CM115A, CM115B Numbers and Functions: Assignment 10

Please take this assignment sheet with you to the tutorial you attend during week 11 of the term. You will be working at this assignment during the tutorial. This will be the last tutorial for this course. There will be no tutorial in which you can hand this sheet in.

Note that some exercises deal with Cauchy sequences which will only be covered in the lectures at the end of week 11, so you might not be able to attempt these exercises at the tutorial. Try them at home later. Solutions to this assignment will be posted on the course web page after the end of week 11, so that you can check your own work.

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1. Find a convergent subsequence of the sequence s_n, where ...
   (a) $s_n = \frac{1+(-1)^n}{2} + \frac{(-1)^n}{n}$
   (b) $s_n = \cos(\pi(\sqrt{n} - n))$
   (c) $s_n = (-1)^{\frac{n(n+1)}{2}}$
2. Find all limit points of the following sequences:
   (a) $s_n = \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \ldots, \frac{1}{2^n}, \frac{2^n-1}{2^n}, \ldots$
   (b) $s_n = 3(1 - \frac{1}{n}) + 2(-1)^n$
   (c) $s_n = 1 + n \sin \frac{n\pi}{2}$
   (d) $s_n = n(1 + (-1)^n)$
   (e) $s_n = \dfrac{2 + (-1)^n - \frac{1}{n}}{2 - (-1)^n + \frac{1}{n}}$
3. Give an example of a sequence s_n which has the limit points $1, 2, 3$ and no other limit points.
4. Give an example of a sequence which has no limit points.
5. For each of the following statements, decide whether it is true or false. Explain your answer; where appropriate, give a reference to a theorem from the notes or construct a counterexample.
   (a) Every bounded sequence has a convergent subsequence.
   (b) For every bounded sequence, all of it subsequences converge.
   (c) If a sequence s_n has a convergent subsequence s_{n_k}, then the whole sequence s_n also converges.
   (d) Every bounded sequence s_n has a subsequence s_{n_k} such that s_{n_k} is a Cauchy sequence.

(e) There exists a convergent sequence s_n which has two subsequences, one of which converges to 1 and another one to -1.

(f) Every sequence has a convergent subsequence.

(g) If a sequence diverges, then any subsequence of it also diverges.

(h) If a sequence is monotone, then every subsequence of it is also monotone.

6. Let $a \in [0, 1]$. Using the theorem about bounded monotone sequences and the discussion of $\sum \frac{1}{k!}$ from the lectures, prove that the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a^k}{k!}$$

exists. Here we use the convention that $0! = 1$. (In fact, this limit equals e^a, but you don't need to prove this.)

7. Let $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$.

   (a) Show that s_n is increasing.

   (b) By induction prove that $s_n \leq 2 - \frac{1}{n}$ for all $n \geq 1$.

   (c) Using the theorem about monotone bounded sequences, deduce that s_n is convergent.

8. Prove that if a sequence diverges to $+\infty$ then any subsequence of this sequence also diverges to $+\infty$.

9. Let $a \in [-1, 0]$. Using the Cauchy convergence criterion and the discussion of $\sum \frac{1}{k!}$ from the lectures, prove that the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a^k}{k!}$$

exists. (In fact, this limit equals e^a, but you don't need to prove this.)

10.* Let $\alpha \in (0, 1)$ and let s_n be a sequence which satisfies $|s_{n+1} - s_n| \leq \alpha^n$ for all $n \in \mathbb{N}$. Using the formula

$$1 + \alpha + \alpha^2 + \cdots + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha},$$

show that for all indices $m > n$ one has

$$|s_n - s_m| \leq \frac{\alpha^n}{1 - \alpha}.$$

Using the Cauchy convergence criterion, prove that s_n converges.

11.* Prove that if a sequence s_n is unbounded, then there exists a subsequence s_{n_k} such that $|s_{n_k}| \to \infty$ as $k \to \infty$.

12.** Using the proof by contradiction, prove Theorem 8.6 from the lecture notes (if a bounded sequence has only one limit point, then it converges to this limit point). Proceed as follows.

(a) Suppose that s_n does not converge to ℓ as $n \to \infty$. Prove that there exists $\varepsilon > 0$ and a subsequence s_{n_k} such that

$$|s_{n_k} - \ell| > \varepsilon \text{ for all } k \in \mathbb{N}. \qquad (**)$$

(b) Using the Bolzano-Weierstrass theorem, prove that there exists a subsubsequence $s_{n_{k_i}}$ which converges to ℓ as $i \to \infty$.

(c) Show that the conclusion of the last step contradicts to $(**)$.

13.** Prove that any $\ell \in [-1, 1]$ is a limit point of the sequence $\cos(\pi\sqrt{n})$. Proceed as follows. Denote $t_n = \pi\sqrt{n}$ and $s_n = \cos(t_n)$. (i) Prove that if $k \in \mathbb{N}$, then for any $n \geq k$ we have $t_{n+1} - t_n < \frac{\pi}{2\sqrt{k}}$. (ii) Using the inequality $|\cos(x) - \cos(y)| \leq |x - y|$, prove that if $k \in \mathbb{N}$, then for any $n \geq k$ we have $|s_{n+1} - s_n| < \frac{\pi}{2\sqrt{k}}$. (iii) Let $k \in \mathbb{N}$. Prove that any interval $[a, b] \subset [-1, 1]$ of length $b - a \geq \frac{\pi}{2\sqrt{k}}$ contains at least one point s_n with $n \in \{k^2, k^2+1, \ldots, (k+1)^2-1\}$. (iv) Let $\ell \in [-1, 1]$. Prove that for any $k \in \mathbb{N}$ there exists an index $n_k \in \{k^2, k^2+1, \ldots, (k+1)^2-1\}$ such that $|s_{n_k} - \ell| \leq \frac{\pi}{2k}$.