4CCM115A and 5CCM115B Numbers and Functions, Summer 2010
Solutions to the Exam paper

SECTION A

	A	B	C	D	E
A1		$\times$			
A2					$\times$
A3			$\times$		
A4		$\times$			
A5	$\times$				
A6	$\times$				
A7			$\times$		
A8			$\times$		
A9					$\times$
A10			$\times$		
A11		$\times$			
A12	$\times$				
A13				$\times$	
A14		$\times$			
A15			$\times$		
A16				$\times$	

SECTION B

In Section B each question carries a maximum of 20 marks.
Question B17

(i) Give an example of an unbounded sequence which neither diverges to $+\infty$ nor diverges to $-\infty$. No proof is required.

Answer (this example was mentioned in the lectures):

The sequence $(-1)^n n$ satisfies these requirements. [10 marks]

(ii) Give an example of sequences s_n, t_n, $n \in \mathbb{N}$, such that $s_n \neq 0$ for all n, s_n converges, t_n diverges, and $s_n t_n$ converges as $n \to \infty$. No proof is required.

Answer (a similar question was offered as an exercise):

The sequences $s_n = 1/n$, $t_n = (-1)^n$ satisfy these requirements. [10 marks]

Question B18 State the definition of the infimum of a set. For the set

$$S = \left\{ 2(-1)^n + \frac{5}{n^2+2} \;\middle|\; n \in \mathbb{N} \right\},$$

find $\inf S$ and prove your claim. Argue directly from the definition of infimum.

Answer (unseen):

If $S \subset \mathbb{R}$ is bounded below, then the number a is called the infimum of S if a is a lower bound for S and any number $a' > a$ is not a lower bound for S. [3 marks]

In this example, $\inf S = -2$. [3 marks] Let us prove this. Clearly, -2 is a lower bound since

$$2(-1)^n + \frac{5}{n^2+2} \geq -2 + \frac{5}{n^2+2} \geq -2$$

for any $n \in \mathbb{N}$. [3 marks] Next, let us prove that for any $\varepsilon > 0$ the number $-2+\varepsilon$ is not a lower bound for S. We need to show that for any given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$2(-1)^n + \frac{5}{n^2+2} < -2 + \varepsilon. \text{ [3 marks]} \tag{1}$$

Take $n = 2\lceil 5^{1/2}\varepsilon^{-1/2}\rceil + 1$. Then n is odd and $n > 5^{1/2}\varepsilon^{-1/2}$. It follows that

$$n > 5^{1/2}\varepsilon^{-1/2} \;\Rightarrow\; n^2 > 5/\varepsilon \;\Rightarrow\; n^2 + 2 > 5/\varepsilon \;\Rightarrow\; \frac{5}{n^2+2} < \varepsilon.$$

Since n is odd we have

$$2(-1)^n + \frac{5}{n^2+2} = -2 + \frac{5}{n^2+2} < -2 + \varepsilon$$

and so (1) is satisfied as required. [8 marks]

Question B19 State the definition of divergence to $+\infty$. Prove that $n^2 - 100n \to +\infty$ as $n \to \infty$. You are not allowed to use any theorems from the course; argue directly from the definition of divergence to $+\infty$.

Answer (unseen):

A sequence s_n diverges to $+\infty$ if for any $M > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have $s_n > M$. [3 marks]

Given $M > 0$, let us take $n_0 = \max\{200, \lceil\sqrt{2M}\rceil\}$. [3 marks]

Then for $n > n_0$ we have $n > 200$ and so $\frac{1}{2}n > 100$ and $\frac{1}{2}n^2 > 100n$. Also, for $n > n_0$ we have $n^2 > 2M$, so $\frac{1}{2}n^2 > M$. Putting this together, we get that for any $n > n_0$ we have

$$n^2 - 100n = \tfrac{1}{2}n^2 + (\tfrac{1}{2}n^2 - 100n) > \tfrac{1}{2}n^2 > M,$$

as required. [14 marks]

Question B20 State the definition of boundedness of a sequence. Prove that if $s_n \to +\infty$ as $n \to \infty$ then the sequence $\sqrt{|s_n|}$ is not bounded. You are not allowed to use any theorems from the course; argue directly from the definition of boundedness and the definition of divergence to $+\infty$.

Answer (unseen):

A sequence is bounded if it has a lower bound and an upper bound. That is, a sequence s_n is bounded if there exist $a, A \in \mathbb{R}$ such that for any $n \in \mathbb{N}$ one has $a \le s_n \le A$. [3 marks]

Suppose $s_n \to +\infty$ as $n \to \infty$; this means that

$$\forall M > 0 \quad \exists n_0 = n_0(M) \text{ such that } \forall n > n_0 \quad \text{one has } s_n > M. \tag{2}$$

Now let us prove that the sequence $\sqrt{|s_n|}$ is not bounded. It suffices to prove that $\sqrt{|s_n|}$ is not bounded above. That is, we need to prove that for any $A \in \mathbb{R}$ there exist $n \in \mathbb{N}$ such that $\sqrt{|s_n|} > A$. [3 marks]

If $A < 0$, the statement is obvious: $\sqrt{|s_n|} > A$ for any $n \in \mathbb{N}$. Suppose $A \geq 0$. Take $M = A^2$ in (2) and let $n = n_0(A^2) + 1$. Then using (2) we have

$$n > n_0(A^2) \;\Rightarrow\; s_n > A^2 \;\Rightarrow\; \sqrt{|s_n|} > A,$$

as required. [14 marks]