King's College London

UNIVERSITY OF LONDON

ATTACH THIS PAPER TO YOUR SCRIPT **USING THE STRING PROVIDED**

Candidate No: .... **Desk No:** ....

BSC AND MSCI EXAMINATION

4CCM115A NUMBERS AND FUNCTIONS

SUMMER 2010

TIME ALLOWED: THREE HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.
SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER.
ALL QUESTIONS IN SECTION B CARRY EQUAL MARKS.
ANSWER ALL QUESTIONS.

YOU ARE PERMITTED TO USE A CALCULATOR.
ONLY CALCULATORS APPROVED BY THE COLLEGE MAY BE USED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

SECTION A

In section A there are 5 marks for a correct answer, −1 for a wrong answer, 0 for a blank answer. Put all your answers in the grid on page 8. Use the booklet provided for any rough work. Remember to put your candidate number and desk number on top of page 8.

A 1. Which one of the following statements is true?

(A) $(0,1) \cap [0,1] = [0,1]$

(B) $[1,2) \setminus (1,2] = \{1\}$

(C) $(-1,1) \cap [0,1] = (0,1)$

(D) $(-1,0) \cup (0,1) = (-1,1)$

(E) None of the above statements is true.

A 2. The set $\cup_{n=1}^{\infty}[\frac{2}{n}, 2+n]$ is equal to ...

(A) $[0,\infty)$

(B) $(0,3)$

(C) $\{2\}$

(D) $[2,3]$

(E) none of the above.

A 3. Which one of the following functions is bijective?

(A) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 - 5x$

(B) $f : (0, \infty) \to \mathbb{R},\ f(x) = (\log x)^2$

(C) $f : (-\pi/2, \pi/2) \to \mathbb{R},\ f(x) = \tan(x)$

(D) $f : [0, \pi] \to [0, 1],\ f(x) = \sin(x)$

(E) None of the above functions is bijective.

A 4. Consider the proposition:
$\forall a > 0\ \exists N \in \mathbb{N}$ *such that* $\forall n \geq N$ *one has* $1/n < a.$
Which one of the following is the negation of this proposition?

(A) $\forall a > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$ one has $1/n \geq a$.

(B) $\exists a > 0$ such that $\forall N \in \mathbb{N}\ \exists n \geq N$ such that $1/n \geq a$.

(C) $\exists a \leq 0$ such that $\forall N \in \mathbb{N}\ \exists n < N$ such that $1/n \geq a$.

(D) $\exists a \leq 0$ such that $\forall N \notin \mathbb{N}\ \exists n < N$ such that $1/n \geq a$.

(E) None of the above is the negation of the given proposition.

A 5. One says that a sequence s_n converges to ℓ as $n \to \infty$, if...

(A) $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$ one has $|s_n - \ell| < \varepsilon$

(B) $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ such that $\forall n > n_0$ one has $|s_n - \ell| < \varepsilon$

(C) $\exists \varepsilon > 0$ such that $\forall n \in \mathbb{N}$ one has $|s_n - \ell| < \varepsilon$

(D) $\exists \varepsilon > 0$ such that $\forall n_0 \in \mathbb{N}$ one has $|s_n - \ell| < \varepsilon$ if $n > n_0$

(E) none of the above is correct

A 6. Let $A = \{(-1)^n - e^{-n} \mid n \in \mathbb{N}\}$. Which one of the following statements holds true?

(A) $\sup A = 1$

(B) $\max A = 1$

(C) $\inf A = -1$

(D) $\min A = -1$

A 7. Which one of the following sets is bounded?

(A) $\cup_{n=1}^{\infty}[2^{-n}, 2^n]$

(B) $\cup_{n=1}^{\infty}[-2^n, 2^n]$

(C) $\cap_{n=1}^{\infty}[2^{-n}, 2^n]$

(D) $\cup_{n=1}^{\infty}[(-2)^n, 2^n]$

(E) None of the above sets is bounded.

A 8. Let $A = \{x \mid x \in \mathbb{R}, |x^2 - 3| < 1\}$ and $B = \{x^2 \mid x \in \mathbb{R}, |x - 3| < 1\}$. Which one of the following statements holds true?

