CM115A, CM115B Numbers and Functions: Assignment 10

Please take this assignment sheet with you to the tutorial you attend during week 11 of the term. You will be working at this assignment during the tutorial. This will be the last tutorial for this course. There will be no tutorial in which you can hand this sheet in.

Note that some exercises deal with Cauchy sequences which will only be covered in the lectures at the end of week 11, so you might not be able to attempt these exercises at the tutorial. Try them at home later. Solutions to this assignment will be posted on the course web page after the end of week 11, so that you can check your own work.

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1. Find a convergent subsequence of the sequence s_n, where ...

   (a) $s_n = \frac{1+(-1)^n}{2} + \frac{(-1)^n}{n}$
   Answer: $s_{2n} \to 1$ and $s_{2n+1} \to 0$ as $n \to \infty$.

   (b) $s_n = \cos(\pi(\sqrt{n} - n))$
   Answer: $s_{4n^2} = 1$ for all $n \in \mathbb{N}$

   (c) $s_n = (-1)^{\frac{n(n+1)}{2}}$
   Answer: $s_{4n} = 1$ for all n and therefore this subsequence converges.
   In fact, $s_n = -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, \ldots$

2. Find all limit points of the following sequences:

   (a) $s_n = \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \ldots, \frac{1}{2^n}, \frac{2^n-1}{2^n}, \ldots$
   Answer: The limit points are $\lim_{n\to\infty} s_{2n} = 1$ and $\lim_{n\to\infty} s_{2n+1} = 0$.

   (b) $s_n = 3(1 - \frac{1}{n}) + 2(-1)^n$
   Answer: The limit points are $\lim_{n\to\infty} s_{2n} = 5$ and $\lim_{n\to\infty} s_{2n+1} = 1$.

   (c) $s_n = 1 + n \sin \frac{n\pi}{2}$
   Answer: The only limit point is $\lim_{n\to\infty} s_{2n} = 1$.

   (d) $s_n = n(1 + (-1)^n)$
   Answer: The only limit point is $\lim_{n\to\infty} s_{2n+1} = 0$.

   (e) $s_n = \dfrac{2 + (-1)^n - \frac{1}{n}}{2 - (-1)^n + \frac{1}{n}}$
   Answer: The limit points are $\lim_{n\to\infty} s_{2n} = 3$ and $\lim_{n\to\infty} s_{2n+1} = 1/3$

3. Give an example of a sequence s_n which has the limit points $1, 2, 3$ and no other limit points.

   Answer: $s_n = 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$

4. Give an example of a sequence which has no limit points.

   Answer: Any sequence which diverges to ∞ satisfies this condition.

5. For each of the following statements, decide whether it is true or false. Explain your answer; where appropriate, give a reference to a theorem from the notes or construct a counterexample.

   (a) Every bounded sequence has a convergent subsequence.

   Answer: true. This is the Bolzano-Weierstrass theorem.

   (b) For every bounded sequence, all of it subsequences converge.

   Answer: false. Counterexample: if $s_n = \sin(\pi n/2)$, then $s_{2n+1} = (-1)^n$ diverges.

   (c) If a sequence s_n has a convergent subsequence s_{n_k}, then the whole sequence s_n also converges.

   Answer: false. Counterexample: if $s_n = \sin(\pi n/2)$, then $s_{4n} = 0$ converges, yet $s_{2n+1} = (-1)^n$ diverges.

   (d) Every bounded sequence s_n has a subsequence s_{n_k} such that s_{n_k} is a Cauchy sequence.

   Answer: true. By the Bolzano-Weierstrass theorem, every bounded sequence has a convergent subsequence. By the Cauchy convergence criterion, this subsequence is a Cauchy sequence.

   (e) There exists a convergent sequence s_n which has two subsequences, one of which converges to 1 and another one to -1.

   Answer: false. If a sequence converges, then all of its subsequences converge to the same limit.

   (f) Every sequence has a convergent subsequence.

   Answer: false. Counterexample: $s_n = n$ has no convergent subsequences.

   (g) If a sequence diverges, then any subsequence of it also diverges.

