is the same as the subspace topology induced from  $X$ .
  - (viii) Show that the inclusion map  $\iota : Y \rightarrow X$  is continuous.
  - (ix) Let  $\mathcal{S}$  be a topology on  $Y$  such that the inclusion map  $\iota : (Y, \mathcal{S}) \rightarrow (X, \mathcal{T})$  is continuous. Show that  $\mathcal{S}$  is finer than  $\mathcal{T}_Y$ . In other words, the subspace topology is the smallest topology on  $Y$  which makes the inclusion map  $\iota : Y \rightarrow X$  continuous.
  - (x) Let  $f : X \rightarrow Z$  be a continuous map between topological spaces. Show that the restriction of  $f$  to  $Y$ , denoted  $f|_Y : Y \rightarrow Z$ , is continuous.
  - (xi) Let  $\mathcal{T}'$  be another topology on  $X$ , and  $\mathcal{T}'_Y$  the subspace topology on  $Y$  induced by  $\mathcal{T}'$ . Show that  $\mathcal{T}'_Y$  is finer than  $\mathcal{T}_Y$  if  $\mathcal{T}'$  is finer than  $\mathcal{T}$ .
15. Show that a product of Hausdorff spaces is Hausdorff.
16. Recall that  $\mathbb{R}^{\mathbb{N}}$  is the set of all sequences of real numbers. It can be thought of as  $\prod_{n \in \mathbb{N}} X_n$  where  $X_n = \mathbb{R}$  for all  $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ .  
Let  $A$  be the subset of  $\mathbb{R}^{\mathbb{N}}$  consisting of all sequences that are “eventually zero”, that is, all  $(x_1, x_2, \dots)$  such that  $x_i = 0$  for all but finitely many values of  $i$ . Find the closure of  $A$  in both the product and box topologies on  $\mathbb{R}^{\mathbb{N}}$ .
17. Let  $I = [0, 1]$  and recall that  $\mathbb{R}^I = \{f : I \rightarrow \mathbb{R}\} = \prod_{x \in I} \mathbb{R}$ . A sequence of functions  $\{f_n\}$  in  $\mathbb{R}^I$  is said to converge to  $f$  point-wise if for each  $x \in I$  the sequence  $\{f_n(x)\}$  converges to  $f(x)$  in  $\mathbb{R}$ . Show that  $\{f_n\}$  converges to  $f$  in the product topology if and only if it converges to  $f$  point-wise.
18. Let  $\{X_\alpha\}$  be a collection of topological spaces. For each  $\alpha$ , let  $A_\alpha \subset X_\alpha$ . Show that  $\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}$ .
19. Let  $X$  be a topological space. Show that
- (i) If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ .
  - (ii) If  $U$  is open in  $X$  and  $A$  is closed in  $X$  then  $U - A$  is open in  $X$  and  $A - U$  is closed in  $X$ .
20. Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Let  $X/\sim$  denote the set of equivalence classes with its quotient topology.
- (i) Assume the quotient map  $q : X \rightarrow X/\sim$  is open. Show that  $X/\sim$  is Hausdorff if and only if  $\{(x, y) \in X \times X | x \sim y\}$  is closed in  $X \times X$ .
  - (ii) Show that  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{(x, x) : x \in X\}$  is closed in  $X \times X$ .

- (iii) Give an example of a Hausdorff  $X$  and an equivalence relation  $\sim$  on  $X$  such that  $X/\sim$  is not Hausdorff.
21. Let  $X = \mathbb{R}^2$  with its standard topology. Define an equivalence relation on  $X$  by  $(x_1, y_1) \sim (x_2, y_2)$  if  $x_1^2 + y_1^2 = x_2^2 + y_2^2$ . Let  $q : X \rightarrow X/\sim$  be the quotient map.
- (i) Show that  $F : X \rightarrow \mathbb{R}$  defined by  $F(x, y) = x^2 + y^2$  induces a continuous map  $f : X/\sim \rightarrow \mathbb{R}$  such that  $F = f \circ q$ .
- (ii) Show that  $f : X/\sim \rightarrow \mathbb{R}$  is a topological embedding and provides us with a homeomorphism between  $X/\sim$  and  $[0, \infty)$ .
22. Prove that the map  $f : S^1 \rightarrow \mathbb{RP}(1)$  defined by  $f(x, y) = [x : y]$  is a homeomorphism.
23. Use induction to prove that  $\mathbb{RP}(n)$  can be written as a disjoint union of subsets

$$X_0 \cup X_1 \cup \cdots \cup X_{n-1}$$

such that each  $X_i$  is homeomorphic to  $\mathbb{R}^i$ . A similar statement holds true for  $\mathbb{CP}(n)$ .

