1 Sets and functions

> Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.
>
> J. W. Goethe

This preliminary chapter sets up some notation and makes sure we have a common language in which to talk mathematically.

1.1 Set Theory

Every object that we encounter in mathematics belongs to a set. It is important therefore to know the language of sets and to know how sets can be manipulated.

1.1.1 Sets

We shall take a *set* to be 'a collection of objects'. The objects will be referred to as *elements.*

We specify a set in one of two ways.

1. By listing its elements. For instance $S = \{1, 2, 3, 4, 5\}$, or $A = \{-2, 0, 3, \pi\}$, or $X = \{\text{Alice}, \text{Bob}, \text{Carol}\}$.

2. By stating a property that determines membership, e.g.

$$T = \{x \text{ integer} : x > 0\} \quad \text{or} \quad T = \{x \text{ integer} \mid x > 0\}. \tag{1.1}$$

   This is to be read as "T is the set of all integers x such that x is greater than zero". The colon ":" or the sign "|" here simply separate different parts of the definition from each other.

We use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the natural numbers, integers, rational numbers, real numbers and complex numbers respectively:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$
$$\mathbb{Q} = \{a/b : a \in \mathbb{Z}, b \in \mathbb{N}\}$$

We will look at $\mathbb{R}$ more carefully soon. We can think of it intuitively as the set of all points on a straight line extending indefinitely in both directions.

In some books you will see $\mathbb{N}$ defined to include 0 too. It does not matter how you define it, provided that once you do define it you stick to your definition.

If x is an element of S we write $x \in S$. If it is not we write $x \notin S$. $x \in S$ may be written as $S \ni x$. With this notation, we can rewrite the definition (1.1) in a more standard way as

$$T = \{x \in \mathbb{Z} : x > 0\}.$$

We say that S is a *subset* of T and write $S \subset T$ if every element of S is an element of T. $S \subset T$ can equally well be written as $T \supset S$. The *empty set*, denoted $\varnothing$, is the set containing no elements. By convention, it is a subset of every set.

Example. Let us list all possible subsets of the set $\{1, 2, 3\}$. These are:

$$\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

Notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

To say that the sets A and B are equal is to say that they have the same elements. In other words, to say that $A = B$ is to say both that if $x \in A$ then also $x \in B$ and if $y \in B$ then also $y \in A$. We can write this as

$$A = B \text{ is the same as } \begin{cases} x \in A \Rightarrow x \in B \\ y \in B \Rightarrow y \in A. \end{cases}$$

Alternatively, using the notation $\subset$, this can be written as

$$A = B \text{ if and only if } (A \subset B \text{ and } B \subset A).$$

1.1.2 Finite and infinite sets

The difference between finite and infinite sets should be intuitively clear. For example, the sets

$$\{1, 2, 3, 4, 5\}, \quad \{-2, 0, 3, \pi\}, \quad \{\text{Alice}, \text{Bob}, \text{Carol}\},$$

are all finite, whereas the sets

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}$$

are all infinite. For any finite set X we can determine the number of elements of X, simply by counting its elements. The number of elements of a set X is called the *cardinality* of X and is denoted by $\#X$. For example,

$$\#\{-2, 0, 3, \pi\} = 4.$$

It should be clear that if $A \subset B$, then $(\#A) \leq (\#B)$. One can often see different notation for cardinality of A, e.g. $|A|$ or $\operatorname{card} A$.

Later on we will discuss a rigourous mathematical approach to the definition of finiteness and cardinality of sets.

1.1.3 Operations with sets

We can form new sets from the given ones using the operations union, intersection and complement. The *union* of two sets A and B, denoted $A \cup B$, is the set of all elements which are either in A or in B (or both). Their *intersection*, $A \cap B$, is the set of those elements that are both in A and in B. Notation $A \backslash B$ stands for the *complement of B in A*, i.e. the set of all elements of A which are not in B.

Example Within the natural numbers $\mathbb{N}$ suppose $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 5\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5\}\,, \quad A \cap B = \{1, 2\}\,, \quad A \backslash B = \{3, 4\}.$$

It is clear that $A \subset A \cup B$ and $B \subset A \cup B$ and also $A \cap B \subset A$ and $A \cap B \subset B$.