   Answer: false. Counterexample: $s_n = n^{(-1)^n}$ diverges yet $s_{2n+1} = \frac{1}{2n+1}$ converges to zero.

   (h) If a sequence is monotone, then every subsequence of it is also monotone.

   Answer: true. Indeed, if, for example, s_n is increasing, and $1 \leq n_1 < n_2 < \ldots$ is a strictly increasing sequence of integers, then for all $k > j$ we have $n_k > n_j$ and so $s_{n_k} \geq s_{n_j}$. A similar argument applies to the case when s_n is decreasing.

6. Let $a \in [0, 1]$. Using the theorem about bounded monotone sequences and the discussion of $\sum \frac{1}{k!}$ from the lectures, prove that the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a^k}{k!}$$

   exists. Here we use the convention that $0! = 1$. (In fact, this limit equals e^a, but you don't need to prove this.)

Answer: Denote $s_n = \sum_{k=0}^{n} \frac{a^k}{k!}$. It is obvious that s_n is increasing. Let us prove that s_n is bounded above. Since $a \le 1$, we have

$$s_n = \sum_{k=0}^{n} \frac{a^k}{k!} \le \sum_{k=0}^{n} \frac{1}{k!} \le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1 - 2^{-n}}{1 - 2^{-1}} \le 3,$$

as discussed in the lectures. Thus, s_n is bounded above by 3. It follows that s_n converges.

7. Let $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$.

   (a) Show that s_n is increasing.

   (b) By induction prove that $s_n \le 2 - \frac{1}{n}$ for all $n \ge 1$.

   (c) Using the theorem about monotone bounded sequences, deduce that s_n is convergent.

   Answer: (a) is obvious. Let us prove that $s_n \le 2 - \frac{1}{n}$ for all $n \ge 1$. For $n = 1$ this is true. Suppose this statement is true for $n = m$. Then for $n = m + 1$ we have

$$s_{m+1} = s_m + \frac{1}{(m+1)^2} \le 2 - \frac{1}{m} + \frac{1}{(m+1)^2} = 2 - \frac{m^2 + m + 1}{m(m+1)^2} < 2 - \frac{1}{m+1},$$

   where the last inequality follows by an elementary calculation. Thus, (b) is proven. From (b) we get $s_n \le 2$ for all n. Since s_n is increasing and bounded above by 2, we get that s_n converges.

8. Prove that if a sequence diverges to $+\infty$ then any subsequence of this sequence also diverges to $+\infty$.

   Answer: Let $s_n \to \infty$ as $n \to \infty$. Then for any $H > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ one has $s_n > H$. Let s_{n_k} be a subsequence of s_n and let $H > 0$ be given. Take $k \ge n_0$. Then, since $n_k \ge k$ for any k (see the proof of Theorem 7.2 from the lecture notes), we have $n_k \ge k \ge n_0$ and so $s_{n_k} \ge H$, as required.

9. Let $a \in [-1, 0]$. Using the Cauchy convergence criterion and the discussion of $\sum \frac{1}{k!}$ from the lectures, prove that the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a^k}{k!}$$

   exists. (In fact, this limit equals e^a, but you don't need to prove this.)

   Answer: Denote $s_n = \sum_{k=0}^{n} \frac{a^k}{k!}$. (note that s_n is not monotone; this makes this case more difficult than the case $a > 0$.) Let us prove that s_n is a Cauchy sequence. For any $n < m$, we have

$$|s_m - s_n| = \left|\sum_{k=n+1}^{m} \frac{a^k}{k!}\right| \leq \sum_{k=n+1}^{m} \frac{|a^k|}{k!} \leq \sum_{k=n+1}^{m} \frac{1}{k!} \leq \sum_{k=n+1}^{m} \frac{1}{2^{k-1}}$$
$$= \frac{1-2^{-m}}{1-2^{-1}} - \frac{1-2^{-n}}{1-2^{-1}} = \frac{2^{-n}-2^{-m}}{1-2^{-1}} \leq \frac{2^{-n}}{1-2^{-1}} = 2^{-n+1}.$$

Given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $2^{-n_0+1} < \varepsilon$. Then for all $n, m \geq n_0$ we have (assuming $m \geq n$):

$$|s_n - s_m| \leq 2^{-n+1} \leq 2^{-n_0+1} \leq \varepsilon.$$

Thus, s_n is a Cauchy sequence and so it converges.