(A) $A \subset B$

(B) $B \subset A$

(C) $A \cap B = \varnothing$

(D) $A \setminus B = \varnothing$

(E) None of the above statements is true.

A 9. The limit $\lim_{n\to\infty} \frac{3^{3n+3} + n^3 9^{9/n}}{9^n + n^9 3^{3/n}}$ equals to ...

(A) 0

(B) 3

(C) 9

(D) 1/3

(E) none of the above

A 10. Which one of the following sequences is bounded?

(A) $n^{(-1)^n}$

(B) $(-n)^n$

(C) n^{-n}

(D) $-n^n$

(E) None of the above sequences is bounded.

A 11. Which one of the following statements holds true?

(A) $n^2 = o(n)$ as $n \to \infty$

(B) $n^{-2} = o(n^{-1})$ as $n \to \infty$

(C) $2^n = o(2n)$ as $n \to \infty$

(D) $2^{-n-2} = o(2^{-n})$ as $n \to \infty$

(E) None of the above statements is true.

A 12. Which one of the following sequences is increasing?

(A) $\dfrac{2^n}{2^n+2}$

(B) $\dfrac{2^{-n}}{2^{-n}+2}$

(C) $|2n-20|$

(D) $(-2)^n$

(E) None of the above sequences is increasing.

A 13. Let $s_n = \sin(2\pi n/3)$, $n \in \mathbb{N}$. Which one of the following statements holds true?

(A) s_n has no limit points

(B) s_n has exactly one limit point

(C) s_n has exactly two limit points

(D) s_n has exactly three limit points

(E) None of the above statements is true.

A 14. Let $s_n = \left(1 + \dfrac{1}{2n}\right)^{2n+2}$. Which one of the following statements holds true?

(A) $s_n \to 1$ as $n \to \infty$

(B) $s_n \to e$ as $n \to \infty$

(C) $s_n \to \infty$ as $n \to \infty$

(D) $s_n \to e^2$ as $n \to \infty$

(E) None of the above statements is true.

A 15. Which one of the following statements holds true?

(A) Any increasing bounded below sequence converges.

(B) Any monotone bounded below sequence converges.

(C) Any monotone bounded sequence converges.

(D) Any monotone sequence is a Cauchy sequence.

(E) None of the above statements holds true.

A 16. Which one of the following statements holds true?

(A) Every sequence has at least one limit point.

(B) A Cauchy sequence can have two distinct limit points.

(C) A convergent sequence can have two distinct limit points.

(D) A bounded sequence has at least one limit point.

(E) None of the above statements holds true.

ANSWER GRID: put a cross in ONE BOX for the correct answer for each question in Section A. If you change your mind and want to correct your answer, obliterate your incorrect answer by shading its box, and put a new cross in the box for the correct answer.

Candidate No: ....

Desk no: ....

	A	B	C	D	E
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SECTION B

In Section B each question carries a maximum of 20 marks. Answer all questions in the booklet provided.

B 17. (i) Give an example of an unbounded sequence which neither diverges to $+\infty$ nor diverges to $-\infty$. No proof is required.

(ii) Give an example of sequences s_n, t_n, $n \in \mathbb{N}$, such that $s_n \neq 0$ for all n, s_n converges, t_n diverges, and $s_n t_n$ converges as $n \to \infty$. No proof is required.

B 18. State the definition of the infimum of a set. For the set

$$S = \left\{ 2(-1)^n + \frac{5}{n^2 + 2} \;\middle|\; n \in \mathbb{N} \right\},$$

find $\inf S$ and prove your claim. Argue directly from the definition of infimum.

B 19. State the definition of divergence to $+\infty$. Prove that $n^2 - 100n \to +\infty$ as $n \to \infty$. You are not allowed to use any theorems from the course; argue directly from the definition of divergence to $+\infty$.

B 20. State the definition of boundedness of a sequence. Prove that if $s_n \to +\infty$ as $n \to \infty$ then the sequence $\sqrt{|s_n|}$ is not bounded. You are not allowed to use any theorems from the course; argue directly from the definition of boundedness and the definition of divergence to $+\infty$.