1.1.4 Intervals

The following subsets of $\mathbb{R}$ are called *intervals.*

(i)	$[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$	(iv)	$(-\infty, b] = \{x \in \mathbb{R} : x \le b\}$
(ii)	$(a, b) = \{x \in \mathbb{R} : a < x < b\}$	(v)	$(a, \infty) = \{x \in \mathbb{R} : a < x\}$
(iii)	$(a, b] = \{x \in \mathbb{R} : a < x \le b\}$	(vi)	$(-\infty, \infty) = \mathbb{R}$

Note that the symbol ∞ is only used as part of the notation. It is not a real number!

∞ is not a real number! There is no such real number as ∞.

(i), (iv) and (vi) are *closed* intervals. (ii), (v) and (vi) are *open* intervals. We shall not define precisely what it means to be 'open' or 'closed' here although you should have an intuitive feel for these concepts.

Example. Which of the following statements are true?
(a) $3 \in (1, 2)$, (b) $2 \in (-\infty, 3]$, (c) $1 \in [1, 2]$, (d) $2 \in (0, 2)$, (e) $[0, 1] \subset (0, 2)$.

We shall often apply the operations of union and intersection to intervals.

Example. One has:

$$\begin{aligned}
(-\infty, 1) \cap [-1, \infty) &= [-1, 1);\\
(-\infty, 1) \cup [-1, \infty) &= \mathbb{R};\\
(-1, 0] \cup [0, 1] &= (-1, 1];\\
(-1, 0] \cap [0, 1] &= \{0\}.
\end{aligned}$$

1.1.5 Multiple and infinite unions and intersections

The notions of union and intersection extend to the situation with more than just two sets. For example,

$$\begin{aligned} A_1 \cup A_2 \cup A_3 &= \{x \mid x \in A_1 \text{ or } x \in A_2 \text{ or } x \in A_3\} \\ &= \{x \mid x \text{ belongs to at least one of the sets } A_1,\ A_2,\ A_3\} \\ &= \{x \mid x \in A_i \text{ for at least one of the indices } i = 1, 2, 3\} \end{aligned}$$

More generally, for n sets A_1, A_2, ..., A_n we have

$$A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid x \in A_i \text{ for at least one of the indices } i = 1, 2, \ldots, n\}. \qquad (1.2)$$

There is a more convenient piece of notation for such a union:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

This notation works similarly to the $\sum$-notation for a sum of several terms. The index i here is a "dummy variable", i.e. it can be replaced by any other letter, e.g.

$$\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{n} A_j = \bigcup_{\ell=1}^{n} A_\ell.$$

Example. Let $A_i = [0, i]$ for $i = 1, \ldots, 10$. Then

$$\bigcup_{i=1}^{10} A_i = [0, 10].$$

Let us rewrite our definition (1.2) as

$$\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for some } i \in \{1, 2, \ldots, n\}\}.$$

Let Λ denote the "index set" $\{1, 2, \ldots, n\}$. This is just the set of labels for the collection of sets we are considering. Then the above can be conveniently written as

$$\bigcup_{i \in \Lambda} A_i = \{x \mid x \in A_i \text{ for some } i \in \Lambda\}.$$

This all makes sense for any non-empty index set Λ. In particular, Λ may be infinite. If $\Lambda = \mathbb{N}$, one often writes $\bigcup\limits_{i=1}^{\infty} A_i$ for $\bigcup\limits_{\lambda \in \Lambda} A_\lambda$.

Example. Let $\Lambda = \mathbb{N}$ and $A_j = [j, j+1]$ for $j \in \mathbb{N}$. Then

$$\bigcup_{j=1}^{\infty} A_j = [1, \infty).$$

A similar discussion can be made regarding intersections:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x \mid x \in A_i \text{ for all } i \in \{1, 2, \ldots, n\}\}$$

and more generally

$$\bigcap_{i \in \Lambda} A_i = \{x \mid x \in A_i \text{ for all } i \in \Lambda\}.$$

Example. 1. Consider the following infinite collection of open intervals:

$$(-1, 1), (-2, 2), (-3, 3), (-4, 4), \ldots$$

We can form the union and intersection of these intervals:

$$\bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}, \qquad \bigcap_{n=1}^{\infty} (-n, n) = (-1, 1).$$