10.* Let $\alpha \in (0, 1)$ and let s_n be a sequence which satisfies $|s_{n+1} - s_n| \leq \alpha^n$ for all $n \in \mathbb{N}$. Using the formula

$$1 + \alpha + \alpha^2 + \cdots + \alpha^n = \frac{1-\alpha^{n+1}}{1-\alpha},$$

show that for all indices $m > n$ one has

$$|s_n - s_m| \leq \frac{\alpha^n}{1-\alpha}.$$

Using the Cauchy convergence criterion, prove that s_n converges.

Answer: To prove the inequality, write

$$|s_n - s_m| \leq |s_n - s_{n+1}| + |s_{n+1} - s_{n+2}| + \cdots + |s_{m-1} - s_m| \leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}$$
$$= \frac{1-\alpha^m}{1-\alpha} - \frac{1-\alpha^n}{1-\alpha} = \frac{\alpha^n - \alpha^m}{1-\alpha} \leq \frac{\alpha^n}{1-\alpha}.$$

Now to prove convergence, it suffices to show that s_n is a Cauchy sequence. Given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $\frac{\alpha^{n_0}}{1-\alpha} < \varepsilon$ (this is easy to do: take $n_0 = \lceil(\log \varepsilon(1-\varepsilon))/\log \alpha\rceil + 1$). Then for all $n, m \geq n_0$ we have (assuming $m \geq n$):

$$|s_n - s_m| \leq \frac{\alpha^n}{1-\alpha} \leq \frac{\alpha^{n_0}}{1-\alpha} \leq \varepsilon,$$

as required.

11.* Prove that if a sequence s_n is unbounded, then there exists a subsequence s_{n_k} such that $|s_{n_k}| \to \infty$ as $k \to \infty$.

Answer: A sequence s_n is bounded if and only if there exists $M > 0$ such that for any $n \in \mathbb{N}$ we have $|s_n| \leq M$. Negating this, we obtain that:

$$\forall M > 0 \quad \exists n \in \mathbb{N} \text{ such that } |s_n| > M.$$

Let us prove a stronger statement. We claim that

$$\forall M > 0 \quad \text{there exist infinitely many indices } n \in \mathbb{N} \text{ such that } |s_n| > M. \qquad (*)$$

Let us prove $(*)$ by contradiction. The following argument is very similar to the proof of the theorem that every convergent sequence is bounded.

Suppose that for some $M > 0$ there exist only finitely many indices n such that $|s_n| > M$. Let us label these indices $n_1, n_2, \dots n_p$. Take $M_0 = \max\{|s_{n_1}|, |s_{n_2}|, \dots, |s_{n_p}|, M\}$. Then $|s_n| \le M_0$ for all n, which contradicts to the assumption that s_n is bounded.

Next, using $(*)$, let us construct the required subsequence s_{n_k} as follows. (The following argument is similar to the proof of the Bolzano-Weierstrass theorem.) Take $M = 1$; then by $(*)$ there exists n_1 such that $|s_{n_1}| > 1$. Next, take $M = 2$; then by $(*)$ the set $S_2 = \{n \in \mathbb{N} \mid |s_n| > 2\}$ is infinite, and therefore we can choose $n_2 \in S_2$ such that $n_2 > n_1$. Further, take $M = 2$; then by $(*)$ the set $S_3 = \{n \in \mathbb{N} \mid |s_n| > 3\}$ is infinite, and therefore we can choose $n_3 \in S_3$ such that $n_3 > n_2$. Continuing in this way, we obtain the infinite sequence of positive integers $n_1 < n_2 < \dots$ such that $|s_{n_k}| > k$ for all k. Thus $|s_{n_k}| \to \infty$ as $k \to \infty$.