24. Let  $X$  and  $Y$  be non-empty topological spaces.
- (a) Show that if  $X \times Y$  is connected, then both  $X$  and  $Y$  are connected. (Hint: Note that  $\pi_1(X \times Y) = X$ ).
- (b) Now assume both  $X$  and  $Y$  are connected. Follow the steps below, to prove that  $X \times Y$  is also connected.
- (i) Let  $x_0 \in X$ . Show that  $\{x_0\} \times Y = \{(x_0, y) : y \in Y\}$  is homeomorphic to  $Y$ . This is called a *vertical slice*. (Draw pictures).
- (ii) Similarly, it follows that for  $y_0 \in Y$ , the *horizontal slice*  $X \times \{y_0\}$  is homeomorphic to  $X$ .
- (iii) Choose a fixed  $y_0 \in Y$ . For any  $x \in X$ , let  $C_x = (\{x\} \times Y) \cup (X \times \{y_0\})$ . Show that  $C_x$  is connected.
- (iv) Show that  $X \times Y = \cup_{x \in X} C_x$ . Note that  $\cap_{x \in X} C_x \neq \emptyset$ .
- (v) Prove that  $X \times Y$  is connected.
25. Let  $X$  be a topological space and  $\{A_\alpha\}_\alpha$  a collection of connected subsets of  $X$ . Let  $A$  be a connected subset of  $X$  such that  $A \cap A_\alpha \neq \emptyset$  for all  $\alpha$ . Show that  $A \cup (\cup_\alpha A_\alpha)$  is connected. (Hint: recall that a union of connected subsets which all have a point in common is connected).
- (i) Show that no two of  $(0, 1)$ ,  $(0, 1]$ , and  $[0, 1]$  are homeomorphic (Hint: use the following general fact: if  $f : X \rightarrow Y$  is a homeomorphism, then for any  $x \in X$ , the restriction  $f : X - \{x\} \rightarrow Y - \{f(x)\}$  is also a homeomorphism.)
- (ii) Show that  $\mathbb{R}^n - \{\text{point}\}$  is connected if  $n > 1$ . Conclude that  $\mathbb{R}$  is not homeomorphic to  $\mathbb{R}^n$  if  $n > 1$ .
26. Give examples of a topological space  $X$  and a subset  $A$  satisfying:
- (i)  $A$  is connected but  $\text{Int}A$  is not connected.
- (ii)  $A$  is connected but  $\text{Bd}A$  is not connected.
- (iii)  $\text{int}A$  and  $\text{Bd}A$  are connected but  $A$  is not connected.
27. Let  $X$  be a topological space, and  $f : X \rightarrow \mathbb{R}$  a locally constant map (i.e., for every point  $x \in X$  there is an open neighborhood  $U$  of  $x$  such that  $f(u) = f(x)$  for all  $u \in U$ ).
- (i) Show that  $f$  is continuous.
- (ii) Show that for every  $r \in \mathbb{R}$ , we have  $f^{-1}(\{r\})$  is open in  $X$ .

- (iii) Show that for every  $r \in \mathbb{R}$ , we have  $f^{-1}(\{r\})$  is closed in  $X$ .
- (iv) Show that if  $X$  is connected, then  $f$  must be constant (by choosing an appropriate  $r$  and using the above parts).
28. Recall that  $S^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$ .
- (i) Let  $f : S^1 \rightarrow \mathbb{R}$  be a continuous map. Show that there is  $x \in S^1$  such that  $f(x) = f(-x)$ . (Hint: first consider  $g : S^1 \rightarrow \mathbb{R}$  defined by  $g(x) = f(x) - f(-x)$  and apply the “intermediate value theorem”).
- (ii) Prove the same statement for  $S^2$ . This implies that, for example, at every given point in time there are two antipodal locations on earth which have the same temperature!
29. Prove that every open connected subset of  $\mathbb{R}^n$  is path connected, via the following steps:
- (i) Let  $X$  be a topological space and  $f : [a, b] \rightarrow X$  and  $g : [c, d] \rightarrow X$  be two continuous maps (or *paths* as we have been calling them). Show that if the endpoint of  $f$  coincides with the beginning point of  $g$  (i.e., if  $f(b) = g(c)$ ), then  $f$  and  $g$  can be used to define a path joining the beginning point of  $f$  (i.e.,  $f(a)$ ) and the endpoint of  $g$  (i.e.  $g(d)$ ). (Hint: first re-scale  $[c, d]$  to modify the domain of the second path to an interval of the form  $[b, e]$ ).
- (ii) Let  $U$  be a non-empty open connected subset of  $\mathbb{R}^n$  and fix a point  $a$  in  $U$ . Show that  $A = \{x \in U : x \text{ can be joined to } a \text{ by a path in } U\}$  is open in  $U$ .
- (iii) Show that  $A$  is closed in  $U$ . Show that together with part (ii) this implies that  $A = U$ . (Use connectedness of  $X$ ).
- (iv) Prove the result.
30. Let  $X$  be a Hausdorff space. Let  $K_1$  and  $K_2$  be two compact subsets of  $X$  that are disjoint. Show there are disjoint open sets  $U_1$  and  $U_2$  in  $X$  such that  $K_1 \subset U_1$ ,  $K_2 \subset U_2$ .
31. A map between topological spaces  $f : X \rightarrow Y$  is called closed if the image of every closed subset of  $X$  under  $f$  is a closed subset of  $Y$ . Assume  $X$  is compact and  $Y$  is Hausdorff. Show that every continuous map  $f : X \rightarrow Y$  is closed.
- Using the above, conclude that any bijective continuous map  $f : X \rightarrow Y$  is automatically a homeomorphism.
32. Let  $X$  be a compact topological space. Let  $\{C_n\}_{n \in \mathbb{N}}$  be a decreasing family of non-empty closed subsets of  $X$ . In other words,  $C_n \neq \emptyset$  is closed and  $C_{n+1} \subseteq C_n$  for all  $n \geq 1$ . Show that  $\bigcap_n C_n \neq \emptyset$ . (Hint: look at the complements of these closed sets).
33. Let  $X$  be a compact topological space. Show that if  $A$  is an infinite subset of  $X$  then  $A$  will have a limit point. (Hint: Suppose this is not true and conclude  $A$  must be closed, and hence compact. Then, find an open covering of  $A$  which has no finite sub-covering).
34. A subset  $A$  of  $\mathbb{R}^n$  is called star convex if there is a point  $a_0 \in A$  such that all the line segments joining  $a_0$  to other points of  $A$  lie in  $A$ .
- (i) Give an example of a star convex subset of  $\mathbb{R}^n$  which is not convex.
- (ii) Prove that any star convex set is simply connected.
- (iii) Prove that if  $A$  is star convex, then any two paths in  $A$  having the same initial and final points are path homotopic.