2. Consider the following infinite sequence of sets, each of which consists precisely of one point:

$$\{1\}, \{2\}, \{3\}, \ldots$$

Then the union and intersection of these sets are:

$$\bigcup_{n=1}^{\infty} \{n\} = \mathbb{N}; \qquad \bigcap_{n=1}^{\infty} \{n\} = \varnothing.$$

3. Consider the following infinite collection of open intervals:

$$(-1, 1), (-1/2, 1/2), (-1/3, 1/3), (-1/4, 1/4), \ldots$$

We can form the union and intersection of these intervals:

$$\bigcup_{n=1}^{\infty} (-1/n, 1/n) = (-1, 1), \qquad \bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}.$$

4. Consider the intervals $(a-1, a+1)$ where $a \in [0, 1]$. We have

$$\bigcup_{a \in [0,1]} [a-1, a+1] = [-1, 2], \qquad \bigcap_{a \in [0,1]} [a-1, a+1] = [0, 1].$$

We will discuss the above examples in more detail soon. You should do some exercises to get used to these concepts.

1.2 Functions

1.2.1 Basics

The concept of function is very important in every branch of mathematics. Here is the formal definition:

Definition 1.1. *Let A and B be sets. Then a* function

$$f : A \to B$$

is a rule which assigns exactly one element of B to each element of A. The set A is called the domain *of the function, the set B is called the* codomain *or* target*.*

If x is an element of A then the element of B assigned to x by the function f is usually denoted $f(x)$. In this context, $x \in A$ is usually called the *argument* of the function f and $f(x) \in B$ is called its *value*. Depending on the situation, various synonyms to the term "function" are often used: *map, mapping, morphism.*

Some functions have standard names, others have names (usually a single letter) defined for a particular piece of work. You should be familiar with the following standard functions from $\mathbb{R}$ to $\mathbb{R}$: sin, cos, exp. Here are some other standard functions:

$$\begin{aligned}
\log\colon (0,\infty) &\to \mathbb{R}, & x &\mapsto \log(x);\\
\tan\colon \mathbb{R}\setminus\{\tfrac{\pi}{2}+\pi n \mid n\in\mathbb{Z}\} &\to \mathbb{R}, & x &\mapsto \tan(x);\\
\text{the ceiling function } \lceil\cdot\rceil\colon \mathbb{R} &\to \mathbb{Z}, & x &\mapsto \lceil x\rceil;\\
\text{the square root } \sqrt{\cdot}\colon [0,\infty) &\to [0,\infty), & x &\mapsto \sqrt{x};\\
\text{the square}\colon \mathbb{R} &\to \mathbb{R}, & x &\mapsto x^2;\\
\text{a linear function }\colon \mathbb{R} &\to \mathbb{R}, & x &\mapsto 2x+3.
\end{aligned}$$

Of couse, in the last example, 2 and 3 can be replaced by any real numbers.

For functions from $\mathbb{R}$ to $\mathbb{R}$, plotting their graph is a very useful tool in analysing them. In this course we will mostly discuss functions $f : \mathbb{R} \to \mathbb{R}$, but in general, the domain and the target sets can be fairly arbitrary:

Example. Let A be the set of students in this class, let $B = \{1, 2, \dots, 100\}$ and for a student $x \in A$ let $f(x)$ be the age of x in years. This sets up a function $f : A \to B$.

Example. Let P be the set of all polygons on the plane. We can define two functions, f and g, on the set P. For $p \in P$, let $f(p)$ be the number of edges of p and let $g(p)$ be the area of p. Thus, we have defined two functions $f : P \to \mathbb{N}$ and $g : P \to \mathbb{R}$.

1.2.2 Setting up a function

A function can be set up, or defined, in various ways. The most common way to define a function of real numbers is to write down an analytic expression for it. For example,

$$f(x) = 2x + 3, \qquad g(x) = \sqrt{x-3}, \quad \text{and} \quad h(x) = \log(2x+4)$$

define functions of a real variable x. In such simple cases, there is usually no need to specify the domain of a function explicitly: the domain is assumed to include all values of x for which the corresponding analytic expression makes sense, i.e. can be computed. For example, one immediately sees that the domain of f is $\mathbb{R}$, the domain of g is $[3, \infty)$ and the domain of h is $(-2, \infty)$.