12.** Using the proof by contradiction, prove Theorem 8.6 from the lecture notes (if a bounded sequence has only one limit point, then it converges to this limit point). Proceed as follows.

(a) Suppose that s_n does not converge to ℓ as $n \to \infty$. Prove that there exists $\varepsilon > 0$ and a subsequence s_{n_k} such that

$$|s_{n_k} - \ell| > \varepsilon \text{ for all } k \in \mathbb{N}. \qquad (**)$$

(b) Using the Bolzano-Weierstrass theorem, prove that there exists a subsubsequence $s_{n_{k_i}}$ which converges to ℓ as $i \to \infty$.

(c) Show that the conclusion of the last step contradicts to $(**)$.

Answer: (a) is simply the negation of the definition of convergence. (b) Apply the Bolzano-Weierstrass theorem to the sequence s_{n_k}, $k \in \mathbb{N}$. We get that there exists a convergent subsequence $s_{n_{k_i}}$. Let ℓ' be the limit of this subsequence. Then ℓ' is a limit point of the sequence s_n and therefore, by the hypothesis, $\ell' = \ell$. (c) Since $s_{n_{k_i}} \to \ell$ as $i \to \infty$, there exists $i_0 \in \mathbb{N}$ such that for all $i \ge i_0$ one has $|s_{n_{k_i}} - \ell| < \varepsilon$. This contradicts $(**)$.

13.** Prove that any $\ell \in [-1, 1]$ is a limit point of the sequence $\cos(\pi\sqrt{n})$. Proceed as follows. Denote $t_n = \pi\sqrt{n}$ and $s_n = \cos(t_n)$. (i) Prove that if $k \in \mathbb{N}$, then for any $n \ge k$ we have $t_{n+1} - t_n < \frac{\pi}{2\sqrt{k}}$. (ii) Using the inequality $|\cos(x) - \cos(y)| \le |x - y|$, prove that if $k \in \mathbb{N}$, then for any $n \ge k$ we have $|s_{n+1} - s_n| < \frac{\pi}{2\sqrt{k}}$. (iii) Let $k \in \mathbb{N}$. Prove that any interval $[a, b] \subset [-1, 1]$ of length $b - a \ge \frac{\pi}{2\sqrt{k}}$ contains at least one point s_n with $n \in \{k^2, k^2+1, \ldots, (k+1)^2-1\}$. (iv) Let $\ell \in [-1,1]$. Prove that for any $k \in \mathbb{N}$ there exists an index $n_k \in \{k^2, k^2+1, \ldots, (k+1)^2-1\}$ such that $|s_{n_k} - \ell| \leq \frac{\pi}{2k}$.

Answer: (i) We have

$$\pi(\sqrt{n+1}-\sqrt{n}) = \pi\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{\pi}{\sqrt{n+1}+\sqrt{n}} < \frac{\pi}{2\sqrt{k}}.$$

(ii) We have

$$|s_{n+1}-s_n| = |\cos(t_{n+1}) - \cos(t_n)| \leq |t_{n+1}-t_n| < \frac{\pi}{2\sqrt{k}}.$$

(iii) Denote $J_k = \{k^2, k^2+1, \ldots, (k+1)^2-1\}$. Assume, to get a contradiction, that for no index $n \in J_k$ we have $s_n \in [a,b]$. Then an easy argument using item (ii) shows that for any two consecutive indices $n, n+1 \in J_k$ we have either both $s_n < a$ and $s_{n+1} < a$ or both $s_n > b$ and $s_{n+1} > b$. Then either all terms s_n, $n \in J_k$, satisfy $s_n < a$ or they all satisfy $s_n > b$. This is inconsistent with $s_{k^2} = (-1)^k$ and $s_{(k+1)^2} = (-1)^{k+1}$.

(iv) For $\ell \in [-1,1]$, choose an interval $[a,b]$ such that $\ell \in [a,b]$ and $b-a \geq \frac{\pi}{2k}$. Then there exists $n_k \in J_k$ such that $s_{n_k} \in [a,b]$. Then $|s_{n_k} - \ell| \leq \frac{\pi}{2k}$.