# CM115A, CM115B Numbers and Functions: Assignment 9

Solutions to this assignment must be handed in at the **start** of the tutorial you attend during week 11 of the term. Assignments handed in late will not normally be marked. It is essential that you write your NAME, STUDENT NUMBER and GROUP NUMBER on your work.

Some harder exercises are included in this sheet; they are marked with an asterisk * and are not compulsory.

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1. Find $\lim_{n\to\infty} a_n$ when $a_n =$

   (a) $\dfrac{3^n - 4}{3^n + 4}$

   (b) $\dfrac{n^3 3^n - n2^n + 1}{4n^3 3^n + 1}$

   (c) $\dfrac{5^{\frac{1}{n}}}{2^{\frac{1}{n}} + 2}$

   (d) $\dfrac{3n^2 + n}{n^2 + 3n + 3}$

   (e) $\dfrac{n + 1 + 2^{2n}}{4^n}$

   (f) $\dfrac{5^n + n^2 3^n}{n5^n - 3^n}$

   (g) $\dfrac{n^4 2^n + 2n^3 3^n + n4^n}{2^n - 3^n + n4^{n+1}}$

   (h) $\dfrac{3^{\frac{1}{n}} n^2 - 2^{\frac{1}{n}} n^2 + 1}{4n^2 + 3}$

   (i)* $\dfrac{5^{\frac{1}{n}} + 1}{3^{\frac{1}{n+1}} + 1}$

   (j)* $\sqrt{n^2 - n + 1} - n$

   [Hint: for (j) use the fact that $\sqrt{a} - \sqrt{b} = (\sqrt{a} - \sqrt{b})\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$.]

   (You may use any of the theorems and standard limits given in the notes.)

2. For each of the following sequences, decide whether it is increasing, decreasing, or neither.

   (a) $s_n = \frac{n+1}{n+2}$

   (b) $s_n = 3 - 3^{-n}$.

(c) $s_n = n + \frac{8}{n}$.

(d) $s_n = \frac{10}{n^{10}} + \frac{1}{n!}$.

(e) $s_n = n + (-1)^n$.

3. For each of the following statements, decide whether it is true or false and prove your claim. You may use any of the theorems from the lecture notes, but you must clearly explain your reasoning and state which theorems you are using.

(a) $10^{-n} = o(1/n)$, as $n \to \infty$.

(b) $n^2 + 5n + 3 = o(2^n)$, as $n \to \infty$.

(c) $(-5)^n = o(4^n)$, as $n \to \infty$.

(d) $(n+1)^3 - n^3 = o(n^3)$, as $n \to \infty$.

(e) $\dfrac{2^{-n} + 3^{-n}}{n - \frac{1}{2}} = o(2^{-n})$, as $n \to \infty$.

(f) $\sqrt{n+1} = O(n)$, as $n \to \infty$.

(g) $\dfrac{n^2 - n}{n^2 + n} = O(1)$, as $n \to \infty$.

(h) $\dfrac{n^2 - n}{n^2 + n} = o(1)$, as $n \to \infty$.

4. Which one of the two given expressions is greater than the other one for all sufficiently large values of $n \in \mathbb{N}$?

(a) $100n + 200$ or $0.01n^2$

(b) $2^n$ or $n^{100}$

(c) $100^n$ or $n!$

(d) $1/2^n$ or $2^{1/n}$

5. Prove that factorials beat exponentials: $\frac{A^n}{n!} \to 0$ as $n \to \infty$ for any $A > 0$. Take $n_0 > 2A$, $n_0 \in \mathbb{N}$; note that $\frac{A^{n+1}}{(n+1)!}\frac{n!}{A^n} \leq 1/2$ for $n \geq n_0$. Deduce that $\frac{A^n}{n!} \leq C(1/2)^n$ for some constant $C$ and use the Sandwich Theorem.

6. Let $s_n$ be a sequence which is non-decreasing and not bounded above. Prove that $s_n \to +\infty$ as $n \to \infty$.

7.* Let $s_n$ be a sequence of positive numbers such that $s_n \to \ell$ as $n \to \infty$, where $\ell > 0$. Prove that $\sqrt{s_n} \to \sqrt{\ell}$ as $n \to \infty$.

8.* Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$. *Hint:* Let $\varepsilon > 0$ be given. Using the theorem "exponentials beat powers", prove that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $n \leq (1+\varepsilon)^n$, and therefore $1 < \sqrt[n]{n} \leq 1 + \varepsilon$.

9.* Let the sequence $a_n$ be defined by $a_1 = 1$ and $a_{n+1} = \frac{a_n^2+2}{2a_n}$ for $n \geq 1$. (Sequences defined in this way are called *iteratively defined.*) Prove that $a_n \to \sqrt{2}$. Proceed as follows:

(a) By induction, prove that $a_n \geq \sqrt{2}$ for all $n > 1$. (Use the inequality $x^2 + y^2 \geq 2xy$).

(b) Using the previous step, by induction, prove that $a_{n+1} \leq a_n$ for all $n > 1$.

(c) Using the previous step, prove that $a_{n+1} \leq a_n$ for all $n > 1$.

(d) Conclude that $a_n$ converges.

(e) Let $\ell = \lim_{n\to\infty} a_n$. Using the definition of $a_n$, prove that $\ell$ must satisfy $\ell = \frac{\ell^2+2}{2\ell}$.

(f) Conclude that $a_n \to \sqrt{2}$ as $n \to \infty$.

(g) Use your calculator to compute $a_2$, $a_3$, $a_4$. Compare with $\sqrt{2}$.