In more complex cases, a function can be defined by a different analytic expression for different values of its argument. For example, the sign function is defined as

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The *modulus*, or *absolute value* function is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

One can define more complex functions for a particular piece of work; for example,

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ \sqrt{1 - x^2} & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 \leq x. \end{cases}$$

In mathematics, one often has to consider very complex and "weird" functions. A classic example is Thomae's function:

$$f(x) = \begin{cases} 1/n & \text{if } x \text{ is rational, } x = m/n,\ n \in \mathbb{N},\ m \text{ and } n \text{ have no common factors;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

One cannot even attempt to construct a graph of this function! However, this is a perfectly well defined function which can be mathematically analysed.

1.2.3 Injective and surjective functions

There is some further terminology for functions which is useful.

Definition 1.2. *A function f is said to be* injective *or* one-to-one *if* $f(x) = f(y) \Rightarrow x = y$ *.*

In words, a function is *injective* if distinct elements of the domain are always mapped to distinct elements of the codomain. (If two different elements of the domain are mapped to the same element of the codomain, then the function is not injective.)

Example. Consider the following functions:

$$
\begin{array}{rcccrcc}
\sin : & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & \sin x \\
\exp : & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & \exp x = e^x \\
\log : & (0,\infty) & \to & \mathbb{R}\,, & x & \mapsto & \log x \\
f : & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & x^2 \\
g : & (3,\infty) & \to & \mathbb{R}\,, & x & \mapsto & \frac{1}{x-3} \\
h : & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & x^3 - x
\end{array}
$$

Which functions in the above examples are injective?

Definition 1.3. *Let $f : A \to B$ be a function. Then $f(A)$, the* image *(or* range*) of A under the function f, is the set*

$$\{y \in B \,|\, y = f(x) \textit{ for some } x \in A\} \subset B\,.$$

Example. What are the images of the functions in the above examples?

It is sometimes the case that the image of a function is equal to the entire codomain. In this case the function is said to be *surjective*.

Definition 1.4. *A function $f : A \to B$ is said to be* surjective *(or* onto*) if*

$$f(A) = B\,.$$

Example. Which functions in the above examples are surjective?

It is of course possible for a function to be both injective and surjective. This situation is sufficiently important to have a name of its own.

Definition 1.5. *A function which is both injective and surjective is said to be* bijective*.*

Example. Which functions in the above examples are bijective?

1.2.4 Composition and inverse

Definition 1.6. *Let A, B, C, be sets and $f : A \to B$ and $g : B \to C$ be functions. Then the* composition *$g \circ f$ is a function defined by*

$$(g \circ f)(x) = g(f(x)), \qquad x \in A.$$

Warning 1: in order to define the composition $g \circ f$, one has to make sure that the image of f is a subset of the domain of g. For example, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$, $g : [0,\infty) \to \mathbb{R}$, $g(x) = \sqrt{x}$. Then $g \circ f$ is NOT defined. Indeed, $g(f(x)) = \sqrt{\sin(x)}$, and if $\sin(x) < 0$, this expression makes no sense!

Warning 2: in general $g \circ f \neq f \circ g$, even if both expressions are well defined. For example, if $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = x + 1$ and $g : \mathbb{R} \to \mathbb{R}$ is $g(x) = x^2$, then $f \circ g(x) = x^2 + 1$ and $g \circ f(x) = (x + 1)^2 = x^2 + 2x + 1$.

An important feature of a bijective function is that it has a (unique) *inverse*.

Definition 1.7. *Let $f : A \to B$ be a bijective function. Then a function $g : B \to A$ is the inverse of the function f if and only if it satisfies*

$$(g \circ f)(x) = x, \quad \forall x \in A \quad \text{and} \quad (f \circ g)(y) = y, \quad \forall y \in B.$$

The function g with these properties is denoted f^{-1}.

Every bijective function has an inverse. On the other hand, if a function is not bijective then it does not have an inverse.

At a practical level, in order to find the inverse of a function f, you need to solve the equation $f(x) = y$ for x.

Example. Find the inverse of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$. Find the inverse of $\exp : \mathbb{R} \to (0, \infty)$.