CM115A, CM115B Numbers and Functions: Christmas Challenge

This exercise is optional. If you have a solution, please submit directly to the lecturer A. Pushnitski by 20 January. Feel free to approach the lecturer discuss it informally.

First, we need a definition and some notation. Let s_n be a sequence and $(a, b) \subset \mathbb{R}$ be an interval. We will say that *the sequence s_n visits the interval (a, b) infinitely often* if the set

$$\{n \in \mathbb{N} \mid s_n \in (a, b)\}$$

is infinite. Further, for a real number x we denote $\operatorname{frac}(x) = x - \lfloor x \rfloor$ the fractional part of x.

CHALLENGE: Let $\alpha > 0$ be irrational. Consider the sequence $s_n = \operatorname{frac}(\alpha n)$, $n \in \mathbb{N}$. Prove that s_n visits every interval $(a, b) \subset (0, 1)$ infinitely often.

As a warm-up, you might want to consider what happens when α is rational.

In fact, it can be proven that the sequence s_n (for irrational α) fills up the interval $(0, 1)$ with a *uniform density*. This means the following. For any interval $(a, b) \subset (0, 1)$, one has

$$\lim_{N \to \infty} \frac{\sharp\{n \leq N \mid s_n \in (a, b)\}}{N} = b - a,$$

where $\sharp A$ is the number of elements in the set A. The proof of the last statement is far beyond the scope of the Numbers and Functions course and belongs to the area of mathematics called ergodic theory.