35. Let  $P : E \rightarrow B$  be a covering map. Let  $\alpha, \beta$  be paths in  $B$  such that  $\alpha(1) = \beta(0)$ . Let  $\tilde{\alpha}, \tilde{\beta}$  be lifts of  $\alpha$  and  $\beta$  such that  $\tilde{\alpha}(1) = \tilde{\beta}(0)$ . Show that the path  $\tilde{\alpha} * \tilde{\beta}$  is a lift of  $\alpha * \beta$ .
36. Let  $p : E \rightarrow B$  be a covering map. Show that  $p$  is an open map. Note that, by definition, it is also continuous and surjective.
37. This exercise shows that every simply connected space has a unique covering space up to homeomorphism: itself.
- (i) Let  $p : E \rightarrow B$  be a covering map,  $k$  a positive integer, and assume that  $B$  is connected. Assume there is a point  $b_0$  in  $B$  such that  $p^{-1}(b_0)$  has  $k$  points. Prove that  $p^{-1}(b)$  has  $k$  points for all  $b \in B$ . This common value,  $k$ , is called the degree of  $p$  and  $E$  is said to be a  $k$ -fold covering of  $B$ . (Hint: define  $U = \{b : p^{-1}(b) \text{ has size } k\}$  and  $V = \{b : p^{-1}(b) \text{ has size different from } k\}$ . Prove that  $B = U \cup V$  is a separation of  $B$ , and from there prove the statement).
- (ii) Show that if  $B$  has a  $k$ -fold covering  $E$  which is path connected then  $\pi_1(B, b_0)$  has at least  $k$  elements.
- (iii) Deduce that if  $B$  is simply connected, then every covering map  $p : E \rightarrow B$  in which  $E$  is path connected is indeed a homeomorphism.
38. Let  $\alpha$  and  $\beta$  be paths in  $S^1$  defined as follows:  $\alpha(s) = (\cos \pi s, \sin \pi s)$  and  $\beta(s) = (\cos(\pi s + \pi), \sin(\pi s + \pi))$ . Prove that  $\alpha$  and  $\beta$  are not path homotopic.

51.1

51.2

51.3

52.1

52.2

52.3

52.4

52.5

55.1

55.3

TG.1 Let  $G$  be a group and a topological space. Prove that  $G$  is a topological group if and only if the map  $\nu : G \times G \rightarrow G$  given by  $\nu(g_1g_2) = g_1^{-1}g_2$  is continuous.

TG.2 Show that the closure of a subgroup of a topological group is a subgroup of  $G$ .

TG.3 Let  $G$  be a topological group and  $H$  a subgroup. Prove that  $H$  is an open subgroup of  $G$  if and only if the quotient topology on  $G/H$  is the discrete topology.

TG.4 Problem 26.12 in the book.

TG 5 Problem 26.13 in the book.

TG.6 Problem 25.9 in the book.

TG.7 Let  $H$  be a normal subgroup of a connected topological group  $G$ , and suppose that the subspace topology on  $H$  is the discrete topology. Show that  $H$  is contained in the centre of  $G$ . (The centre of  $G$  is the subgroup  $Z(G) = \{g \in G \mid gg' = g'g \text{ for all } g' \in G\}$ .)

TG.8. Let  $G$  and  $H$  be topological groups. Prove that  $G \times H$ , with the direct product group structure  $((g, h).(g', h') = (gg', hh'))$  and the product topology, is a topological group.